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M. Wakayama & Y. Yamasaki

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Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

Hecke's zeros and higher depth determinants

Masato WAKAYAMA* and Yoshinori YAMASAKI†

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Abstract

We establish "higher depth" analogues of regularized determinants due to Milnor for the zeros of Hecke L-functions. This is an extension of the result of Deninger about the regularized determinant for the zeros of the Riemann zeta function.

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1 Introduction

Let K be an algebraic number field of degree n and of discriminant d_K, $\mathcal{O}_K$ the ring of integers of K, and r_1 and r_2 the number of real and complex places of K, respectively. Let χ be a Hecke grössencharacter with conductor $\mathfrak{f}$ and

$$L_K(s;\chi) := \prod_{\mathfrak{p}}\Bigl(1-\frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\Bigr)^{-1} = \sum_{\mathfrak{a}}\frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s} \qquad (\mathrm{Re}\,(s)>1)$$

be the Hecke L-function associate with χ. Here, $\mathfrak{p}$ runs over all prime ideals of $\mathcal{O}_K$ and $\mathfrak{a}$ over all integral ideals of $\mathcal{O}_K$ (we understand that $\chi(\mathfrak{p})=0$ if $\mathfrak{p}$ and $\mathfrak{f}$ are not coprime). It is known that $L_K(s;\chi)$ admits a meromorphic continuation to the whole complex plane $\mathbb{C}$ with a possible simple pole at $s=1$ and has a functional equation

$$\Lambda_K(1-s;\overline{\chi}) = W_K(\chi)\Lambda_K(s;\chi),$$

where $\Lambda_K(s;\chi)$ is the entire function defined by

$$\Lambda_K(s;\chi) := \Bigl(\frac{1}{2}s(s-1)\Bigr)^{\varepsilon_\chi}\Bigl(\frac{N(\mathfrak{f})|d_K|}{2^{2r_2}\pi^n}\Bigr)^{\frac{s}{2}}L_K(s;\chi)\prod_{v\in S_\infty(K)}\Gamma\Bigl(\frac{N_v(s+i\varphi_v)+|m_v|}{2}\Bigr), \tag{1.1}$$

and $W_K(\chi)$ is a constant with $|W_K(\chi)|=1$. Here, $S_\infty(K)$ is the set of all archimedean places of K, $\varepsilon_\chi=1$ if χ is principal and 0 otherwise. Moreover, for $v\in S_\infty(K)$, $N_v=1$ if v is real and 2 otherwise, and $\varphi_v=\varphi(\chi)\in\mathbb{R}$ with $\sum_{v\in S_\infty(K)}N_v\varphi_v=0$ and $m_v=m(\chi)\in\mathbb{Z}$ are determined by

$$\chi\bigl((\alpha)\bigr) = \prod_{v\in S_\infty(K)}|\alpha_v|^{-iN_v\varphi_v}\Bigl(\frac{\alpha_v}{|\alpha_v|}\Bigr)^{m_v} \qquad (\alpha\in O_K \text{ with } \alpha\equiv 1 \bmod^\times \mathfrak{f}),$$

where $\mathrm{mod}^\times$ indicates the multiplicative congruence and α_v is the image of α of the embedding $K\hookrightarrow K_v$ with $K_v=\mathbb{R}$ or $\mathbb{C}$. When $\varphi_v=m_v=0$ for all $v\in S_\infty(K)$, χ is called a class character.

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For $T>0$, let $\mathcal{R}_K(T;\chi)$ be the set of non-trivial zeros of $L_K(s;\chi)$ (that is, the zeros whose real part is in $(0,1)$) with $|\mathrm{Im}\,(\rho)|<T$ and $\mathcal{R}_K(\chi):=\lim_{T\to\infty}\mathcal{R}_K(T;\chi)$. In this paper, we study the function

$$\xi_K(s,z;\chi):=\sum_{\rho\in\mathcal{R}_K(\chi)}\Big(\frac{z-\rho}{2\pi}\Big)^{-s}:=\lim_{T\to\infty}\sum_{\rho\in\mathcal{R}_K(T;\chi)}\Big(\frac{z-\rho}{2\pi}\Big)^{-s} \tag{1.2}$$

and, for a positive integer r, compute the function

$$\Xi_{K,r}(z;\chi):=\exp\Big(-\frac{d}{ds}\xi_K(s,z;\chi)\Big|_{s=1-r}\Big). \tag{1.3}$$

Remark that, when $\mathrm{Re}\,(z)>1$, the function $\Xi_{K,r}(z;\chi)$ can be defined because it will be shown that $\xi_K(s,z;\chi)$ admits a meromorphic continuation to $\mathbb{C}$ as a function of s and, in particular, is holomorphic at $s=1-r$ for any $r\in\mathbb{N}$ (Proposition 2.2). When $r=1$, the right-hand side of (1.3) coincides with the so-called the zeta-regularized product of the sequence $\{(\frac{z-\rho}{2\pi})^{-s}\}_{\rho\in\mathcal{R}_K(\chi)}$ and is denoted by

$$\prod_{\rho\in\mathcal{R}_K(\chi)}\Big(\frac{z-\rho}{2\pi}\Big)=\exp\Big(-\frac{d}{ds}\xi_K(s,z;\chi)\Big|_{s=0}\Big).$$

Hence one may call $\Xi_{K,r}(z;\chi)$ a "*higher depth (or depth r) determinants (regularized product)*" of the sequence $\{(\frac{z-\rho}{2\pi})^{-s}\}_{\rho\in\mathcal{R}_K(\chi)}$. Such a higher depth object was first studied by Milnor in [Mi]. Actually, from the viewpoint of the Kubert identity which plays an important role in the study of Iwasawa theory, he introduced an higher depth gamma function $\mathbf{\Gamma}_r(z)$, which we call a "Milnor-gamma function" of depth r, defined by

$$\mathbf{\Gamma}_r(z):=\exp\Big(\frac{d}{ds}\zeta(s,z)\Big|_{s=1-r}\Big)$$

with $\zeta(s,z):=\sum_{m=0}^{\infty}(m+z)^{-s}$ being the Hurwitz zeta function, and studied functional relations among them (see [KOW] for some analytic properties of $\mathbf{\Gamma}_r(z)$). Notice that, by the Lerch formula $\frac{d}{ds}\zeta(s,z)=\log\frac{\Gamma(s)}{\sqrt{2\pi}}$, we have $\mathbf{\Gamma}_1(z)=\frac{\Gamma(s)}{\sqrt{2\pi}}$, whence $\mathbf{\Gamma}_r(z)$ indeed gives a generalization of the usual gamma function.

When $K=\mathbb{Q}$ and $r=1$, by Deninger [D, Theorem 3.3] (see also [SS, V]), it is shown that, as an entire function,

$$\Xi(z)=\prod_{\rho\in\mathcal{R}}\Big(\frac{z-\rho}{2\pi}\Big)=2^{-\frac{1}{2}}(2\pi)^{-2}\pi^{-\frac{z}{2}}\Gamma\Big(\frac{z}{2}\Big)\zeta(z)z(z-1)=\frac{1}{2^{\frac{3}{2}}\pi^2}\Lambda(z). \tag{1.4}$$

Here, when χ is the trivial character $\mathbf{1}$, we write $L_K(s;\mathbf{1})=\zeta_K(s)$ (that is, $\zeta_K(s)$ is the Dedekind zeta function of K), $\Lambda_K(s;\mathbf{1})=\Lambda_K(s)$, $W_K(\mathbf{1})=W_K$, $\mathcal{R}_K(T;\mathbf{1})=\mathcal{R}_K(T)$ and $\mathcal{R}_K(\mathbf{1})=\mathcal{R}_K$, respectively. Moreover, we omit the symbol K when $K=\mathbb{Q}$ and r when $r=1$.

The aim of the present paper is to extend the result (1.4) of Deninger to general r and algebraic number fields. Namely, we calculate the function $\Xi_{K,r}(z;\chi)$ explicitly for any χ and $r\in\mathbb{N}$. To state our main result, let us introduce a "poly-Hecke L-function" $L_K^{(r)}(s;\chi)$. Let $Li_r(z):=\sum_{m=1}^{\infty}\frac{z^m}{m^r}$ be the polylogarithm of degree r and $H_r(z):=\exp(-Li_r(z))$. Then, the function $L_K^{(r)}(s;\chi)$ is defined by the following Euler product;

$$L_K^{(r)}(s;\chi):=\prod_{\mathfrak{p}}H_r\Big(\frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\Big)^{-(\log N(\mathfrak{p}))^{1-r}}. \tag{1.5}$$

Notice that, since $\sum_{\mathfrak{p}}\big|\log(H_r(\frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s})^{-(\log N\mathfrak{p})^{1-r}})\big|\le\log\zeta_K(\mathrm{Re}\,(s))$, the infinite product converges absolutely for $\mathrm{Re}\,(s)>1$, whence the right-hand side of (1.5) defines a holomorphic function on the region. It is obvious to see that this is a poly-analogue of the Hecke L-function. Actually, when $r=1$, since $Li_1(z) = -\log(1-z)$ and hence $H_1(z) = (1-z)$, we have $L_K^{(1)}(s;\chi) = L_K(s;\chi)$. We study several analytic properties of $L_K^{(r)}(s;\chi)$ in Section 3.

The main theorem of the paper is the following.

Theorem 1.1. *For* $\mathrm{Re}\,(z) > 1$, *it holds that*

$$\begin{aligned}\Xi_{K,r}(z;\chi) &= \Bigl(\frac{z}{2\pi}\Bigr)^{\varepsilon_\chi(\frac{z}{2\pi})^{r-1}} \Bigl(\frac{z-1}{2\pi}\Bigr)^{\varepsilon_\chi(\frac{z-1}{2\pi})^{r-1}} L_K^{(r)}(z;\chi)^{(-1)^{r-1}(r-1)!(2\pi)^{1-r}} \\ &\quad\times \prod_{v\in S_\infty(K)} (N_v\pi)^{-\frac{(N_v\pi)^{1-r}}{r}B_r(\frac{N_v(z+i\varphi_v)+|m_v|}{2})} \boldsymbol{\Gamma}_r\Bigl(\frac{N_v(z+i\varphi_v)+|m_v|}{2}\Bigr)^{(N_v\pi)^{1-r}},\end{aligned} \tag{1.6}$$

where $B_r(z)$ *is the* r*th Bernoulli polynomial.*

2 Proof of Theorem 1.1

To prove our main theorem, we employ a refined version of the Weil explicit formula due to Barner [Ba]. For a function F of bounded variation (i.e., $V_{\mathbb{R}}(F) < \infty$ where $V_{\mathbb{R}}(F)$ is the total variation of F on $\mathbb{R}$), we define the function $\Phi_F(s)$ $(s \in \mathbb{C})$ by

$$\Phi_F(s) := \int_{-\infty}^{\infty} F(x) e^{(s-\frac{1}{2})x}\,dx.$$

Moreover, for a Hecke character χ and $v \in S_\infty(K)$, define

$$F_v(x;\chi) := F(x)e^{-i\varphi_v x}.$$

Lemma 2.1 ([Ba, Theorem 1])**.** *Let* χ *be a Hecke character and* $F : \mathbb{R} \to \mathbb{C}$ *be a function of bounded variation satisfying the following three conditions:*

(a) *There is a positive constant* b *such that*

$$V_{\mathbb{R}}(F(x)e^{(\frac{1}{2}+b)|x|}) < \infty.$$

(b) F *is "normalized", that is,*

$$2F(x) = F(x+0) + F(x-0) \qquad (x \in \mathbb{R}).$$

(c) *For any* $v \in S_\infty(K)$, *it holds that*

$$F_v(x;\chi) + F_v(-x;\chi) = 2F(0) + O(|x|) \quad (|x| \to 0).$$

Then, the following equation holds:

$$\begin{aligned}\lim_{T\to\infty} \sum_{\rho\in\mathcal{R}_K(T;\chi)} \Phi_F(\rho) &= \varepsilon_\chi\bigl(\Phi_F(0)+\Phi_F(1)\bigr) + F(0)\log\frac{N(\mathfrak{f})|d_K|}{2^{2r_2}\pi^n} \\ &\quad - \sum_{\mathfrak{p}}\sum_{l=1}^{\infty} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{\frac{l}{2}}}\bigl(\chi(\mathfrak{p}^l)F(\log N(\mathfrak{p})^l) + \overline{\chi}(\mathfrak{p}^l)F(-\log N(\mathfrak{p})^l)\bigr) \\ &\quad + \sum_{v\in S_\infty(K)} W_v(F;\chi),\end{aligned} \tag{2.1}$$

where

$$W_v(F;\chi) := \int_0^\infty \left(\frac{N_vF(0)}{x} - \bigl(F_v(x;\chi)+F_v(-x;\chi)\bigr)\frac{e^{(\frac{2-|m_v|}{N_v}-\frac{1}{2})x}}{1-e^{-\frac{2x}{N_v}}}\right)e^{-\frac{2x}{N_v}}\,dx.$$

□

Using the explicit formula (2.1), we first show the following

Proposition 2.2. *For* $\mathrm{Re}\,(z) > 1$, *we have*

$$\begin{aligned}\xi_K(s,z;\chi) &= \varepsilon_\chi\Big(\Big(\frac{2\pi}{z}\Big)^s + \Big(\frac{2\pi}{z-1}\Big)^s\Big) + \frac{(2\pi)^s}{2\pi i}\int_{L_-}\frac{L_K'}{L_K}(z-t;\chi)t^{-s}dt \\ &\quad - \sum_{v\in S_\infty(K)}(N_v\pi)^s\zeta\Big(s,\frac{N_v(z+i\varphi_v)+|m_v|}{2}\Big),\end{aligned} \tag{2.2}$$

where L_- *is the contour consisting of the lower edge of the cut from* $-\infty$ *to* $-\delta$, *the circle* $t=\delta e^{i\psi}$ *for* $-\pi \le \psi \le \pi$ *and the upper edge of the cut from* $-\delta$ *to* $-\infty$. *This gives a meromorphic continuation of* $\xi_K(s,z;\chi)$ *as a function of* s *to the whole plane* $\mathbb{C}$ *with a simple pole at* $s=1$.

Proof. Suppose $\mathrm{Re}\,(z) > 1$ and $\mathrm{Re}\,(s) > 1$. Then, it is shown that the function

$$F(x) := \begin{cases} x^{s-1}e^{-(z-\frac{1}{2})x} & (x\ge 0),\\ 0 & (x<0)\end{cases}$$

satisfies the conditions (a), (b) and (c) in Lemma 2.1. Notice that

$$\Phi_F(w) = \frac{\Gamma(s)}{(z-w)^s},\quad \text{whence}\quad \Phi_F(0) = \frac{\Gamma(s)}{z^s},\ \Phi_F(1) = \frac{\Gamma(s)}{(z-1)^s},$$

and

$$\begin{aligned} W_v(F;\chi) &= -\int_0^\infty\Bigg(x^{s-1}e^{-(z-\frac{1}{2}+i\varphi_v)x}\frac{e^{(\frac{2-|m_v|}{N_v}-\frac{1}{2})x}}{1-e^{-\frac{2x}{N_v}}}\Bigg)e^{-\frac{2x}{N_v}}dx \\ &= -\int_0^\infty x^{s-1}\frac{e^{-(z+i\varphi_v+\frac{|m_v|}{N_v})x}}{1-e^{-\frac{2x}{N_v}}}dx \\ &= -\Gamma(s)\Big(\frac{N_v}{2}\Big)^s\zeta\Big(s,\frac{N_v(z+i\varphi_v)+|m_v|}{2}\Big).\end{aligned}$$

In the last equality, we have used the formula

$$\Gamma(s)\zeta(s,z) = \int_0^\infty x^{s-1}\frac{e^{-zx}}{1-e^{-x}}dx \qquad (\mathrm{Re}\,(s)>1).$$

Therefore the explicit formula (2.1) reads

$$\begin{aligned}(2\pi)^{-s}\Gamma(s)\xi_K(s,z;\chi) &= \varepsilon_\chi\Big(\frac{\Gamma(s)}{z^s}+\frac{\Gamma(s)}{(z-1)^s}\Big) - \sum_{\mathfrak{p}}\sum_{l=1}^\infty\frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{lz}}\chi(\mathfrak{p}^l)(\log N(\mathfrak{p})^l)^{s-1} \\ &\quad - \Gamma(s)\sum_{v\in S_\infty(K)}\Big(\frac{N_v}{2}\Big)^s\zeta\Big(s,\frac{N_v(z+i\varphi_v)+|m_v|}{2}\Big).\end{aligned} \tag{2.3}$$

Moreover, from the formula

$$\frac{L_K'}{L_K}(s;\chi) = -\sum_{\mathfrak{p}}\sum_{l=1}^\infty \log N(\mathfrak{p})\chi(\mathfrak{p}^l)N(\mathfrak{p})^{-ls} \tag{2.4}$$

together with

$$\frac{a^{s-1}}{\Gamma(s)} = \frac{1}{2\pi i}\int_{L_-}t^{-s}e^{at}dt \qquad (a>0),$$

a standard manipulation shows

$$-\sum_{\mathfrak{p}}\sum_{l=1}^{\infty}\frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{lz}}\chi(\mathfrak{p}^l)(\log N(\mathfrak{p})^l)^{s-1}=\frac{\Gamma(s)}{2\pi i}\int_{L_-}\frac{L'_K}{L_K}(z-t;\chi)t^{-s}dt. \tag{2.5}$$

By the same argument performed in [D], we see that the integral on the right-hand side converges absolutely for any $s\in\mathbb{C}$, whence it defines an entire function as a function of s. Therefore, substituting the formula (2.5) into (2.3) and multiplying $(2\pi)^s\Gamma(s)^{-1}$ to the both-hand sides of (2.3), we obtain the expression (2.2). Now it is easy to see that (2.2) gives a meromorphic continuation of $\xi_K(s,z;\chi)$ to the whole plane $\mathbb{C}$ with only a simple pole at $s=1$. This completes the proof of the proposition. □

We now give a proof of our main result.

Proof of Theorem 1.1. Let us caluculate the derivative of $\xi_K(s,z;\chi)$ at $s=1-r$ for $r\in\mathbb{N}$. Write $\xi_K(s,z;\chi)=A_1(s,z)+A_2(s,z)+A_3(s,z)$ where

$$\begin{aligned}
A_1(s,z)&:=\varepsilon_\chi\Bigl(\Bigl(\frac{2\pi}{z}\Bigr)^s+\Bigl(\frac{2\pi}{z-1}\Bigr)^s\Bigr),\\
A_2(s,z)&:=\frac{(2\pi)^s}{2\pi i}\int_{L_-}\frac{L'_K}{L_K}(z-t;\chi)t^{-s}dt,\\
A_3(s,z)&:=-\sum_{v\in S_\infty(K)}(N_v\pi)^s\zeta\Bigl(s,\frac{N_v(z+i\varphi_v)+|m_v|}{2}\Bigr).
\end{aligned}$$

At first, it is easy to see that

$$\exp\Bigl(-\frac{d}{ds}A_1(s,z)\Big|_{s=1-r}\Bigr)=\Bigl(\frac{z}{2\pi}\Bigr)^{\varepsilon_\chi(\frac{z}{2\pi})^{r-1}}\Bigl(\frac{z-1}{2\pi}\Bigr)^{\varepsilon_\chi(\frac{z-1}{2\pi})^{r-1}}. \tag{2.6}$$

We next calculate the derivative of $A_2(s,z)$ at $s=1-r$ by the same way in [D]. It is clear that

$$\frac{d}{ds}A_2(s,z)\Big|_{s=1-r}=-\frac{(2\pi)^{1-r}}{2\pi i}\int_{L_-}\frac{L'_K}{L_K}(z-t;\chi)t^{r-1}\log\frac{t}{2\pi}dt.$$

It holds that

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{L_-}\frac{L'_K}{L_K}(z-t;\chi)t^{r-1}\log\frac{t}{2\pi}dt\\
&\quad=\frac{1}{2\pi i}\int_{\infty}^{0}\frac{L'_K}{L_K}(z-xe^{-\pi i};\chi)(xe^{-\pi i})^{r-1}\log\frac{xe^{-\pi i}}{2\pi}e^{-\pi i}dx\\
&\qquad+\frac{1}{2\pi i}\int_{0}^{\infty}\frac{L'_K}{L_K}(z-xe^{\pi i};\chi)(xe^{\pi i})^{r-1}\log\frac{xe^{\pi i}}{2\pi}e^{\pi i}dx\\
&\quad=\frac{1}{2\pi i}\int_{0}^{\infty}\frac{L'_K}{L_K}(z+x;\chi)(-1)^{r-1}x^{r-1}\Bigl(\log\frac{x}{2\pi}-\pi i\Bigr)dx\\
&\qquad-\frac{1}{2\pi i}\int_{0}^{\infty}\frac{L'_K}{L_K}(z+x;\chi)(-1)^{r-1}x^{r-1}\Bigl(\log\frac{x}{2\pi}+\pi i\Bigr)dx\\
&\quad=(-1)^r\int_{0}^{\infty}\frac{L'_K}{L_K}(z+x;\chi)x^{r-1}dx.
\end{aligned}$$

Moreover, using the formula (2.4), we see that the right-hand side above is equal to

$$\begin{aligned}&(-1)^{r-1}\sum_{\mathfrak{p}}\sum_{l=1}^{\infty}\log N(\mathfrak{p})\cdot\chi(\mathfrak{p}^l)\cdot N(\mathfrak{p})^{-lz}\int_0^{\infty}x^{r-1}e^{-lx\log N(\mathfrak{p})}dx\\&=(-1)^{r-1}\sum_{\mathfrak{p}}\sum_{l=1}^{\infty}\log N(\mathfrak{p})\cdot\chi(\mathfrak{p})^l\cdot N(\mathfrak{p})^{-lz}\frac{\Gamma(r)}{(l\log N(\mathfrak{p}))^r}\\&=(-1)^{r-1}(r-1)!\sum_{\mathfrak{p}}(\log N(\mathfrak{p}))^{1-r}Li_r\Big(\frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^z}\Big)\\&=(-1)^{r-1}(r-1)!\log L_K^{(r)}(s;\chi).\end{aligned}$$

This shows that
$$\frac{d}{ds}A_2(s,z)\Big|_{s=1-r}=(-1)^r(r-1)!(2\pi)^{1-r}\log L_K^{(r)}(s;\chi),$$
whence

$$\exp\Big(-\frac{d}{ds}A_2(s,z)\Big|_{s=1-r}\Big)=L_K^{(r)}(s;\chi)^{(-1)^{r-1}(r-1)!(2\pi)^{1-r}}. \tag{2.7}$$

Finally, using the fact $\zeta(1-r,z)=-\frac{B_r(z)}{r}$ where $B_r(z)$ is the Bernoulli polynomial, we have

$$\begin{aligned}&\frac{d}{ds}A_3(s,z)\Big|_{s=1-r}\\&\quad=\sum_{v\in S_\infty(K)}(N_v\pi)^{1-r}\Big[\frac{\log(N_v\pi)}{r}B_r\Big(\frac{N_v(z+i\varphi_v)+|m_v|}{2}\Big)-\log\mathbf{\Gamma}_r\Big(\frac{N_v(z+i\varphi_v)+|m_v|}{2}\Big)\Big],\end{aligned}$$

whence

$$\begin{aligned}&\exp\Big(-\frac{d}{ds}A_3(s,z)\Big|_{s=1-r}\Big)\\&\quad=\prod_{v\in S_\infty(K)}(N_v\pi)^{-\frac{(N_v\pi)^{1-r}}{r}B_r(\frac{N_v(z+i\varphi_v)+|m_v|}{2})}\mathbf{\Gamma}_r\Big(\frac{N_v(z+i\varphi_v)+|m_v|}{2}\Big)^{(N_v\pi)^{1-r}}.\end{aligned} \tag{2.8}$$

Combining three equations (2.6), (2.7) and (2.8), we obtain the desired formula (1.6). This completes the proof of the theorem. □

Corollary 2.3. *We have*

$$\prod_{\rho\in\mathcal{R}_K(\chi)}\Big(\frac{z-\rho}{2\pi}\Big)=\frac{(N(\mathfrak{f})|d_K|)^{-\frac{z}{2}}}{2^{\varepsilon_\chi+\frac12r_1+i\varphi_{\mathbb{C}}+\frac12m_{\mathbb{C}}}\pi^{2\varepsilon_\chi+m}}\Lambda_K(z;\chi), \tag{2.9}$$

where $\varphi_{\mathbb{C}}:=\sum_{v\,:\,complex}\varphi_v$, $m_{\mathbb{C}}:=\sum_{v\,:\,complex}|m_v|$ *and* $m:=\sum_{v\in S_\infty(K)}|m_v|$. *In particular, if* χ *is a class character, then we have*

$$\prod_{\rho\in\mathcal{R}_K(\chi)}\Big(\frac{z-\rho}{2\pi}\Big)=\frac{(N(\mathfrak{f})|d_K|)^{-\frac{z}{2}}}{2^{\varepsilon_\chi+\frac12r_1}\pi^{2\varepsilon_\chi}}\Lambda_K(z;\chi). \tag{2.10}$$

Proof. Let $r=1$ in (1.6). Then, noting that $L_K^{(1)}(z;\chi)=L_K(z;\chi)$, $\mathbf{\Gamma}_1(z)=\frac{\Gamma(z)}{\sqrt{2\pi}}$ and $B_1(z)=z-\frac12$, and recalling the definition (1.1) of $\Lambda_K(z;\chi)$, one easily obtains the expression (2.9). The formula (2.10) immediately follows from (2.9) since $\varphi_{\mathbb{C}}=m_{\mathbb{C}}=m=0$. □

Example 2.4. Let $\chi = \mathbf{1}$. Then, from the equation (2.10), we obtain the regularized determinant expression of the Dedekind zeta function $\zeta_K(z)$;

$$\prod_{\rho \in \mathcal{R}_K} \Big(\frac{z-\rho}{2\pi}\Big) = \frac{|d_K|^{-\frac{z}{2}}}{2^{\frac{1}{2}r_1+1}\pi^2} \Lambda_K(z).$$

This yields the equation (1.4) of Deninger by letting $K = \mathbb{Q}$.

Remark 2.5. As analogues of Theorem 1.1, "higher depth determinants" of the Laplacian on compact Riemann surfaces of genus $g \geq 2$ are investigated in [KWY] (see [Y] for the corresponding results on higher dimensional spheres). We notice that these are defined like (1.3) but we employ the spectral zeta functions for surfaces instead of $\xi_K(s, z; \chi)$, whence the determination of gamma factors is involved.

3 Analytic properties of the Poly-Hecke L-function

Let $\Omega_K(\chi)$ be the set of all complex numbers which are not of the form $\rho - \lambda$ for $\rho \in \mathcal{R}_K(\chi)$ and for $\lambda \geq 0$ or, if χ is principal, $1 - \lambda$ for $\lambda \geq 0$ (See Figure 1). We now give an analytic continuation of $L_K^{(r)}(s; \chi)$ to the region $\Omega_K(\chi)$.

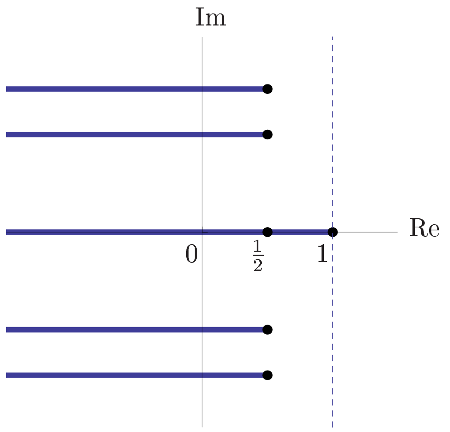

Figure 1: The region $\Omega_K(\chi)$ (if χ is principal)

Lemma 3.1. *It holds that*

$$\frac{d^{r-1}}{ds^{r-1}} \log L_K^{(r)}(s; \chi) = (-1)^{r-1} \log L_K(s; \chi) \qquad (\mathrm{Re}\,(s) > 1). \tag{3.1}$$

Proof. The case $r = 1$ is trivial. Assume $r \geq 2$. Then, using the differential equation

$$\frac{d}{dz} Li_r(z) = z^{-1} Li_{r-1}(z)$$

of the polylogarithm, we have

$$\begin{aligned}\frac{d}{ds}\log L_K^{(r)}(s;\chi) &= \sum_{\mathfrak{p}}(\log N(\mathfrak{p}))^{1-r}\frac{d}{ds}Li_r\Big(\frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\Big)\\ &= \sum_{\mathfrak{p}}(\log N(\mathfrak{p}))^{1-r}\Big(\frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\Big)^{-1}Li_{r-1}\Big(\frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\Big)\frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}(-\log N(\mathfrak{p}))\\ &= -\sum_{\mathfrak{p}}(\log N(\mathfrak{p}))^{1-(r-1)}Li_{r-1}\Big(\frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\Big)\\ &= -\log L_K^{(r-1)}(s;\chi).\end{aligned}$$

Therefore we inductively obtain the formula (3.1).

□

Corollary 3.2. *Let* $\mathrm{Re}\,(a) > 1$*. Then, for* $r \geq 2$*, we have*

$$L_K^{(r)}(s;\chi) = Q_K^{(r)}(s,a)\exp\Bigg(\underbrace{\int_a^s\int_a^{\xi_{r-1}}\cdots\int_a^{\xi_2}}_{r-1}\log L_K(\xi_1;\chi)d\xi_1\cdots d\xi_{r-1}\Bigg)^{(-1)^{r-1}}. \tag{3.2}$$

Here $Q_K^{(r)}(s,a) := \prod_{k=0}^{r-2} L_K^{(r-k)}(a;\chi)^{\frac{(-1)^k}{k!}(s-a)^k}$ *and the path for each integral is contained in* $\Omega_K(\chi)$*. The expression shows an analytic continuation of* $L_K^{(r)}(s;\chi)$ *to the region* $\Omega_K(\chi)$*.*

Proof. By induction on r, (3.2) follows from (3.1). Since $\log L_K(s;\chi)$ is a (single-valued) holomorphic function in $\Omega_K(\chi)$, (3.2) in fact gives an analytic continuation of $L_K^{(r)}(s;\chi)$ to $\Omega_K(\chi)$. This proves the corollary. □

Remark 3.3. Let $\Delta_K(\chi)$ be the set of all complex numbers which are not of the form $-\frac{|m_v|}{N_v}-i\varphi_v-\lambda$ for $v \in S_\infty(K)$ and for $\lambda \geq 0$. Then, since the Milnor-gamma function $\mathbf{\Gamma}_r(z)$ is holomorphic in the region $\mathbb{C}\setminus(-\infty,0]$, from Corollary 3.2, one sees that the expression (1.6) is valid for all $z \in \Omega_K(\chi)\cap\Delta_K(\chi)$. We notice that $\Omega_K(\chi)\cap\Delta_K(\chi) = \Omega_K(\chi)$ when χ is a class character.

Remark 3.4. Let $\widetilde{L}_K^{(r)}(s;\chi) := \prod_{\mathfrak{p}} H_r\big(\frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\big)^{-1}$ for $\mathrm{Re}\,(s) > 1$. Then we have also $\widetilde{L}_K^{(1)}(s;\chi) = L_K(s;\chi)$. It does not, however, seem to have an analytic continuation to the whole plane $\mathbb{C}$. In fact, in [KW], it was shown that $\widetilde{\zeta}^{(r)}(s) := \widetilde{L}_{\mathbb{Q}}^{(r)}(s;\mathbf{1})$ has an analytic continuation to the region $\mathrm{Re}\,(s) > 0$ but has a natural boundary at the imaginary axis $\mathrm{Re}\,(s) = 0$.

We finally show a relation between $L_K^{(r)}(s;\chi)$ and the extended Riemann hypothesis for $L_K(s;\chi)$. Recall that the extended Riemann hypothesis asserts that $\mathrm{Re}\,(\rho) = \frac{1}{2}$ for any $\rho \in \mathcal{R}_K(\chi)$.

Corollary 3.5. *The extended Riemann hypohesis for* $L_K(s;\chi)$ *is equivalent to say that the function* $(s-1)^{-(s-1)}L_K^{(2)}(s;\chi)$ *is a single-valued holomorphic function in* $\mathrm{Re}\,(s) > \frac{1}{2}$*.*

Proof. Let $r = 2$ in (3.2). Then, from (3.2), we have

$$L_K^{(2)}(s;\chi) = L_K^{(2)}(a;\chi)\exp\Big(-\int_a^s \log L_K(\xi;\chi)d\xi\Big) \quad (s \in \Omega_K(\chi),\ \mathrm{Re}\,(a) > 1). \tag{3.3}$$

Here the path is taken in $\Omega_K(\chi)$. Notice that, since

$$\int_a^s \log(\xi-1)d\xi = (s-1)\log(s-1) - s - ((a-1)\log(a-1) - a),$$

we have

$$e^s(s-1)^{-(s-1)} = e^a(a-1)^{-(a-1)} \exp\Bigl(-\int_a^s \log(\xi-1)d\xi\Bigr).$$

Hence

$$e^s(s-1)^{-(s-1)}L_K^{(2)}(s;\chi) = e^a(a-1)^{-(a-1)}L_K^{(2)}(a;\chi) \exp\Bigl(-\int_a^s \log(\xi-1)L_K(\xi;\chi)d\xi\Bigr).$$

Now the statement follows immediately from the fact that $(\xi-1)L_K(\xi;\chi)$ is holomorphic at $\xi = 1$. □

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Masato WAKAYAMA
Faculty of Mathematics, Kyushu University,
Motooka, Nishiku, Fukuoka, 819-0395, JAPAN.
`wakayama@math.kyushu-u.ac.jp`

Yoshinori YAMASAKI
Graduate School of Science and Engineering, Ehime University,
Bunkyo-cho, Matsuyama, 790-8577 JAPAN.
`yamasaki@math.sci.ehime-u.ac.jp`

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