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A comprehensive view of Lagrangian invariants of hydrodynamics, ideal and Hall magnetohydrodynamics on three-dimensional Riemannian manifold

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Abstract. Lagrangian invariants of hydrodynamic, magnetohydrodynamic (MHD) and Hall MHD fluids are reviewed in a general viewpoint of differential topology. It is shown that, introducing the particle trajectory map (PTM) and its inverse (back-to-labels map, BLM) and utilizing their spatial derivatives, one can easily derive the conservation laws along the Lagrangian trajectories. All the invariants are derived as composite of such elementary invariants as entropy per unit mass, impulse, mass density, and electromagnetic vector potential and their derivatives. Treating the spatial derivatives of PTM and BLM as kinds of Lagrangian invariants formally, one can understand the following conservation laws as Lagrangian invariants: Cauchy's formula, Weber's transformation, Ertel's theorem, Ertel-Rossby's theorem (i.e. helicity density), magnetic-helicity and cross-helicity in a MHD fluid, hybrid-helicity in a Hall MHD fluid.

Keywords. Lagrangian invariants, conservation laws, differential topology, barotropic fluids

1. Introduction

Direct numerical simulation (DNS) is nowadays the most basic tool for understanding the nature of fluid motions. Recently Constantin developed such a numerical scheme that is a kind of hybrid of the Lagrangian and the Eulerian specifications of flow[1], and numarical studies with this scheme were carried out for an incompressible neutral fluid by Ohkitani and Constantin[2] and an incompressible MHD fluid[3]. By the way, particle methods, for example, moving-particle semi-implicit method (MPS) [4] or smoothed particle hydrodynamics (SPH) [5], are mainly based on the Lagrangian specification.

For the check of accuracy of DNS, the conservation laws that the model equation satisfies in the disspationless limit provide important benchmarks. It may be very useful if there is a list of the Lagrangian invariants, the quantities that is conserved on each fluid element for the accuracy check of the Lagrangian specification based numerical schemes.

Recently, Fukumoto has discussed the nature of "topological invariants" from a general analytical mechanical viewpoint[6]. In this paper we focus our discussion on the more formal and mathematical aspects of these Lagrangian invariants. One of such formal discussions has been given by Tur and Yanovsky[7]. They showed a general way to construct Lagrangian invariants though their notations and presentating way of invariants seem to be slightly confusing. In this paper we extend their methodology in somewhat systematic way. It is shown that introduction of the maps for tracking the Lagrangian trajectories of fluid motion into the construction of Lagrangian invariants enables us to simplify some mathematical calculations and to discuss wider classes of invariants, which includes the well-known hydrodynamic formulae such as Cauchy's formula and Weber's transformation.

2. Lagrangian description of fluid motion

Let $\mathcal{M}$ be a "container of a fluid", a three dimensional Riemannian manifold with metric $g = (g_{ij})$. It is assumed that a global coordinate system $\vec{q} = (q^i) = (q^1, q^2, q^3)$ is defined on it.

2.1. Remarks on mathematical notations

It should be remarked here about the mathematical expressions used in the present work.

In this article we use the notation with arrow on top to denote point on $\mathcal{M}$. It should be emphasized that the point on $\mathcal{M}$ is not a vector in its strict sense unless $\mathcal{M}$ is the Euclidean space $\mathcal{M} = E^3$. For example, the addition of components of two points $\vec{p}+\vec{q} = (p^1 + q^1, p^2 + q^2, p^3 + q^3)$ may indicate outside of $\mathcal{M}$ unless one of them has sufficiently small magnitude. The subtraction of components of two points $\vec{p}-\vec{q} = (p^1 - q^1, p^2 - q^2, p^3 - q^3)$ makes sense when these two positions are sufficiently close to each other.

On the other hand, we will use boldface fonts, for example $\boldsymbol{u}$, $\boldsymbol{A}$, etc., to denote vector or tensor fields on $\mathcal{M}$, each of which constitutes a genuine vector space.

In many literatures of hydrodynamics, the letters $\vec{a}$ and $\vec{x}$ are often used to denote the Lagrangian label of a fluid particle and the Eulerian coordinate, respectively. However, this distinction has only *physical* implication, since *mathematically* a unique coordinate system (q^i) is assigned on the fluid container $\mathcal{M}$ and specific values of $\vec{a}$ and $\vec{x}$ are measured in the unit of (q^i). Thus, the independent variables of a function on $\mathcal{M}$ (say F) are q^i's so that only the notation $\partial F/\partial q^i$ appears to denote the partial derivatives of F. Notations such as $\partial F/\partial a^i$, $\partial F/\partial x^i$, $\nabla_a F$, $\nabla_x F$, etc. do not appear in the following.

It should be emphasized that, when a function is written with its arguments, it denotes its *value* at an assigned position and a time which are given in the arguments. For example, the notation $\vec{X}(\vec{a},t)$, which appears in the following, does not imply that $\vec{X}$ is a function of variable $\vec{a}$, but denotes *the value of function* $\vec{X}$ *at the position* $\vec{a}$ where $\vec{a}$ physically implies the label of a fluid particle. Similarly, the notation $\frac{\partial f}{\partial q^i}(\vec{X}(\vec{a},t),t)$ does not imply that f is a function of $\vec{X}$, but denotes *the value of derivative* $\frac{\partial f}{\partial q^i}$ *at the position given by* $\vec{X}(\vec{a},t)$. On the other hand, the notation without argument indicate function itself, whose variables are obviously q^i and t.

In this notation convention, one should be careful about the treatment of partial derivatives when the arguments of functions are different. For example, when $f(\vec{a},0) = f(\vec{X}(\vec{a},t),t)$, the *values* of their partial derivatives satisfy the following relation:

$$
\begin{aligned}
&\frac{\partial f}{\partial q^i}(\vec{a},0)\\
&:= \lim_{\epsilon\to 0}\frac{f(...,a^i+\epsilon,...,0)-f(...,a^i,...,0)}{\epsilon}\\
&= \lim_{\epsilon\to 0}\frac{f(\vec{X}(...,a^i+\epsilon,...,t),t)-f(\vec{X}(...,a^i,...,t),t)}{\epsilon}\\
&= \frac{\partial}{\partial a^i}f(\vec{X}(\vec{a},t),t) = \frac{\partial f}{\partial q^j}(\vec{X}(\vec{a},t),t)\frac{\partial X^j}{\partial q^i}(\vec{a},t).
\end{aligned}
$$

In such cases we will partly use partial differentiation *operator* notation with respect to a^i for convenience.

2.2. Particle trajectory map and back-to-labels map

In order to describe fluid motion which may be a solution of the Euler or the Navier-Stokes equation or be a prescribed flow, we introduce here particle trajectory map (PTM)

$$
\begin{aligned}
\vec{X} &= \vec{X}(\vec{q},t)\\
&= (X^1(q^1,q^2,q^3,t), X^2(q^1,q^2,q^3,t), X^3(q^1,q^2,q^3,t))
\end{aligned}
$$

which is such a triplet of functions that maps the initial position or the "label" of a fluid particle $\vec{a}$ to the position $\vec{X}(\vec{a},t)$ at the time t[1].

[1] Only the mathematical expressions in the explanations of PTM and BLM are the exceptions of the argument writing rule discussed in §2.1.

The velocity of a fluid particle labeled by $\vec{a}$ is given by the tangent vector of its trajectory[2]

$$
\begin{aligned}
\frac{\partial \vec{X}}{\partial t}(\vec{a},t) &:= \lim_{\tau\to 0}\frac{\vec{X}(\vec{a},t+\tau)-\vec{X}(\vec{a},t)}{\tau}\\
&= \frac{\partial X^i}{\partial t}(\vec{a},t)\left(\frac{\partial}{\partial q^i}\right)_{\vec{X}(\vec{a},t)}
\end{aligned}
\tag{1}
$$

where subindex of the vector field basis $\partial/\partial q^i$ is the position where the velocity vector stems. This is the exact explicit expression of the *Lagrangian velocity* in conventional notation of differential geometry. It should be emphasized that, since the argument of component functions $\vec{a}$ and that of basis $\vec{X}(\vec{a},t)$ do not agree with each other, the Lagrangian velocity is not any contravariant or covariant vector field in its strict sense and is inconvenient to calculate some properties in terms of differential topology[3]. In order to match the argument of components and basis, we introduce, therefore, such a contravariant vector field (say $\boldsymbol{u} = u^i\partial/\partial q^i$) whose components satisfy

$$
u^i\big(\vec{X}(\vec{a},t),t\big) := \frac{\partial X^i}{\partial t}(\vec{a},t). \tag{2}
$$

The vector field $\boldsymbol{u}$ is the *Eulerian velocity* associated with the PTM $\vec{X}$.

Let us introduce here "back-to-labels" map (BLM, cf. Ref.[1])

$$
\begin{aligned}
\vec{\Lambda} &= \vec{\Lambda}(\vec{q},t)\\
&= (\Lambda^1(q^1,q^2,q^3,t), \Lambda^2(q^1,q^2,q^3,t), \Lambda^3(q^1,q^2,q^3,t))
\end{aligned}
$$

which is the inverse of the PTM $\vec{X}$:

$$
\vec{X}(\vec{\Lambda}(\vec{q},t),t) = \vec{q}, \qquad \vec{\Lambda}(\vec{X}(\vec{q},t),t) = \vec{q}. \tag{3}
$$

Differentiating the second equation in Eq.(3) with respect to t and substituting Eq.(2), we obtain the following partial differential equation (PDE) that each component of BLM obeys:

$$
\frac{\partial \Lambda^i}{\partial t} + u^j\frac{\partial \Lambda^i}{\partial q^j} = 0. \tag{4}
$$

Differentiating Eqs.(3) and substituting the Eulerian position $\vec{x}$ and the Lagrangian label $\vec{a}$, respectively, one obtains the following inverse matrix relations:

$$
\begin{aligned}
&\frac{\partial X^i}{\partial q^k}(\vec{\Lambda}(\vec{x},t),t)\frac{\partial \Lambda^k}{\partial q^j}(\vec{x},t) = \delta^i_j,\\
&\frac{\partial \Lambda^k}{\partial q^j}(\vec{X}(\vec{a},t),t)\frac{\partial X^i}{\partial q^k}(\vec{a},t) = \delta^i_j.
\end{aligned}
\tag{5}
$$

In terms of the cofactor and the determinant the inverse matrix relations can be written as

$$
\frac{\mathrm{cof}\left(\frac{\partial X^i}{\partial q^j}(\vec{a},t)\right)}{\det\left(\frac{\partial X^i}{\partial q^j}(\vec{a},t)\right)} = \frac{\partial \Lambda^j}{\partial q^i}(\vec{X}(\vec{a},t),t), \tag{6}
$$

[2] In this paper Einstein's summation convention is used.

[3] It seems somewhat misleading to use $\partial\boldsymbol{X}/\partial t = \boldsymbol{v}(\boldsymbol{a},t)$ to denote the Lagrangian velocity because $\boldsymbol{v}$ is not a proper object of the differential topology.

where $\mathrm{cof}(M_i^j)$ and $\det(M_i^j)$ stand for the cofactor of the component M_i^j and the determinant of matrix (M_i^j), respectively.

In the next section we will see that the derivatives of PTM and BLM act as "transformation matrices" for the frozen-in fields.

3. Frozen-in fields

In this section we summarize "frozen-in" scalar, vector and tensor fields generally. By $\tilde{X}_t$ we denote the transformation of functions, vectors and tensors induced by the PTM $\vec{X}$. The condition that a field (say $\boldsymbol{A}_t = \boldsymbol{A}(\vec{q}, t)$) is frozen-in is defined by $\boldsymbol{A}_t = \tilde{X}_t \boldsymbol{A}_0$. Differentiating this equation with respect to t, one obtains $\partial \boldsymbol{A}/\partial t_t = -L_{\boldsymbol{u}} \boldsymbol{A}_t$, where L is the Lie derivative[8]. It should be remarked that, as we will see in the following, the specific expressions of the Lie derivatives vary depending on the tensor type.

Our consideration starts with the "frozen-in" function or, in the terminology of hydrodynamics, dissipationless passive scalar. Then the advection of a contravariant vector field is discussed. Finally we derive the formulae for differential n-forms.

3.1. Advection of a function and a contravariant vector field

The advection of a function and a vector field constitute building blocks for Lagrangian invariants. The advection of a function (say f) is defined by

$$f(\vec{X}(\vec{a}, t), t) = f(\vec{a}, 0). \tag{7}$$

Physically this equation implies that the value of function is conveyed by the fluid motion without diffusion. Differentiating this equation with respect to t and substituting the relation Eq.(2), we obtain the following PDE of Eulerian fields[4]:

$$\frac{\partial f}{\partial t} + u^j \frac{\partial f}{\partial q^j} = 0. \tag{8}$$

It should be remarked that each component of BLM is formally a "frozen-in" function.

Advection of a "frozen-in" vector field (say $\boldsymbol{\xi} = \xi^i \partial/\partial q^i$) is defined by the difference between two very close particles that are conveyed by the flow $\vec{X}$. That is, the value of a "frozen-in" vector field at $\vec{X}(\vec{a}, t)$ is given by the following differentiation:

$$\xi^i(\vec{X}(\vec{a}, t), t) \left(\frac{\partial}{\partial q^i}\right)_{\vec{X}(\vec{a}, t)} := \lim_{\epsilon \to 0} \frac{\vec{X}(\vec{b}, t) - \vec{X}(\vec{a}, t)}{\epsilon} \tag{9}$$

where $\vec{b}$ is the position of a fluid particle given in terms of initial vector field as

$$b^i := a^i + \epsilon \xi^i(\vec{a}, 0) + o(\epsilon). \tag{10}$$

Substituting Eq.(10) into the RHS of Eq.(9), one can derive the following relation between the components at the initial time and the time t:

$$\xi^i(\vec{X}(\vec{a}, t), t) = \xi^k(\vec{a}, 0) \frac{\partial X^i}{\partial q^k}(\vec{a}, t). \tag{11}$$

Multiplying $(\partial \Lambda^k/\partial q^i)(\vec{X}(\vec{a}, t), t)$ on both sides of Eq.(11), one obtain

$$\xi^i(\vec{X}(\vec{a}, t), t) \frac{\partial \Lambda^k}{\partial q^i}(\vec{X}(\vec{a}, t), t) = \xi^k(\vec{a}, 0). \tag{12}$$

Differentiating this equation with respect to t and substituting Eqs.(2) and (4), we obtain the following well-known PDE for each component of a frozen-in vector field[5]:

$$\frac{\partial \xi^i}{\partial t} + u^j \frac{\partial \xi^i}{\partial q^j} - \xi^j \frac{\partial u^i}{\partial q^j} = 0. \tag{13}$$

In other words Eq.(12) is the integral of Eq.(13) over time interval [0,t] for the flow history given by $\vec{X}$.

3.2. Advection of differential n-forms

The advection of differential n-form is obtained by composition of the basic mathematical building blocks. Discussion on the frozen-in tensors begins with the commutativity of the following basic operations with the transformation $\tilde{X}_t$ [8]:

1. product of a function and a tensor and tensor product $\otimes$ of the tensors of arbitrary order: $\tilde{X}_t(f\boldsymbol{A}) = \tilde{X}_t f\, \tilde{X}_t \boldsymbol{A}$, $\tilde{X}_t(\boldsymbol{A} \otimes \boldsymbol{B}) = \tilde{X}_t \boldsymbol{A} \otimes \tilde{X}_t \boldsymbol{B}$,
2. contraction between contravariant and covariant components, symmetrization and skew-symmetrization of tensors: $\tilde{X}_t(\mathrm{Tr}(\boldsymbol{A} \otimes \boldsymbol{B})) = \mathrm{Tr}(\tilde{X}_t \boldsymbol{A} \otimes \tilde{X}_t \boldsymbol{B})$, etc.,
3. exterior differentiation d on the differential forms (skew-symmetric covariant tensors): $\tilde{X}_t(d\boldsymbol{A}) = d(\tilde{X}_t \boldsymbol{A})$.

Combination of the operations 1 and 2 leads to the commutativity of the wedge product: $\tilde{X}_t(\boldsymbol{A} \wedge \boldsymbol{B}) = \tilde{X}_t \boldsymbol{A} \wedge \tilde{X}_t \boldsymbol{B}$. These relations hold a key to understand the general method to construct Lagrangian invariants given by Tur and Yanovsky[7].

3.2.1. Commutativity of tensor operations and transformation

The first example of application of the commutative relations is the advection of a differential 1-form (or a covariant vector field). It is defined through the fact that differential 1-form is a linear functional of a vector field. That is, the inner product between a vector field and a 1-form (say $\boldsymbol{\eta} = \eta_i dq^i$) is a function (0-form) on $\mathcal{M}$ so that it is advected by the rule Eq.(7):

$$\eta_j(\vec{X}(\vec{a}, t), t)\, \xi^j(\vec{X}(\vec{a}, t), t) = \eta_i(\vec{a}, 0)\, \xi^i(\vec{a}, 0). \tag{14}$$

[4]This equation gives the expression of the Lie derivative for a function as $L_{\boldsymbol{u}} f = u^i \partial f/\partial q^i$.

[5]This equation gives the expression of the Lie derivative for a contravariant vector field as $L_{\boldsymbol{u}} \boldsymbol{\xi} = (u^j \partial \xi^i/\partial q^j - \xi^j \partial u^i/\partial q^j)\partial/\partial q^i$.

Substituting the transformation rule for a contravariant vector field Eq.(11), one obtains the transformation relation of the covariant components

$$\eta_j(\vec{X}(\vec{a},t),t)\frac{\partial X^j}{\partial q^i}(\vec{a},t) = \eta_i(\vec{a},0) \tag{15}$$

Differentiating this equation with respect to t and substituting Eq.(2), we obtain the PDE for the "frozen-in" 1-form[6]:

$$\frac{\partial \eta_j}{\partial t} + u^k\frac{\partial \eta_j}{\partial q^k} + \eta_k\frac{\partial u^k}{\partial q^j} = 0. \tag{16}$$

As a consequence of the commutativity, the components of "frozen-in" differential 2- and 3-forms (say $\boldsymbol{\omega} = \omega_{ij}dq^i \wedge dq^j$ and $\boldsymbol{\rho} = \rho_{ijk}dq^i \wedge dq^j \wedge dq^k$, respectively) are found to obey the following transformation rules:

$$\omega_{kl}(\vec{X}(\vec{a},t),t)\frac{\partial X^k}{\partial q^i}(\vec{a},t)\frac{\partial X^l}{\partial q^j}(\vec{a},t) = \omega_{ij}(\vec{a},0), \tag{17}$$

$$\rho_{lmn}(\vec{X}(\vec{a},t),t)\frac{\partial X^l}{\partial q^i}(\vec{a},t)\frac{\partial X^m}{\partial q^j}(\vec{a},t)\frac{\partial X^n}{\partial q^k}(\vec{a},t) = \rho_{ijk}(\vec{a},0). \tag{18}$$

For three dimensional case these transformation rules can be rewritten as

$$\sum_{i=1}^{3}\underline{\omega}^i(\vec{X}(\vec{a},t),t)\,\mathrm{cof}\left(\frac{\partial X^i}{\partial q^j}(\vec{a},t)\right) = \underline{\omega}^j(\vec{a},0), \tag{19}$$

$$\underline{\rho}(\vec{X}(\vec{a},t),t)\,\det\left(\frac{\partial X^i}{\partial q^j}(\vec{a},t)\right) = \underline{\rho}(\vec{a},0), \tag{20}$$

where $\underline{\omega}^j := \frac{1}{2}\epsilon^{jkl}\omega_{kl}$, $\underline{\rho} := \frac{1}{6}\epsilon^{lmn}\rho_{lmn}$ and ϵ^{ijk} is the Levi-Civita density[7].

It should be remarked that, taking into account Eq.(6), we found that the quotient of these coefficients obeys the following transformation rule, which is well known as Cauchy's formula when $\underline{\omega}$ and $\underline{\rho}$ are the vorticity and the mass density, respectively:

$$\begin{aligned}\frac{\underline{\omega}^j(\vec{a},0)}{\underline{\rho}(\vec{a},0)} &= \frac{\sum_{i=1}^{3}\underline{\omega}^i(\vec{X}(\vec{a},t),t)\,\mathrm{cof}\left(\frac{\partial X^i}{\partial q^j}(\vec{a},t)\right)}{\underline{\rho}(\vec{X}(\vec{a},t),t)\,\det\left(\frac{\partial X^i}{\partial q^j}(\vec{a},t)\right)}\\ &= \frac{\underline{\omega}^i(\vec{X}(\vec{a},t),t)}{\underline{\rho}(\vec{X}(\vec{a},t),t)}\frac{\partial \Lambda^j}{\partial q^i}(\vec{X}(\vec{a},t),t).\end{aligned} \tag{21}$$

Comparing this relation with Eq.(12), we observe that the quotients of frozen-in 2-forms coefficient to the 3-forms one work as components of a frozen-in vector field.

[6]This equation gives the expression of the Lie derivative for a covariant vector field as $L_{\boldsymbol{u}}\boldsymbol{\eta} = (u^k\partial\eta_j/\partial q^k + \eta_k\partial u^k/\partial q^j)dq^j$.

[7]Differentiating the equations (19) and (20) with respect to the time t, one obtains the PDE's for 2- and 3-forms as

$$\frac{\partial \underline{\omega}^i}{\partial t} + u^j\frac{\partial \underline{\omega}^i}{\partial q^j} - \underline{\omega}^j\frac{\partial u^i}{\partial q^j} + \underline{\omega}^i\frac{\partial u^j}{\partial q^j} = 0,$$
$$\frac{\partial \underline{\rho}}{\partial t} + u^j\frac{\partial \underline{\rho}}{\partial q^j} + \underline{\rho}\frac{\partial u^j}{\partial q^j} = 0.$$

3.2.2. Commutativity of exterior differentiation and transformation

Another important way to obtain a frozen-in differential form is to operate the exterior differentiation d on a given frozen-in differential form. We comment here the correspondence between exterior differentiation d and grad, curl and div operations and check the commutativity of d and $\tilde{X}$ in terms of components of fields.

Exterior differentiation of 0-form is the gradient of a function. The gradient of frozen-in 0-form is given by

$$\begin{aligned}\frac{\partial f}{\partial q^i}(\vec{a},0) &= \frac{\partial}{\partial a^i}f(\vec{X}(\vec{a},t),t)\\ &= \frac{\partial f}{\partial q^j}(\vec{X}(\vec{a},t),t)\frac{\partial X^j}{\partial q^i}(\vec{a},t)\end{aligned} \tag{22}$$

and it is easy to see this relation is the same as the transformation rule Eq.(15).

When the spatial dimension is three, exterior diffentiation of 1-form coincides with the operation of curl. So taking the curl of the frozen-in covariant vector field, one obtains the frozen-in relation for the derivatives of components, which agree with the Eq.(19):

$$\begin{aligned}&\epsilon^{ijk}\frac{\partial \eta_k}{\partial q^j}(\vec{a},0)\\ &= \epsilon^{ijk}\frac{\partial}{\partial a^j}\left(\eta_n(\vec{X}(\vec{a},t),t)\frac{\partial X^n}{\partial q^k}(\vec{a},t)\right)\\ &= \epsilon^{ijk}\frac{\partial \eta_n}{\partial q^m}(\vec{X}(\vec{a},t),t)\frac{\partial X^m}{\partial q^j}(\vec{a},t)\frac{\partial X^n}{\partial q^k}(\vec{a},t).\end{aligned} \tag{23}$$

It is known that divergence of a vector field (say $(\underline{\omega}^i)$) is given by the exterior differentiation of the 2-form whose components are given by $\omega_{jk} := \frac{1}{2}\epsilon_{ijk}\underline{\omega}^i$. So the frozen-in 2-form induces the following relation that agrees with Eq.(18):

$$\begin{aligned}&\epsilon^{ijk}\frac{\partial \omega_{jk}}{\partial q^i}(\vec{a},0)\\ &= \epsilon^{ijk}\frac{\partial}{\partial a^i}\left(\omega_{mn}(\vec{X}(\vec{a},t),t)\frac{\partial X^m}{\partial q^j}(\vec{a},t)\frac{\partial X^n}{\partial q^k}(\vec{a},t)\right)\\ &= \epsilon^{ijk}\frac{\partial \omega_{mn}}{\partial q^l}(\vec{X}(\vec{a},t),t)\frac{\partial X^l}{\partial q^i}(\vec{a},t)\frac{\partial X^m}{\partial q^j}(\vec{a},t)\frac{\partial X^n}{\partial q^k}(\vec{a},t).\end{aligned} \tag{24}$$

4. Construction of Lagrangian invariants

In the previous section we observed that the wedge product and the exterior differentiation of "frozen-in" differential forms are also "frozen-in". Therefore, if we have a list of basic conservation laws, we can construct a wide variety of Lagrangian invariants from these basic ones. In this section we will firstly list up the basic conservation laws. Then the combinations of basic frozen-in fields are considered.

4.1. Basic kinematic and dynamic building blocks for Lagrangian invariants

In this subsection we list up the basic frozen-in fields. Firstly, we list such frozen-in fields with which neutral, MHD and Hall MHD fluids have in common. Then we discuss the frozen-in fields that are specific to each kind of fluid, all of which are given by differential 1-forms.

4.1.1. Common frozen-in 0- and 3-form

There are three kinds of frozen-in fields with which neutral, MHD and Hall MHD fluids have in common. From Eq.(4) one can see that each component of BLM Λ^i for i =1,2,3 is 0-form invariant. Since we treat the dissipationless systems throughout this paper, entropy per unit mass of a fluid s is one of the 0-form Lagrangian invariant[9]: $(\partial/\partial t + L_{\boldsymbol{u}})s = 0$. The gradient of Λ^i and s are the frozen-in 1-forms that are used as building blocks for 3-form invariants.

The mass $M := \rho\sqrt{|g|}\,dq^1 \wedge dq^2 \wedge dq^3$ is an another basic frozen-in field which is a 3-form, i.e. $(\frac{\partial}{\partial t} + L_{\boldsymbol{u}})M = 0$, where ρ and $|g|$ are the mass density distribution and the determinant of the Riemannian metric tensor. The coefficient $\rho\sqrt{|g|}$ is used as the denominator of the 3-form quotient type Lagrangian invariants.

4.1.2. Frozen-in 1-form of a perfect barotropic fluid

In terms of the Lie derivative and exterior derivatives the Euler equation is written as

$$(\frac{\partial}{\partial t} + L_{\boldsymbol{u}})\underline{\boldsymbol{u}} = -\frac{dp}{\rho} - \frac{d|\boldsymbol{u}|^2}{2}$$

where $\boldsymbol{u} = u^i\partial/\partial q^i$, $\underline{\boldsymbol{u}} = u_i dq^i = g_{ij}u^i dq^i$ and $|\boldsymbol{u}|^2 = u^i u_i$. If the fluid is barotropic $\rho = \rho(p)$, the equation of motion can be rewritten as

$$(\frac{\partial}{\partial t} + L_{\boldsymbol{u}})\tilde{\boldsymbol{u}} = 0, \tag{25}$$

where the variable $\tilde{\boldsymbol{u}}$ is the impulse defined by $\tilde{\boldsymbol{u}} = \tilde{u}_j dq^j = (u_j + \partial\psi/\partial q^j)dq^j$ and ψ is such a function that is defined by the integral

$$\psi(\vec{X}(\vec{a},t),t) = \int_0^t \left(\int \frac{dp}{\rho(p)} - \frac{|\boldsymbol{u}|^2}{2}\right)_{(\vec{q},t)=(\vec{X}(\vec{a},\tau),\tau)} d\tau \tag{26}$$

(see Ref.[10]). Applying Eq.(15), we formally obtain the integral of this equation that is known as *Weber's transformation* [11]:

$$\tilde{u}_j(\vec{X}(\vec{a},t),t)\frac{\partial X^j}{\partial q^i}(\vec{a},t) = \tilde{u}_i(\vec{a},0). \tag{27}$$

The impulse 1-form $\tilde{\boldsymbol{u}}$ and its exterior differentiation $d\tilde{\boldsymbol{u}} = d\boldsymbol{u}$, i.e., the vorticity 2-form constitute the building blocks for the Lagrangian invariants of an ideal barotropic fluid.

4.1.3. Frozen-in 1-form of an ideal MHD fluid

The ideal MHD imposes an assumption that the conductivity of a plasma is approximately infinite so that Ohm's law has the following simple form (Ref.[12] §8.4);

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \boldsymbol{0}$$

where $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{u}$ are the electric and magnetic fields of plasma and the mean velocity of macroscopic ion and electron flow, respectively.

Substituting Ohm's law into Faraday's law of induction in potential form $\boldsymbol{E} = -\nabla\phi - \partial\boldsymbol{A}/\partial t$ where ϕ and $\boldsymbol{A}$ are the scalar and vector potentials of electromagnetic fields and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, we obtain the ideal MHD equation for the vector potential 1-form $\boldsymbol{A} = A_i dq^i$ as

$$(\frac{\partial}{\partial t} + L_{\boldsymbol{u}})\boldsymbol{A} = -d\phi + d(\boldsymbol{A}\cdot\boldsymbol{u}). \tag{28}$$

where $\boldsymbol{A}\cdot\boldsymbol{u} = A_i u^i$. This equation can be written in a frozen-in form as

$$(\frac{\partial}{\partial t} + L_{\boldsymbol{u}})\tilde{\boldsymbol{A}} = 0. \tag{29}$$

where $\tilde{\boldsymbol{A}} = (A_i + \partial\tilde{\phi}/\partial q^i)dq^i$ and $\tilde{\phi}$ is such a function given by

$$\tilde{\phi}(\vec{X}(\vec{a},t),t) = \int_0^t (\phi - \boldsymbol{A}\cdot\boldsymbol{u})_{(\vec{q},t)=(\vec{X}(\vec{a},\tau),\tau)} d\tau. \tag{30}$$

The integral of Eq.(29) is obtained in an analogous way as for the impulse of the barotropic fluid:

$$\tilde{A}_j(\vec{X}(\vec{a},t),t)\frac{\partial X^j}{\partial q^i}(\vec{a},t) = \tilde{A}_i(\vec{a},0). \tag{31}$$

It should be remarked here that the impulse is no longer a frozen-in 1-form since the equation of an ideal MHD fluid motion has the Lorentz force term:

$$(\frac{\partial}{\partial t} + L_{\boldsymbol{u}})\tilde{\boldsymbol{u}} = \frac{\boldsymbol{j}\times\boldsymbol{B}}{\rho} \tag{32}$$

where $\boldsymbol{j}$ is the current density of the MHD plasma $\boldsymbol{j} = \mu^{-1}\nabla \times \boldsymbol{B}$.

4.1.4. Frozen-in 1-form of a Hall MHD fluid

Recently Shivamoggi has shown that both the incompressible and the compressible Hall MHD systems can be written in the impulse form[13]. Hall MHD is such an approximation that Ohm's law has the Hall current term:

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \frac{1}{en}\boldsymbol{j} \times \boldsymbol{B}$$

where e, n are the elementary charge and number density of ions and electrons, respectively. The evolution equation of vector potential is given by

$$(\frac{\partial}{\partial t} + L_{\boldsymbol{u}})\tilde{\boldsymbol{A}} = -\frac{1}{en}\boldsymbol{j} \times \boldsymbol{B}. \tag{33}$$

It is easy to see that the Lorentz force term in Eq.(32) and the Hall term in Eq.(33) cancel each other if the mass density is approximated by $\rho = mn$ where m is the ion mass. Thus we obtain the following conservation law for the sum of impulse and vector potential 1-forms:

$$(\frac{\partial}{\partial t} + L_{\boldsymbol{u}})\left(\tilde{\boldsymbol{u}} + \frac{e}{m}\tilde{\boldsymbol{A}}\right) = 0. \tag{34}$$

The bluiding blocks for a Hall MHD fluid, therefore, are $\boldsymbol{w} := \tilde{\boldsymbol{u}} + (e/m)\tilde{\boldsymbol{A}}$ and its exterior derivative $d\boldsymbol{w} := \boldsymbol{\omega} + (e/m)\boldsymbol{B}$.

4.2. Tables of Lagrangian invariants

Now we construct Lagrangian invariants, i.e., frozen-in 0-forms as combinations of basic frozen-in fields. In the first four subsections we list such Lagrangian invariants each of which is given by a quotient of the frozen-in 3-form coefficients. For simplicity the denominator of quotients is fixed to the coeffcient of mass 3-form $\tilde{\rho} := \rho\sqrt{|g|}$.

When the building blocks are, for example, ds, $d\Lambda^i$ and $d\Lambda^j$, the numerator 3-form of corresponding Lagrangian invariant is given by

$$ds \wedge d\Lambda^i \wedge d\Lambda^j = \left(\frac{\partial s}{\partial q^l}dq^l\right) \wedge \left(\frac{\partial \Lambda^i}{\partial q^m}dq^m\right) \wedge \left(\frac{\partial \Lambda^j}{\partial q^n}dq^n\right)$$
$$= \epsilon^{lmn}\frac{\partial s}{\partial q^l}\frac{\partial \Lambda^i}{\partial q^m}\frac{\partial \Lambda^j}{\partial q^n}dq^1 \wedge dq^2 \wedge dq^3.$$

Thus one obtains the Lagrangian invariant, i.e. the frozen-in 0-form[8]: $\dfrac{\epsilon^{lmn}}{\tilde{\rho}}\dfrac{\partial s}{\partial q^l}\dfrac{\partial \Lambda^i}{\partial q^m}\dfrac{\partial \Lambda^j}{\partial q^n} = \dfrac{\nabla s \cdot (\nabla\Lambda^i \times \nabla\Lambda^j)}{\tilde{\rho}}$.

Then we disscuss Weber's transformation type invariants and its relation to Kelvin's circulation theorem. Finally, the exponential map type invariants is presented as an example of recursive use of invariants formula.

4.2.1. Kinematic type Lagrangian invariants

We list in Table 1 such a type of invariants which the neutral, the MHD and the Hall MHD fluids have in common. The invariants are given by combinations of BLM, entropy per unit mass and mass density.

Table 1: Lagrangian invariants with which neutral, MHD and Hall MHD fluids have in common.

blocks			invariant	comments
ds	$d\Lambda^i$	$d\Lambda^j$	$\dfrac{\nabla s \cdot (\nabla\Lambda^i \times \nabla\Lambda^j)}{\tilde{\rho}}$	$i, j = 1, 2, 3$, $i \neq j$
$d\Lambda^i$	$d\Lambda^j$	$d\Lambda^k$	$\dfrac{\nabla\Lambda^i \cdot (\nabla\Lambda^j \times \nabla\Lambda^k)}{\tilde{\rho}}$	conservation of mass

[8] In the following tables, we use the ordinary notation of vector analysis. For example, $\nabla s = \left(\dfrac{\partial s}{\partial q^1}, \dfrac{\partial s}{\partial q^2}, \dfrac{\partial s}{\partial q^3}\right)$.

4.2.2. Lagrangian invariants of a perfect barotropic fluid

For a barotropic perfect fluid the building blocks for Lagrangian invariants are the impulse $\tilde{\boldsymbol{u}}$, the exterior differentiation of entropy and each component BLM ds, $d\Lambda^i$ for $i = 1, 2, 3$ and the vorticity $\boldsymbol{\omega} := d\tilde{\boldsymbol{u}} = d\boldsymbol{u}$. The invariants that are given by combinations of 1-forms are listed in Table 2 and those of 1- and 2-forms are listed in Table 3.

Table 2: Lagrangian invariants of a barotropic fluid. Combinations of 1-forms are listed.

blocks			invariant	comments
$\tilde{\boldsymbol{u}}$	ds	$d\Lambda^i$	$\dfrac{\tilde{\boldsymbol{u}} \cdot (\nabla s \times \nabla\Lambda^i)}{\tilde{\rho}}$	$i = 1, 2, 3$
$\tilde{\boldsymbol{u}}$	$d\Lambda^i$	$d\Lambda^j$	$\dfrac{\tilde{\boldsymbol{u}} \cdot (\nabla\Lambda^i \times \nabla\Lambda^j)}{\tilde{\rho}}$	$i, j = 1, 2, 3,\ i \neq j$

Table 3: Lagrangian invariants of a barotropic fluid. Combinations of a 1-form and a 2-form are listed.

blocks		invariant	comments
$\tilde{\boldsymbol{u}}$	$d\boldsymbol{u}$	$\dfrac{\tilde{\boldsymbol{u}} \cdot \boldsymbol{\omega}}{\tilde{\rho}}$	Ertel-Rossby theorem ([14], [15])
ds	$d\boldsymbol{u}$	$\dfrac{\nabla s \cdot \boldsymbol{\omega}}{\tilde{\rho}}$	Ertel's theorem ([16], [11])
$d\Lambda^i$	$d\boldsymbol{u}$	$\dfrac{\nabla\Lambda^i \cdot \boldsymbol{\omega}}{\tilde{\rho}}$	Cauchy's formula ([17], [11])

As is listed in Table 3, well known three theorems are obtained as combinations of frozen-in 1- and 2-forms. It should be remarked that in many literatures the equation

$$\frac{\omega^i(\vec{X}(\vec{a},t),t)}{\tilde{\rho}(\vec{X}(\vec{a},t),t)} = \frac{\omega^j(\vec{a},t)}{\tilde{\rho}(\vec{a},t)}\frac{\partial X^i}{\partial q^j}(\vec{a},t)$$

is usually referred to as Cauchy's formula. By muliplying $(\partial\Lambda^j/\partial q^i)(\vec{X}(\vec{a},t),t)$ on both sides, one obtains the formula appears in Table 3. The formula given in Table 3 clealy shows that Cauchy's formula is able to be regarded as one of the more general Lagrangian invariants.

4.2.3. Lagrangian invariants of ideal MHD

For an ideal barotropic MHD fluid, instead of the impulse $\tilde{\boldsymbol{u}}$ and the vorticity $d\boldsymbol{u}$, the vector potential $\tilde{\boldsymbol{A}}$ and the magnetic field $\boldsymbol{B} = d\boldsymbol{A}$ are the basic building blocks for Lagrangian invariants. The impulse is not a frozen-in 1-form because of the Lorentz force term: $(\partial/\partial t + L_{\boldsymbol{u}})\tilde{\boldsymbol{u}} = \tilde{\rho}^{-1}\boldsymbol{j} \times \boldsymbol{B}$. Since the other blocks are the same as those of a neutral fluid, the obtained Lagrangian invariants are almost quite analogous to those of a neutral fluid except for the cross helicity. The invariants that are given by the

Table 4: Lagrangian invariants specific to an MHD fluid. Combinations of 1-forms are listed.

blocks			invariant	comments
$\boldsymbol{A}$	ds	$d\Lambda^i$	$\dfrac{\boldsymbol{A}\cdot(\nabla s\times\nabla\Lambda^i)}{\tilde{\rho}}$	$i=1,2,3$
$\boldsymbol{A}$	$d\Lambda^i$	$d\Lambda^j$	$\dfrac{\boldsymbol{A}\cdot(\nabla\Lambda^i\times\nabla\Lambda^j)}{\tilde{\rho}}$	$i,j=1,2,3,\ i\neq j$

Table 5: Lagrangian invariants of an MHD fluid. Combinations of a 1-form and a 2-form are listed.

blocks		invariant	comments
$\tilde{\boldsymbol{A}}$	$\boldsymbol{B}$	$\dfrac{\tilde{\boldsymbol{A}}\cdot\boldsymbol{B}}{\tilde{\rho}}$	magnetic helicity
ds	$\boldsymbol{B}$	$\dfrac{\nabla s\cdot\boldsymbol{B}}{\tilde{\rho}}$	counterpart of Ertel's theorem
$d\Lambda^i$	$\boldsymbol{B}$	$\dfrac{\nabla\Lambda^i\cdot\boldsymbol{B}}{\tilde{\rho}}$	counterpart of Cauchy's formula
$\tilde{\boldsymbol{u}}$	$\boldsymbol{B}$	$\dfrac{\boldsymbol{u}\cdot\boldsymbol{B}}{\tilde{\rho}}$	cross helicity

combinations of 1-forms are listed in Table 4 and those of 1- and 2-forms are listed in Table 5.

The reason for the invariance of the cross helicity is different from other invariants. While the other invariants are obtained by the wedge product of conservation laws, conservation of the cross helicity stems from the fact that the Lorentz force term and the magnetic field are always perpendicular to each other: $(\partial/\partial t+L_{\boldsymbol{u}})(\tilde{\boldsymbol{u}}\wedge\boldsymbol{B})=\tilde{\rho}^{-1}(\boldsymbol{j}\times\boldsymbol{B})\cdot\boldsymbol{B}=0$.

4.2.4. Lagrangian invariants of Hall MHD

For a barotropic ideal Hall MHD fluid the 1-form type building block for Lagrangian invariants is the linear combination of the impulse and the vector potential $\boldsymbol{w}:=\tilde{\boldsymbol{u}}+\frac{e}{m}\tilde{\boldsymbol{A}}$. The invariants that are given by the combinations of 1-forms are listed in Table 6 and those of 1- and 2-forms are listed in Table 7.

Table 6: Lagrangian invariants of a Hall MHD fluid. Combinations of 1-forms are listed.

blocks			invariant	comments
$\boldsymbol{w}$	ds	$d\Lambda^i$	$\tilde{\rho}^{-1}(\tilde{\boldsymbol{u}}+\dfrac{e}{m}\tilde{\boldsymbol{A}})\cdot(\nabla s\times\nabla\Lambda^i)$	$i=1,2,3$
$\boldsymbol{w}$	$d\Lambda^i$	$d\Lambda^j$	$\tilde{\rho}^{-1}(\tilde{\boldsymbol{u}}+\dfrac{e}{m}\tilde{\boldsymbol{A}})\cdot(\nabla\Lambda^i\times\nabla\Lambda^j)$	$i,j=1,2,3,\ i\neq j$

Table 7: Lagrangian invariants of a Hall MHD fluid. Combinations of a 1-form and a 2-form are listed.

blocks		invariant	counterpart
$\boldsymbol{w}$	$d\boldsymbol{w}$	$\tilde{\rho}^{-1}(\boldsymbol{\omega}+\dfrac{e}{m}\boldsymbol{B})\cdot(\tilde{\boldsymbol{u}}+\frac{e}{m}\tilde{\boldsymbol{A}})$	Ertel-Rossby theorem
ds	$d\boldsymbol{w}$	$(\boldsymbol{\omega}+\dfrac{e}{m}\boldsymbol{B})\cdot\nabla s/\tilde{\rho}$	Ertel's theorem
$d\Lambda^i$	$d\boldsymbol{w}$	$(\boldsymbol{\omega}+\dfrac{e}{m}\boldsymbol{B})\cdot\nabla\Lambda^i/\tilde{\rho}$	Cauchy's formula

4.3. Lagrangian invariants of Weber's transformation type and their relation to Kelvin's circulation

In this subsection we discuss such Lagrangian invariants that are given by Weber's transformation and its physical implication. Invariants of Weber's transformation type are listed in Table 8. Keivin's circulation theorem is discussed as well as an application of this type of Lagrangian invariance.

Table 8: Lagrangian invariants of Weber's transformation type

1-form	invariant	comments
ds	$\nabla s\cdot\dfrac{\partial\boldsymbol{X}}{\partial q^i}$	unknown invariant
$d\Lambda^i$	$\nabla\Lambda^i\cdot\dfrac{\partial\boldsymbol{X}}{\partial q^j}$	Eq.(6)
$\tilde{\boldsymbol{u}}$	$\boldsymbol{u}\cdot\dfrac{\partial\boldsymbol{X}}{\partial q^i}$	Weber transformation
$\boldsymbol{A}$	$\boldsymbol{A}\cdot\dfrac{\partial\boldsymbol{X}}{\partial q^j}$	ideal MHD counterpart
$\boldsymbol{w}$	$(\tilde{\boldsymbol{u}}+\dfrac{e}{m}\boldsymbol{A})\cdot\dfrac{\partial\boldsymbol{X}}{\partial q^j}$	Hall MHD counterpart

All the Lagrangian invariants given above are quotients of frozen-in 3-forms. Weber's transformation type invariants, on the other hand, are regarded as the inner product of a frozen-in 1-form and a frozen-in vector field (see Eqs.(14) and (15)) by the following reason. The spatial derivatives of PTM themselves are not the components of any proper vector field or 1-form, but have corresponding frozen-in vector fields (say $\boldsymbol{\xi}_{(i)}=\xi^j_{(i)}\partial/\partial q^j$, i=1,2,3) whose components are defined by

$$\xi^j_{(i)}(\vec{X}(\vec{a},t),t)=\frac{\partial X^j}{\partial q^i}(\vec{a},t). \tag{35}$$

Differentiating this equation with respect to t, one can easily check that the induced vector fields $\boldsymbol{\xi}_{(i)}$ satisfy the PDE Eq.(13). Using the relations at the initial time

$$\xi^j_{(i)}(\vec{X}(\vec{a},0),0)=\xi^j_{(i)}(\vec{a},0)=(\partial X^j/\partial q^i)(\vec{a},0)=\delta^j_i,$$

we can rewrite Weber's transformation in an inner product form as

$$\tilde{u}_j(\vec{X}(\vec{a},t),t)\,\xi^j_{(i)}(\vec{X}(\vec{a},t),t) = \tilde{u}_k(\vec{a},0)\,\xi^k_{(i)}(\vec{a},0). \tag{36}$$

Physical implication of the triplet of fields $(\boldsymbol{\xi}_{(1)}, \boldsymbol{\xi}_{(2)}, \boldsymbol{\xi}_{(3)})$ is that they constitute such a coordinate frame on $\mathcal{M}$ that is initially given by the frame associated with the coordinate system (q^1, q^2, q^3) and then advected and deformed by the fluid flow given by $\vec{X}$ (see Figure 1). The physical implication of Weber's transformation is, therfore, that each projection of the impulse $\tilde{\boldsymbol{u}}$ on the advected frames $\boldsymbol{\xi}_{(i)}$ is conserved along the fluid flow.

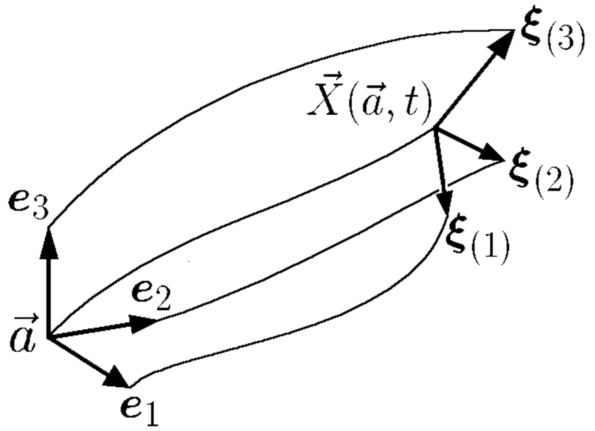

Figure 1: physical implication of the field $\boldsymbol{\xi}_{(i)}$.

From the mathematical viewpoint, Kelvin's circulation theorem and Weber's transformation are both based on the inner product type Lagrangian invariant. The circulation theorem is a composite of the following three facts:

1. The integral contour is a frozen-in material line so that tangential vectors of the contour are frozen-in contravariant vectors;

2. The impulse 1-form of a barotropic fluid is frozen-in so that the inner product of impulse and path's tangent vector is a Lagrangian invariant;

3. The integral contour is a loop, i.e., has no boundary so that, by virtue of Stokes' theorem[18], only the solenoidal part of impulse 1-form is relevant to the integral value.

The third condition is specific to the circulation theorem. Since there are frozen-in 1-forms for the ideal MHD and the Hall MHD, they have counterparts of Kelvin's theorem:

$$\oint_{\vec{x}\in c} \boldsymbol{A}(\vec{x})\cdot d\boldsymbol{l}(\vec{x}) \tag{37}$$

for the ideal MHD and

$$\oint_{\vec{x}\in c} \left(\boldsymbol{u}(\vec{x}) + \frac{e}{m}\boldsymbol{A}(\vec{x})\right)\cdot d\boldsymbol{l}(\vec{x}) \tag{38}$$

for the Hall MHD where c, $d\boldsymbol{l}(\vec{x})$ are the integral loop and its tangential vector at $\vec{x}$, respectively.

4.4. Recursive use of Lagrangian invariant formula

It seems to be an interesting attempt to use the obtained formula recursively. Since Ertel's invariant $\dfrac{\omega^i}{\tilde{\rho}}\dfrac{\partial s}{\partial q^i}$, for example, is a frozen-in 0-form of a barotropic fluid, one can substitute it into s recursively. The resulting function $\dfrac{\omega^i}{\tilde{\rho}}\dfrac{\partial}{\partial q^i}\left(\dfrac{\omega^i}{\tilde{\rho}}\dfrac{\partial s}{\partial q^i}\right)$ is also a Lagrangian invariant[7]. Thus the derivatives $\left(\dfrac{\omega^i}{\tilde{\rho}}\dfrac{\partial}{\partial q^i}\right)^n s$ are conserved. Furthermore one can formally construct the exponential map type Lagrangian invariant given by

$$\exp\left(\tau\frac{\boldsymbol{\omega}}{\tilde{\rho}}\right)s := \sum_{n=0}^{\infty}\frac{\tau^n}{n!}\left(\frac{\omega^i}{\tilde{\rho}}\frac{\partial}{\partial q^i}\right)^n s.$$

The implication of this invariant is quite natural. Mathematically the exponential map is a diffeomorphism on $\mathcal{M}$ generated by the vector field $\boldsymbol{\omega}/\tilde{\rho}$. In other words, this operation implies the advection of s for a finite "time" τ by the "flow" $\boldsymbol{\omega}/\tilde{\rho}$. Since $\boldsymbol{\omega}/\tilde{\rho}$ is a frozen-in vector field, the modified function $\exp(\boldsymbol{\omega}/\tilde{\rho})s$ is advected by $\vec{X}$ with retaining its value for each fluid particle.

Proof of the conservation law is straightforward. Since the transformation of the exponential map is given by

$$\exp\left(\tau\frac{\boldsymbol{\omega}_t}{\tilde{\rho}_t}\right) = \tilde{X}_t \exp\left(\tau\frac{\boldsymbol{\omega}_0}{\tilde{\rho}_0}\right)\tilde{X}_t^{-1},$$

where the subindex stands for the time (cf. Proposition 1.7 in Ref.[8]), the transformation of the modified function $\exp(\boldsymbol{\omega}/\tilde{\rho})s$ is given by

$$\begin{aligned}\tilde{X}_t \exp\left(\tau\frac{\boldsymbol{\omega}_0}{\tilde{\rho}_0}\right)s_0 &= \tilde{X}_t \exp\left(\tau\frac{\boldsymbol{\omega}_0}{\tilde{\rho}_0}\right)\tilde{X}_t^{-1}\tilde{X}_t s_0\\ &= \exp\left(\tau\frac{\boldsymbol{\omega}_t}{\tilde{\rho}_t}\right)s_t.\end{aligned}$$

In place of $\boldsymbol{\omega}/\tilde{\rho}$, analogous conservation laws are generated by $\boldsymbol{B}/\tilde{\rho}$ for the MHD, $(\boldsymbol{\omega} + \frac{e}{m}\boldsymbol{B})/\tilde{\rho}$ for the Hall MHD and for each of the advected coordinate frame fields $\boldsymbol{\xi}_{(i)}$ for all cases.

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References

[1] Constantin, P.: An Eulerrian-Lagrangian Approach to the Navier-Stokes Equations, *Comm. Math. Phys.* **216** (2001) 663–686.

[2] Ohkitani, K., and Constantin, P.: Numerical study of the Eulerian-Lagrangian formulation of the Navier-Stokes equations, *Phys. Fluids* **15** (2003) 3251–3254.

[3] Ohkitani, K., and Constantin, P.: Two- and three-dimensional magnetic reconnection observed in the Eulerian-Lagrangian analysis of magnetohydrodynamics equations, *Phys. Rev. E* **78** (2008) 066315.

[4] Koshizuka, S., and Oka, Y.: Moving-particle semi-implicit method for fragmentation of incompressible fluid, *Nucl. Sci. Eng.* **123** (1996) 421–434.

[5] Monaghan, J. J.: An introduction to SPH, *Comput. Phys. Comm.* **48** (1988) 89–96.

[6] Fukumoto, Y.: A Unified View of Topological Invariants of Fluid Flows, *Topologica* **1** (2008) 003.

[7] Tur, A. V. and Yanovsky, V. V.: Invariants in dissipationless hydrodynamic media, *J. Fluid Mech.* **248** (1993) 67–106.

[8] Kobayashi, S. and Nomizu, K.: *Foundations of differential geometry*, Vol.I, Wiley, New York, 1963.

[9] Landau, L. D., and E. M. Lifshits: *Course of Theoretical Physics*, Vol. 6 Pergamon, Oxford, 1987.

[10] Kuzmin, G. A.: Ideal incompressible hydrodynamics in terms of the vortex momentum density, *Phys. Lett.* **96 A** (1983) 88–90.

[11] Serrin, J.: Mathematical principles of classical fluid mechanics, in: *Handbuch der Physik* Vol.8 *Strömungsmechanik I*, Springer, Berlin, 1959, 125–263.

[12] Goldston, R. J., and Rutherford, P. H.: *Introduction to Plasma Physics*, Institute of Physics Press, Bristol, 1995.

[13] Shivamoggi, B. K.: Impulse Formulations of Hall Magnetohydrodynamic Equations, *Phys. Lett.* **373 A** (2009), 708-710.

[14] Ertel, H. and Rossby, C. G., Sitzgsber. Akad. Wiss. Berlin, Math.-nat. Kl., p.1-11 (1949).

[15] Oswatitsch, K.: Physikalische Grundlagen der Strömungslehre, in: *Handbuch der Physik* Vol.8 *Strömungsmechanik I*, Springer, Berlin, 1959, 1–124.

[16] Ertel, H., Meteor. Z., Vol.59, pp.277-281 (1942).

[17] A.-L. Cauchy, Mem. Divers Savants (2) **1**, (Oeuvres (1) **1**, pp.5 to 318), especially 1st part, §4.

[18] Flanders, H.: *Differential forms with applications to the physical sciences*, Dover, New York, 1989.

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