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A numerical verification method for solutions of nonlinear parabolic problems

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Abstract. By using the finite element approximation and constructive a priori error estimates, a new formulation for proving the existence of solutions for nonlinear parabolic problems is presented. We present a method to estimate the norm of the linearized inverse operator for concerned nonlinear problem. Then we formulate a verification principle for solutions by using the Newton-type operator incorporating with Schauder's fixed point theorem.

Keywords. Numerical verification, Guaranteed error bounds, Parabolic problem

1. Introduction

In this paper, we consider a numerical method to verify the existence of solutions for the following nonlinear parabolic problems:

$$\text{(1.1)}\quad \begin{array}{rcll} \frac{\partial}{\partial t}u - \Delta u &=& f(u;x,t) & (x,t) \in Q, \\ u(x,0) &=& 0 & x \in \Omega, \\ u(x,t) &=& 0 & (x,t) \in \partial\Omega \times J. \end{array}$$

where $\Omega \subset R^n$ is a bounded and convex domain ($n = 1,2,3$), $J = (0,T]$ with $T > 0$ and $Q \equiv \Omega \times J$. Here, the nonlinear map f will be prescribed later.

1.1. Functional spaces

Here we denote the usual k-th order L^2 Sobolev space on Ω by $H^k(\Omega)$, and also denote the L^2-inner product and norm on Q by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. Moreover, we introduce the following Sobolev spaces:

$$H(Q) \equiv \{\phi \in H_0(Q) \cap H^1(J;L^2(\Omega)) \ ; \ \phi(x,0) = 0 \text{ in } \Omega\}.$$

where $H_0(Q) \equiv L^2(J;H_0^1(\Omega))$ and $H_0^1(\Omega) \equiv \{\phi \in H^1(\Omega) \ ; \ \phi = 0 \text{ on } \partial\Omega\}$. We define the H_0-norm and H-norm by $\|\phi\|_X \equiv (\nabla\phi, \nabla\phi)^{\frac{1}{2}}$ for $\phi \in H_0(Q)$ and

$$\|\phi\|_{H(Q)} \equiv \left((\frac{\partial}{\partial t}\phi, \frac{\partial}{\partial t}\phi) + (\nabla\phi,\nabla\phi) \right)^{\frac{1}{2}},$$

for $\phi \in H(Q)$, respectively.

1.2. Finite element subspaces and projections

We introduce the finite element subspaces S_h of $H_0^1(\Omega)$ and S^k of $L^2(J)$ depending on the parameter h and k with nodal functions $\{\phi_i\}_{1\le i\le N_h}$ and $\{\psi_i\}_{1\le i\le N_k}$, respectively. Moreover, we denote the finite element subspace $S_h^k := S_h \otimes S^k$ of $H(Q)$ with nodal functions $\{\varphi_i\}_{1\le i\le N}$.

For an arbitrary $u \in H_0^1(\Omega)$, we define the H_0^1-projection $P_x : H_0^1(\Omega) \longrightarrow S_h \subset H_0^1(\Omega)$ by $(\nabla u - \nabla P_x u, \nabla\phi_h)_{L^2(\Omega)} = 0$, for all $\phi_h \in S_h$. Moreover, for an arbitrary $v \in L^2(J)$, we define the L^2-projection $P_t : L^2(J) \longrightarrow S^k \subset L^2(J)$ by $(v - P_t v, \phi^k)_{L^2(J)} = 0$, for all $\phi^k \in S^k$. Also for an arbitrary $u \in H(Q)$, we define the parabolic-projection $P_h^k : H(Q) \longrightarrow S_h^k \subset H(Q)$ by

$$(\frac{\partial}{\partial t}(u - P_h^k u), \varphi_h) + (\nabla u - \nabla P_h^k u, \nabla\varphi_h) = 0, \quad \forall\varphi_h \in S_h^k.$$

Now for each $\psi \in L^2(Q)$, let u be a solution of the following basic parabolic problem

$$\text{(1.2)}\quad \begin{array}{rcll} \frac{\partial}{\partial t}u - \Delta u &=& \psi & (x,t) \in Q, \\ u(x,0) &=& 0 & x \in \Omega, \\ u(x,t) &=& 0 & (x,t) \in \partial\Omega \times J. \end{array}$$

And we denote $u \equiv \Delta_t^{-1}\psi$. Then notice that $P_h^k u$ satisfies the flowing weak form in S_h^k, which implies that $P_h^k u$ coincides with the usual finite element approximation of the problem (1.2).

$$(\frac{\partial}{\partial t}P_h^k u, \varphi_h) + (\nabla P_h^k u, \nabla\varphi_h) = (\psi, \varphi_h), \quad \forall\varphi_h \in S_h^k.$$

The following assumption is natural and our starting point[4].

Assumption 1. There exist positive constants c_0 and c_1 independent of h and k such that, for any $u \in H_0^1(\Omega) \cap H^2(\Omega)$ and $v \in H^1(J)$,

$$\begin{array}{rcl} \|u - P_x u\|_{L^2(\Omega)} &\le& (c_1 h)^2 \|\Delta u\|_{L^2(\Omega)}, \\ \|\nabla u - \nabla P_x u\|_{L^2(\Omega)} &\le& c_1 h \|\Delta u\|_{L^2(\Omega)}, \\ \|v - P_t v\|_{L^2(J)} &\le& c_0 k \|\frac{\partial}{\partial t} v\|_{L^2(J)}. \end{array}$$

Here, h and k correspond to the maximum mesh size in space and time directions, respectively.

2. Constructive a priori error estimates

In this section, suppose that u is a solution of (1.2). Then note that $u \in H(Q) \cap L^2(J; H^2(\Omega))$.

Lemma 1. *[1] For an arbitrary $\varphi_h \in S_h^k$, we have*

$$\|u - P_h^k u\|_X^2 \leq 2\|\tfrac{\partial}{\partial t}(u - P_h^k u)\|\|u - \varphi_h\| + \|\nabla u - \nabla \varphi_h\|^2.$$

Lemma 2. *We have the following estimates.*

$$\begin{aligned} \|u - P_t P_x u\| &\leq C_0(h^2, k)\|\psi\|, \\ \|\nabla u - \nabla P_t P_x u\| &\leq C_1(h, \sqrt{k})\|\psi\|, \end{aligned}$$

where $C_0(h^2, k) := \sqrt{4(c_1 h)^4 + (c_0 k)^2}$ and $C_1(h, \sqrt{k}) := \sqrt{4(c_1 h)^2 + 2c_0 k}$.

Proof. By simple computations and Assumption 1, it implies that

$$\begin{aligned} \|u - P_t P_x u\|^2 &= \|u - P_t u + P_t(u - P_x u)\|^2 \\ &\leq \|u - P_t u\|^2 + \|P_t(u - P_x u)\|^2 \\ &\leq (c_0 k)^2 \|\frac{\partial}{\partial t} u\|^2 + \|u - P_x u\|^2 \\ &\leq (c_0 k)^2 \|\frac{\partial}{\partial t} u\|^2 + (c_1 h)^4 \|\Delta u\|^2, \\ \|\nabla u - \nabla P_t P_x u\|^2 &= \|\nabla u - \nabla P_x u + \nabla P_x(u - P_t u)\|^2 \\ &\leq \|\nabla u - \nabla P_x u\|^2 + \|\nabla P_x(u - P_t u)\|^2 \\ &\leq (c_1 h)^2 \|\Delta u\|^2 + \|\nabla(u - P_t u)\|^2 \\ &= (c_1 h)^2 \|\Delta u\|^2 - (u - P_t u, \Delta u) \\ &\leq (c_1 h)^2 \|\Delta u\|^2 + c_0 k \|\frac{\partial}{\partial t} u\|\|\Delta u\|. \end{aligned}$$

Hence using inequalities $\|\frac{\partial}{\partial t} u\| \leq \|\psi\|$ and $\|\Delta u\| \leq 2\|\psi\|$, we can obtain

$$\begin{aligned} \|u - P_t P_x u\|^2 &\leq \left((c_0 k)^2 + 4(c_1 h)^4\right)\|\psi\|^2, \\ \|\nabla u - \nabla P_t P_x u\|^2 &\leq \left(4(c_1 h)^2 + 2c_0 k\right)\|\psi\|^2. \end{aligned}$$

Therefore, this proof is completed. □

Using Lemmas 1 and 2, we obtain the following constructive a priori error estimation.

Theorem 2. *The following estimates hold true.*

$$\|u - P_h^k u\|_X \leq C(h, \sqrt{k})\|\psi\|,$$

where

$$C(h, \sqrt{k}) \equiv \sqrt{2C_0(h^2, k)(1 + \sigma) + C_1(h, \sqrt{k})^2}$$

and $\sigma > 0$ is a constant satisfying $\|\frac{\partial}{\partial t} P_h^k u\| \leq \sigma\|\psi\|$. Note that σ can be numerically determined by solving some matrix eigenvalue problems.

3. Norm of the linearized inverse operator

In order to formulate a verification algorithm by using an infinite dimensional Newton-like method, we need the norm estimation for the linearized inverse operator of the original nonlinear parabolic problems.

First, we consider the solvability of the linear parabolic problem of the form

$$(3.1) \quad \begin{aligned} \mathcal{L}u \equiv \tfrac{\partial}{\partial t} u - \Delta u + b \cdot \nabla u + cu &= g \quad (x,t) \in Q, \\ u(x, 0) &= 0 \quad x \in \Omega, \\ u(x, t) &= 0 \quad (x,t) \in \partial\Omega \times J, \end{aligned}$$

where $g \in L^2(Q)$. We assume that $b \in L^\infty(J; W_\infty^1(\Omega)^n)$, $c \in L^\infty(Q)$. It is well-known that the operator $\mathcal{L}$ defined by (3.1) is invertible. Thus we show a numerical method to estimate the norm for $\mathcal{L}^{-1}$ in the below.

Now according to the usual verification principle, e.g.,[2][3], we formulate a sufficient condition for which the equation (3.1) has a unique solution. As the preliminary, letting

$$a_h(v_h, w_h) \equiv (\frac{\partial}{\partial t} v_h, w_h) + (\nabla v_h, \nabla w_h),$$

for $v_h, w_h \in S_h^k$, we define the matrices $\mathbf{G} = (\mathbf{G}_{i,j})$, $\mathbf{L} = (\mathbf{L}_{i,j})$ and $\mathbf{D} = (\mathbf{D}_{i,j})$ by : for $1 \leq i, j \leq N$

$$\begin{aligned} \mathbf{G}_{i,j} &= a_h(\varphi_j, \varphi_i) + (b \cdot \nabla\varphi_j, \varphi_i) + (c\varphi_j, \varphi_i), \\ \mathbf{D}_{i,j} &= (\nabla\varphi_j, \nabla\varphi_i), \\ \mathbf{L}_{i,j} &= (\varphi_j, \varphi_i). \end{aligned}$$

Let $\mathbf{D}^{\frac{1}{2}}$ and $\mathbf{L}^{\frac{1}{2}}$ be lower triangular matrices satisfying the Cholesky decomposition: $\mathbf{D} = \mathbf{D}^{\frac{1}{2}}\mathbf{D}^{\frac{T}{2}}$ and $\mathbf{L} = \mathbf{L}^{\frac{1}{2}}\mathbf{L}^{\frac{T}{2}}$, respectively. And we denote the matrix norm by $\|\cdot\|_E$ induced from the Euclidean norm $|\cdot|_E$ in R^N. Also we define the following constants:

$$K_b := \|\, |b|_E \,\|_{L^\infty(Q)}, \quad K_c := \|c\|_{L^\infty(Q)},$$

where $\|\cdot\|_{L^\infty(Q)}$ means L^∞-norm on Q. Here, e.g. for $N = 2$, $|b|_E = \sqrt{b_1(x,t)^2 + b_2(x,t)^2}$. Let $c_p > 0$ be a Poincaré constant such that $\|\phi\| \leq c_p\|\phi\|_X$ for each $\phi \in H_0(Q)$. Then we have the following main result of this paper.

Theorem 3. *Let $\gamma \equiv C(h, \sqrt{k})\tau(M\tau + 1)$, where $M \equiv \|\mathbf{D}^{\frac{T}{2}}\mathbf{G}^{-1}\mathbf{L}^{\frac{1}{2}}\|_E$ and $\tau \equiv K_b + c_p K_c$. If $\gamma < 1$ then for any $g \in L^2(Q)$, a unique solution $u \in H_0(Q)$ of the equation $\mathcal{L}u = g$ satisfies*

$$\|u\|_X \leq \mathcal{M}\|g\|,$$

where $\mathcal{M} \equiv \left(M + C(h, \sqrt{k})(\kappa_1 + \kappa_2)\right)$ and

$$\kappa_1 := \frac{1}{1 - \gamma}(M\tau + 1),\ \kappa_2 := M\tau\kappa_1.$$

Proof. Let $\psi := \Delta_t^{-1} g \in H(Q) \cap L^2(J; H^2(\Omega))$. Then we can rewrite the equation $\mathcal{L}u = g$ as $u = Au + \psi$, where the compact operator $A : H_0(Q) \longrightarrow H_0(Q)$ is defined by

$Au := -\Delta_t^{-1}(b \cdot \nabla u + cu)$. As in [3], we decompose the equation $u = Au + \psi$ as

$$\begin{aligned} P_h^k u &= P_h^k Au + P_h^k \psi, \\ (I - P_h^k)u &= (I - P_h^k)Au + (I - P_h^k)\psi, \end{aligned}$$

where I implies the identity map on $H_0(Q)$. Here we define two operators by

$$\begin{aligned} N_h u &\equiv P_h^k u - [I - A]_{S_h^k}^{-1}(I - P_h^k)Au + [I - A]_{S_h^k}^{-1}\psi, \\ Tu &\equiv N_h u + (I - P_h^k)Au + (I - P_h^k)\psi, \end{aligned}$$

respectively, where $[I - A]_{S_h^k}^{-1}$ means the inverse of $P_h^k(I - A)|_{S_h^k} : S_h^k \longrightarrow S_h^k$. The existence of the operator $[I - A]_{S_h^k}^{-1}$ can be verified by some guaranteed numerical computations in computer. Then the equation $u = Au + \psi$ is equivalent to $u = Tu$. Setting $u_* := (I - P_h^k)u$, we have

$$\begin{aligned} N_h u &= P_h^k u - [I - A]_{S_h^k}^{-1} P_h^k(u - Au) + [I - A]_{S_h^k}^{-1} P_h^k \psi \\ &= [I - A]_{S_h^k}^{-1} P_h^k Au_* + [I - A]_{S_h^k}^{-1} P_h^k \psi. \qquad (3.2) \end{aligned}$$

Since $P_h^k Au_* = -P_h^k \Delta_t^{-1}(b \cdot \nabla u_* + cu_*) \in S_h^k$, the equation (3.2) implies that

$$\begin{aligned} a_h(N_h u, \varphi_h) &+ (b \cdot \nabla N_h u + cN_h u, \varphi_h) \\ &= a_h(P_h^k Au_* + P_h^k \psi, \varphi_h) \\ &= (\frac{\partial}{\partial t}(Au_* + \psi) - \Delta(Au_* + \psi), \varphi_h) \\ &= (-b \cdot \nabla u_* - cu_* + g, \varphi_h) \\ &= (\varphi, \varphi_h) \\ &= (P_0 \varphi, \varphi_h), \end{aligned}$$

for all $\varphi_h \in S_h^k$, where $\varphi := -b \cdot \nabla u_* - cu_* + g \in L^2(J; L^2(\Omega))$ and $P_0 : L^2(Q) \longrightarrow S_h^k$ is the L^2-projection such that $(\varphi - P_0\varphi, \phi_h) = 0$ for all $\phi_h \in S_h^k$. Note that $\|P_0\varphi\| \leq \|\varphi\|$. Now denoting

$$N_h u := \sum_{j=1}^{N} w_j \phi_j \quad \text{and} \quad P_0 \varphi := \sum_{j=1}^{N} v_j \phi_j,$$

for the basis $\{\phi_j\}_{1 \leq j \leq N}$ of S_h^k, we have a matrix equation of the form

$$\mathbf{G}\vec{w} = \mathbf{L}\vec{v}.$$

Here $\vec{w} = (w_1, w_2, \cdots, w_N)^T$ and $\vec{v} = (v_1, v_2, \cdots, v_N)^T$ are coefficient vectors of $N_h u$ and $P_0\varphi$, respectively. Thus it implies that

$$\begin{aligned} \|N_h u\|_X^2 &= \vec{w}^T \mathbf{D} \vec{w} \\ &= \vec{w}^T \mathbf{D} \mathbf{G}^{-1} \mathbf{L} \vec{v} \\ &= (\vec{w}^T \mathbf{D}^{\frac{1}{2}})(\mathbf{D}^{\frac{T}{2}} \mathbf{G}^{-1} \mathbf{L}^{\frac{1}{2}})(\mathbf{L}^{\frac{T}{2}} \vec{v}) \\ &\leq \|\mathbf{D}^{\frac{T}{2}} \vec{w}\|_E \|\mathbf{D}^{\frac{T}{2}} \mathbf{G}^{-1} \mathbf{L}^{\frac{1}{2}}\|_E \|\mathbf{L}^{\frac{T}{2}} \vec{v}\|_E \\ &= \|N_h u\|_X \|\mathbf{D}^{\frac{T}{2}} \mathbf{G}^{-1} \mathbf{L}^{\frac{1}{2}}\|_E \|P_0 \varphi\|. \end{aligned}$$

By some simple calculations, it holds that

$$\begin{aligned} \|\varphi\| &= \| - b \cdot \nabla u_* - cu_* + g\| \\ &\leq K_b \|u_*\|_X + K_c \|u_*\| + \|g\| \\ &\leq (K_b + c_p K_c)\|u_*\|_X + \|g\|, \end{aligned}$$

where we have used the fact that $\|u_*\| \leq c_p \|u_*\|_X$. Thus defining $M \equiv \|\mathbf{D}^{\frac{T}{2}} \mathbf{G}^{-1} \mathbf{L}^{\frac{1}{2}}\|_E$, we obtain

$$\begin{aligned} \|N_h u\|_X \leq M\|P_0\varphi\| &\leq M\|\varphi\| \\ &\leq M\left(\tau\|u_*\|_X + \|g\|\right), \end{aligned}$$

where $\tau \equiv K_b + c_p K_c$. Therefore, by the triangle inequality, we have

$$\begin{aligned} \|(I - P_h^k)(Au + \psi)\|_X &\leq C(h, \sqrt{k})\left(\|b \cdot \nabla u + cu\| + \|g\|\right) \\ &\leq C(h, \sqrt{k})\left(\tau\|u\|_X + \|g\|\right) \\ &\leq C(h, \sqrt{k})\left(\tau\|P_h^k u\|_X + \tau\|u_*\|_X + \|g\|\right). \end{aligned}$$

Since the unique solution $u \in H_0(Q)$ of (3.1) satisfies $u = Tu$, it implies that

$$P_h^k u = N_h u, \quad (I - P_h^k)u = (I - P_h^k)Au + (I - P_h^k)\psi.$$

Hence we can obtain

$$\begin{aligned} \|P_h^k u\|_X &\leq M\tau\|(I - P_h^k)u\|_X + M\|g\|, \\ \|(I - P_h^k)u\|_X &\leq C(h, \sqrt{k})\left(\tau\|P_h^k u\|_X + \tau\|(I - P_h^k)u\|_X + \|g\|\right). \end{aligned}$$

If $\gamma \equiv C(h, \sqrt{k})\tau(M\tau + 1) < 1$ then substituting the estimate of $\|P_h^k u\|_X$ into the right-hand side of $\|(I - P_h^k)u\|_X$ and solving it with respect to $\|(I - P_h^k)u\|_X$, we get

$$\begin{aligned} \|(I - P_h^k)u\|_X &\leq \frac{C(h, \sqrt{k})}{1 - \gamma}(M\tau + 1)\|g\| \\ &= C(h, \sqrt{k})\kappa_1\|g\|, \end{aligned}$$

where $\kappa_1 = (M\tau + 1)/(1 - \gamma)$. Thus setting $\kappa_2 = M\tau\kappa_1$, we also have

$$\begin{aligned} \|P_h^k u\|_X &\leq MC(h, \sqrt{k})\tau\kappa_1\|g\| + M\|g\| \\ &= \left(M + C(h, \sqrt{k})\kappa_2\right)\|g\|. \end{aligned}$$

Therefore, this proof is completed by $\|u\|_X \leq \|P_h^k u\|_X + \|(I - P_h^k)u\|_X$. □

4. Verification algorithms for nonlinear problems

In this section, we mention about the actual applications of the results obtained in the previous section to the verification of solutions for nonlinear parabolic problem (1.1). We assume that the nonlinear map $f(u; x, t)$ from $H(Q)$ into $L^2(Q)$ is continuous and bounded.

Usually, we transform the original parabolic problem (1.1) into the so-called residual equation by using an approximate solution $u_h \in S_h^k \subset H(Q) \cap L^2(J; H^2(\Omega))$ defined by

$$(4.1)\qquad (\frac{\partial}{\partial t}u_h, \varphi_h) + (\nabla u_h, \nabla \varphi_h) = (f(u_h; x,t), \varphi_h) \quad \text{for} \quad \forall \varphi_h \in S_h^k.$$

Setting $w := u - u_h$, concerned problem is reduced to the following residual form

$$(4.2)\qquad \begin{aligned} \tfrac{\partial}{\partial t}w - \Delta w &= f(w+u_h; x,t) - \left(\tfrac{\partial}{\partial t}u_h - \Delta u_h\right) && \textit{in } Q,\\ & w(x,0) = 0 && x \in \Omega,\\ & w(x,t) = 0 && \textit{in } \partial\Omega \times J. \end{aligned}$$

Hence denoting the Fréchet derivative at u_h by $f'(u_h)$, the Newton-type residual equation for (4.2) is written as:

$$(4.3)\qquad \begin{aligned} \mathcal{L}w \equiv \tfrac{\partial}{\partial t}w - \Delta w - f'(u_h)w &= g(w) && (x,t) \in Q,\\ w(x,0) &= 0 && x \in \Omega,\\ w(x,t) &= 0 && (x,t) \in \partial\Omega \times J, \end{aligned}$$

where $g(w) \equiv f(w+u_h; x,t) - \left(\frac{\partial}{\partial t}u_h - \Delta u_h\right) - f'(u_h)w$. Then the equation (4.3) is rewritten as the fixed point form

$$w = F(w) \left(\equiv \mathcal{L}^{-1}g(w)\right).$$

We consider the set, which we often refer as the *candidate set*, of the form

$$W_{\alpha,\beta} \equiv \{w \in H(Q) : \|w\|_X \le \alpha,\ \|\frac{\partial}{\partial t}w\| \le \beta\}.$$

Then the Newton-like operator $F : H_0(Q) \to H_0(Q)$ becomes compact on $W_{\alpha,\beta}$, and is expected to be a contraction map on some neighborhood of zero.

First for the existential condition of solutions, we need to choose the set $W_{\alpha,\beta}$, which is equivalent to determine positive numbers α and β, satisfying the following criterion based on Schauder's fixed point theorem:

$$(4.4)\qquad F(W_{\alpha,\beta}) \subset W_{\alpha,\beta}.$$

Next for the proof of local uniqueness within $W_{\alpha,\beta}$, the following contraction property is needed on the same set $W_{\alpha,\beta}$ in (4.4):

$$(4.5)\qquad \begin{aligned} \|F(w_1) &- F(w_2)\|_{H(Q)}\\ &\le \lambda\|w_1 - w_2\|_{H(Q)}, \forall w_1, w_2 \in W_{\alpha,\beta}, \end{aligned}$$

for some constant $0 < \lambda < 1$. Notice that, in the above case, Schauder's fixed point theorem can be replaced by Banach's fixed point theorem.

For (4.4), by using the same constant $\mathcal{M}$ in the theorem 3, a sufficient condition can be written as

$$\begin{aligned} \sup_{w \in W_{\alpha,\beta}} \|F(w)\|_X &\le \mathcal{M} \sup_{w \in W_{\alpha,\beta}} \|g(w)\| < \alpha,\\ \sup_{w \in W_{\alpha,\beta}} \|\frac{\partial}{\partial t}F(w)\| &\le \sup_{w \in W_{\alpha,\beta}} \|g(w) + f'(u_h)F(w)\|\\ &\le \mathcal{N} \sup_{w \in W_{\alpha,\beta}} \|g(w)\| < \beta, \end{aligned}$$

where $\mathcal{N} \equiv 1 + \mathcal{M}\tau$. Here, we assumed the equality $f'(u_h)\phi = -b \cdot \nabla\phi - c\phi$ holds for the coefficient functions b and c in (3.1).

On the other hand, for the verification of local uniqueness condition (4.5) on $W_{\alpha,\beta}$, in general, we use the following deformation:

$$g(w_1) - g(w_2) = \Phi(w_1, w_2)(w_1 - w_2),$$

where $\Phi(w_1, w_2)$ denotes a function in w_1 and w_2, for example, if $g(w) = w^2$, then $\Phi(w_1, w_2) = w_1 + w_2$. Therefore, the condition (4.5) reduces to find a constant $0 < \lambda < 1$ satisfying the inequalities of the form

$$\begin{aligned} \mathcal{M}\|\Phi(w_1, w_2)(w_1 - w_2)\| &\le \lambda\|w_1 - w_2\|_X,\\ \mathcal{N}\|\Phi(w_1, w_2)(w_1 - w_2)\| &\le \lambda\|\frac{\partial}{\partial t}(w_1 - w_2)\|, \end{aligned}$$

for all $w_1, w_2 \in W_{\alpha,\beta}$.

Concluding remarks: We derived a constructive a priori error estimates for the finite element approximation defined on the whole domain of space and time of the basic linear parabolic problems. By using this result, we presented a verification principle based on a Newton-like method for the solutions of nonlinear parabolic problems. In general, some constants included in the error estimates seem to be not necessarily effective when the time interval J is large. Therefore, in order to apply the method for more realistic problem than the prototype example, e.g., in [1], we would need to develop a technique based on the step by step method in time.

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