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## Lifting Galois representations over arbitrary number fields

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## Abstract

It is proved that every two-dimensional residual Galois representation of the absolute Galois group of an arbitrary number field lifts to a characteristic zero  $p$ -adic representation, if local lifting problems at places above  $p$  are unobstructed.

## 1 Introduction

Let  $\mathbf{k}$  be a finite field of characteristic  $p \geq 3$ . Let  $K$  be a number field of finite degree over  $\mathbb{Q}$  and  $G_K$  its absolute Galois group  $\text{Gal}(\bar{K}/K)$ . We consider continuous representations

$$\bar{\rho} : G_K \rightarrow \text{GL}_2(\mathbf{k}).$$

The central question that we study in this paper is the existence of a lift of  $\bar{\rho}$  to  $W(\mathbf{k})$ , the ring of Witt vectors of  $\mathbf{k}$ . This question has been motivated by a conjecture of Serre ([S1]), that is, all odd absolutely irreducible continuous representations  $\bar{\rho} : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathbf{k})$  are modular of prescribed weight, level and character. This predicts the existence of a lift to characteristic zero. This conjecture was proved by Khare and Wintenberger in [KW1, KW2]. In [K], Khare proved the existence of lifts to  $W(\mathbf{k})$  for any  $\bar{\rho} : G_K \rightarrow \text{GL}_2(\mathbf{k})$  which are reducible. Ramakrishna proved under very general conditions on  $\bar{\rho}$  that there exist lifts to  $W(\mathbf{k})$  for  $K = \mathbb{Q}$  in [R1, R2]. Gee's results ([G]) imply that there exist lifts to  $W(\mathbf{k})$  for  $p \geq 5$  and  $K$  satisfying  $[K(\mu_p) : K] \geq 3$ , where  $\mu_p$  is the group of  $p$ -th roots of unity. Böckle and Khare have proved the general  $n$ -dimensional case for function field in [BK]. In this paper, we extend Theorem 1 of [R1] to arbitrary number fields. In particular, we will omit the condition  $[K(\mu_p) : K] \geq 3$ . Hence we can take the field  $K$  to be  $\mathbb{Q}(\mu_p)^+$ , the totally real subfield of  $\mathbb{Q}(\mu_p)$ .

For a place  $v$  of  $K$ , let  $K_v$  be the completion of  $K$  at  $v$ , and let  $G_v$  be its absolute Galois group  $\text{Gal}(\bar{K}_v/K_v)$ . Let  $\text{Ad}^0 \bar{\rho}$  be the set of all trace zero two-by-two matrices over  $\mathbf{k}$  with Galois action through  $\bar{\rho}$  by conjugation. Our main result is the following:

**Theorem.** *Let  $K$  be a number field, and let  $\bar{\rho} : G_K \rightarrow \text{GL}_2(\mathbf{k})$  be a continuous representation with coefficients in a finite field  $\mathbf{k}$  of characteristic  $p \geq 7$ . Assume that  $H^2(G_v, \text{Ad}^0 \bar{\rho}) = 0$  for each places  $v \mid p$ . Then  $\bar{\rho}$  lifts to a continuous*

representation  $\rho : G_K \rightarrow \mathrm{GL}_2(W(\mathbf{k}))$  which is unramified outside a finite set of places of  $K$ .

Our method used in the proof is essentially that of Ramakrishna [R1,R2]. In this paper, we follow the more axiomatic treatment presented in [T]. In Section 2, we recall a criterion of Ramakrishna [R2] and Taylor [T] for lifting problems. In Section 3, we define good local lifting problems at certain unramified places and ramified places not dividing  $p$ , which will be used in Section 4. In Section 4, we prove Theorem by using the criterion in Section 2 and local lifting problems in Section 3.

Throughout this paper, we assume that  $p$  is a prime  $\geq 7$ .

## 2 A criterion for lifting problems

In this section we recall a criterion of Ramakrishna [R2] and Taylor [T] for a lifting from a fixed residual Galois representation to a  $p$ -adic Galois representation.

Let  $\mathbf{k}$  be a finite field of characteristic  $p$ . Throughout this paper, we consider a continuous representation

$$\bar{\rho} : G_K \rightarrow \mathrm{GL}_2(\mathbf{k}).$$

Let  $S$  denote a finite set of places of  $K$  containing the places above  $p$ , the infinite places and the places at which  $\bar{\rho}$  is ramified, and let  $K_S$  denote the maximal algebraic extension of  $K$  unramified outside  $S$ . Thus  $\bar{\rho}$  factors through  $\mathrm{Gal}(K_S/K)$ . Put  $G_{K,S} = \mathrm{Gal}(K_S/K)$ . For each place  $v$  of  $K$ , we fix an embedding  $\bar{K} \subset \bar{K}_v$ . This gives a corresponding continuous homomorphism  $G_v \rightarrow G_{K,S}$ .

Let  $\mathcal{A}$  be the category of complete noetherian local rings  $(R, \mathfrak{m}_R)$  with residue field  $\mathbf{k}$  where the morphisms are homomorphisms that induce the identity map on the residue field.

Fix a continuous homomorphism  $\delta : G_{K,S} \rightarrow W(\mathbf{k})^\times$ , and for every  $(R, \mathfrak{m}_R) \in \mathcal{A}$  let  $\delta_R$  be the composition  $\delta_R : G_{K,S} \rightarrow W(\mathbf{k})^\times \rightarrow R^\times$ . Suppose  $\bar{\rho} : G_{K,S} \rightarrow \mathrm{GL}_2(\mathbf{k})$  has  $\det \bar{\rho} = \delta_{\mathbf{k}}$ .

By a  $\delta$ -lift (resp.  $\delta|_{G_v}$ -lift) of  $\bar{\rho}$  (resp.  $\bar{\rho}|_{G_v}$ ) we mean a continuous representation  $\rho : G_{K,S} \rightarrow \mathrm{GL}_2(R)$  (resp.  $\rho_v : G_v \rightarrow \mathrm{GL}_2(R)$ ) for some  $(R, \mathfrak{m}_R) \in \mathcal{A}$  such that  $\rho \pmod{\mathfrak{m}_R} = \bar{\rho}$  (resp.  $\rho_v \pmod{\mathfrak{m}_R} = \bar{\rho}|_{G_v}$ ) and  $\det \rho = \delta_R$  (resp.  $\det \rho_v = \delta_R|_{G_v}$ ). Let  $\mathrm{Ad}^0 \bar{\rho}$  be the set of all trace zero two-by-two matrices over  $\mathbf{k}$  with Galois action through  $\bar{\rho}$  by conjugation.

**Definition 1.** For a place  $v$  of  $K$ , we say that a pair  $(\mathcal{C}_v, L_v)$ , where  $\mathcal{C}_v$  is a collection of  $\delta|_{G_v}$ -lifts of  $\bar{\rho}|_{G_v}$  and  $L_v$  is a subspace of  $H^1(G_v, \mathrm{Ad}^0 \bar{\rho})$ , is *locally admissible* if it satisfies the following conditions:

- (P1)  $(\mathbf{k}, \bar{\rho}|_{G_v}) \in \mathcal{C}_v$ .
- (P2) The set of  $\delta|_{G_v}$ -lifts in  $\mathcal{C}_v$  to a fixed ring  $(R, \mathfrak{m}_R) \in \mathcal{A}$  is closed under conjugation by elements of  $1 + M_2(\mathfrak{m}_R)$ .
- (P3) If  $(R, \rho) \in \mathcal{C}_v$  and  $f : R \rightarrow S$  is a morphism in  $\mathcal{A}$  then  $(S, f \circ \rho) \in \mathcal{C}_v$ .

- (P4) Suppose that  $(R_1, \rho_1)$  and  $(R_2, \rho_2) \in \mathcal{C}_v$ , and  $I_1$  (resp.  $I_2$ ) is an ideal of  $R_1$  (resp.  $R_2$ ) and that  $\phi : R_1/I_1 \xrightarrow{\sim} R_2/I_2$  is an isomorphism such that  $\phi(\rho_1 \pmod{I_1}) = \rho_2 \pmod{I_2}$ . Let  $R_3$  be the fiber product of  $R_1$  and  $R_2$  over  $R_1/I_1 \xrightarrow{\sim} R_2/I_2$ . Then  $(R_3, \rho_1 \oplus \rho_2) \in \mathcal{C}_v$ .
- (P5) If  $((R, \mathfrak{m}_R), \rho)$  is a  $\delta|_{G_v}$ -lift of  $\bar{\rho}|_{G_v}$  such that each  $(R/\mathfrak{m}_R^n, \rho \pmod{\mathfrak{m}_R^n}) \in \mathcal{C}_v$  then  $(R, \rho) \in \mathcal{C}_v$ .
- (P6) For  $(R, \mathfrak{m}_R) \in \mathcal{A}$ , suppose that  $I$  is an ideal of  $R$  with  $\mathfrak{m}_R I = (0)$ . If  $(R/I, \rho) \in \mathcal{C}_v$  then there is a  $\delta|_{G_v}$ -lift  $\tilde{\rho}$  of  $\bar{\rho}|_{G_v}$  to  $R$  such that  $(R, \tilde{\rho}) \in \mathcal{C}_v$  and  $\tilde{\rho} \pmod{I} = \rho$ .
- (P7) Suppose that  $((R, \mathfrak{m}_R), \rho_1)$  and  $(R, \rho_2)$  are  $\delta|_{G_v}$ -lifts of  $\bar{\rho}$  with  $(R, \rho_1) \in \mathcal{C}_v$ , and that  $I$  is an ideal of  $R$  with  $\mathfrak{m}_R I = (0)$  and  $\rho_1 \pmod{I} = \rho_2 \pmod{I}$ . We shall denote by  $[\rho_2 - \rho_1]$  an element of  $H^1(G_v, \text{Ad}^0 \bar{\rho}) \otimes_{\mathbf{k}} I$  defined by  $\sigma \mapsto \rho_2(\sigma)\rho_1(\sigma)^{-1} - 1$ . Then  $[\rho_2 - \rho_1] \in L_v \otimes_{\mathbf{k}} I$  if and only if  $(R, \rho_2) \in \mathcal{C}_v$ .

**Remark 1.** Note that we do regard  $\mathcal{C}_v$  as a functor from  $\mathcal{A}$  to the category of sets.

Let  $S_f$  be the subset of  $S$  consisting of finite places. Throughout this section, suppose that for each  $v \in S_f$  a locally admissible pair  $(\mathcal{C}_v, L_v)$  is given.

Let  $\bar{\chi}_p : G_K \rightarrow \mathbf{k}^\times$  be the mod  $p$  cyclotomic character. For the  $\mathbf{k}[G_K]$ -module  $\text{Ad}^0 \bar{\rho}$ , by  $\text{Ad}^0 \bar{\rho}(i)$  for  $i \in \mathbb{Z}$  we denote the twist of  $\text{Ad}^0 \bar{\rho}$  by the  $i$ th tensor power of  $\bar{\chi}_p$ , and by  $\text{Ad}^0 \bar{\rho}^* := \text{Hom}(\text{Ad}^0 \bar{\rho}, \mathbf{k})$  we denote its dual representation. The  $G_K$ -equivariant trace pairing  $\text{Ad}^0 \bar{\rho} \times \text{Ad}^0 \bar{\rho} \rightarrow \mathbf{k} : (A, B) \mapsto \text{Trace}(AB)$  is perfect. In particular,  $\text{Ad}^0 \bar{\rho} \cong \text{Ad}^0 \bar{\rho}^*$  as representations. Thus  $\text{Ad}^0 \bar{\rho}(1) \cong \text{Ad}^0 \bar{\rho}^*(1)$  as representations. By the Tate local duality this induces a perfect pairing

$$H^1(G_v, \text{Ad}^0 \bar{\rho}) \times H^1(G_v, \text{Ad}^0 \bar{\rho}(1)) \rightarrow H^2(G_v, \mathbf{k}(1)) \cong \mathbf{k}.$$

**Definition 2.** A  $\delta$ -lift of type  $(\mathcal{C}_v)_{v \in S_f}$  is a  $\delta$ -lift such that  $\rho|_{G_v} \in \mathcal{C}_v$  for all  $v \in S_f$ .

**Definition 3.** We define the Selmer group  $H^1_{\{L_v\}}(G_{K,S}, \text{Ad}^0 \bar{\rho})$  to be the kernel of the map

$$H^1(G_{K,S}, \text{Ad}^0 \bar{\rho}) \rightarrow \bigoplus_{v \in S_f} H^1(G_v, \text{Ad}^0 \bar{\rho})/L_v$$

and the dual Selmer group  $H^1_{\{L_v^\perp\}}(G_{K,S}, \text{Ad}^0 \bar{\rho}(1))$  to be the kernel of the map

$$H^1(G_{K,S}, \text{Ad}^0 \bar{\rho}(1)) \rightarrow \bigoplus_{v \in S_f} H^1(G_v, \text{Ad}^0 \bar{\rho}(1))/L_v^\perp$$

where  $L_v^\perp \subset H^1(G_v, \text{Ad}^0 \bar{\rho}(1))$  is the annihilator of  $L_v \subset H^1(G_v, \text{Ad}^0 \bar{\rho})$  under the above pairing.

**Proposition 1.** *Keep the above notation and assumptions. If*

$$H^1_{\{L_v^\perp\}}(G_{K,S}, \text{Ad}^0 \bar{\rho}(1)) = 0,$$

*then there exists a  $\delta$ -lift of  $\bar{\rho}$  to  $W(\mathbf{k})$  of type  $(\mathcal{C}_v)_{v \in S_f}$ .*

*Proof.* By Theorem 4.50 of [H] we have the exact sequence

$$\begin{aligned} H^1(G_{K,S}, \text{Ad}^0 \bar{\rho}) &\xrightarrow{\alpha} \bigoplus_{v \in S_f} H^1(G_v, \text{Ad}^0 \bar{\rho})/L_v \rightarrow H^1_{\{L_v^\perp\}}(G_{K,S}, \text{Ad}^0 \bar{\rho}(1))^* \\ &\rightarrow H^2(G_{K,S}, \text{Ad}^0 \bar{\rho}) \xrightarrow{\beta} \bigoplus_{v \in S_f} H^2(G_v, \text{Ad}^0 \bar{\rho}). \end{aligned}$$

Consequently, we see that the map  $\alpha$  is surjective and the map  $\beta$  is injective. Now we construct  $\delta$ -lifts  $\rho_n$  of  $\bar{\rho}$  to  $W(\mathbf{k})/p^n$  of type  $(\mathcal{C}_v)_{v \in S_f}$  inductively. By the condition (P1), there is nothing to prove for  $n = 1$ . Assume that there is a  $\delta$ -lift  $\rho_{n-1}$  of  $\bar{\rho}$  to  $W(\mathbf{k})/p^{n-1}$  of type  $(\mathcal{C}_v)_{v \in S_f}$ . By the condition (P6), for each  $v \in S_f$  we can lift  $\rho_{n-1}|_{G_v}$  to a continuous homomorphism  $\rho_v : G_v \rightarrow \text{GL}_2(W(\mathbf{k})/p^n)$  such that  $(W(\mathbf{k})/p^n, \rho_v) \in \mathcal{C}_v$ . Thus we can lift  $\rho_{n-1}$  to a continuous homomorphism  $\rho : G_{K,S} \rightarrow \text{GL}_2(W(\mathbf{k})/p^n)$  by injectivity of the map  $\beta$ . By surjectivity of the map  $\alpha$  we may find a class  $\phi \in H^1(G_{K,S}, \text{Ad}^0 \bar{\rho})$  mapping to

$$([\rho_v - \rho|_{G_v}] \bmod L_v)_{v \in S_f} \in \bigoplus_{v \in S_f} H^1(G_v, \text{Ad}^0 \bar{\rho})/L_v.$$

We define  $\rho_n := (1 + \phi)\rho$ . By the condition (P7) the representation  $\rho_n$  is a  $\delta$ -lift of  $\bar{\rho}$  to  $W(\mathbf{k})/p^n$  of type  $(\mathcal{C}_v)_{v \in S_f}$ . The induction is now complete. Then we have a  $\delta$ -lift of  $\bar{\rho}$  to  $W(\mathbf{k})$  of type  $(\mathcal{C}_v)_{v \in S_f}$  by the condition (P5) and the proposition is proved.  $\square$

### 3 Local lifting problems

For a place  $v$  of  $K$ , consider a continuous homomorphism

$$\bar{\rho}_v : G_v \rightarrow \text{GL}_2(\mathbf{k}).$$

We denote by  $\widehat{\varepsilon} : G_v \rightarrow W(\mathbf{k})^\times$  the Teichmüller lift for any character  $\varepsilon : G_v \rightarrow \mathbf{k}^\times$  and  $\widehat{\mu} \in W(\mathbf{k})$  the Teichmüller lift for any element  $\mu$  of  $\mathbf{k}$ . Let  $\chi_p$  be the  $p$ -adic cyclotomic character.

In this section, for ramified places not dividing  $p$  and certain unramified places, we construct a good locally admissible pairs  $(\mathcal{C}_v, L_v)$  with the  $\delta_v := \widehat{\det \bar{\rho}_v} \widehat{\chi}_p^{-1} \chi_p$ , which will be used in Section 4. Let  $I_v$  be the inertia subgroup of  $G_v$ . We distinguish following three cases.

#### 3.1 Case I

Suppose  $\bar{\rho}_v$  is unramified and  $v \nmid p$ . Suppose that

$$\bar{\rho}_v(s) = \begin{pmatrix} \lambda & \lambda \\ 0 & \lambda \end{pmatrix}$$

and  $q_v \equiv 1 \pmod{p}$ , where  $\lambda$  is an element of  $\mathbf{k}^\times$  and  $s$  is a lift of the Frobenius automorphism in  $G_v/I_v$  and  $q_v$  is the order of the residue field of  $K_v$ . Note that any  $\delta_v$ -lift of  $\bar{\rho}_v$  factors through the Galois group  $\text{Gal}(K_v^t/K_v)$  of the maximal tamely ramified extension  $K_v^t$  of  $K_v$ . Let  $P_v$  be the wild inertia subgroup of

$G_v$ . Let  $t$  be a topological generator of  $I_v/P_v$ . The Galois group  $\text{Gal}(K_v^t/K_v)$  is generated topologically by  $s$  and  $t$  with the relation  $sts^{-1} = t^{q_v}$ . We now define a homomorphism  $\rho_v : G_v \rightarrow \text{Gal}(K_v^t/K_v) \rightarrow \text{GL}_2(W(\mathbf{k})[[X]])$  by

$$s \mapsto \begin{pmatrix} \widehat{\lambda}q_v & \widehat{\lambda} \\ 0 & \widehat{\lambda} \end{pmatrix}$$

and

$$t \mapsto \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}.$$

The images of  $s$  and  $t$  satisfy the relation  $sts^{-1} = t^{q_v}$ . We define a pair  $(\mathcal{C}_v, L_v)$ . The functor  $\mathcal{C}_v : \mathcal{A} \rightarrow \mathbf{Sets}$  is given by

$$\mathcal{C}_v(R) := \{ \rho : G_v \rightarrow \text{GL}_2(R) \mid \text{there are } \alpha \in \text{Hom}_{\mathcal{A}}(W(\mathbf{k})[[X]], R) \text{ and } M \in 1 + \text{M}_2(\mathfrak{m}_R) \text{ such that } \rho = M(\alpha \circ \rho_v)M^{-1} \}.$$

Moreover, if  $\rho_0 : G_v \rightarrow \text{GL}_2(\mathbf{k}[X]/(X^2))$  denotes the trivial lift of  $\bar{\rho}_v$ , we define a subspace  $L_v \subset H^1(G_v, \text{Ad}^0 \bar{\rho}_v)$  to be the set

$$\{ [c] \in H^1(G_v, \text{Ad}^0 \bar{\rho}_v) \mid (1 + Xc)\rho_0 \in \mathcal{C}_v(\mathbf{k}[X]/(X^2)) \}.$$

**Lemma 1.** *We have*

- (i)  $\dim_{\mathbf{k}} L_v = \dim_{\mathbf{k}} H^1(G_v/I_v, \text{Ad}^0 \bar{\rho}_v) = 1$ .
- (ii) *The pair  $(\mathcal{C}_v, L_v)$  satisfies the conditions (P1)-(P7) of Definition 1.*

*Proof.* (i) First we prove that  $\dim_{\mathbf{k}} H^1(G_v/I_v, \text{Ad}^0 \bar{\rho}_v) = 1$ . By Proposition 18 of [S2] the dimension of  $H^1(G_v/I_v, \text{Ad}^0 \bar{\rho}_v)$  is the same as that of  $H^0(G_v, \text{Ad}^0 \bar{\rho}_v)$ . Thus it suffices to show that  $H^0(G_v, \text{Ad}^0 \bar{\rho}_v)$  is one-dimensional. This follows from

$$\begin{pmatrix} \lambda & \lambda \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} 1/\lambda & -1/\lambda \\ 0 & 1/\lambda \end{pmatrix} = \begin{pmatrix} a+c & -2a+b-c \\ c & -(a+c) \end{pmatrix},$$

where  $a, b, c \in \mathbf{k}$ .

Next we prove that  $\dim_{\mathbf{k}} L_v = 1$ . Let  $f_1 : W[[X]] \rightarrow \mathbf{k}[X]/(X^2)$  be the morphism in  $\mathcal{A}$  determined by  $f_1(X) = X$ . We define  $\rho_1 : G_v \rightarrow \text{GL}_2(\mathbf{k}[X]/(X^2))$  by the composition  $f_1 \circ \rho_v$ . The images of  $s$  and  $t$  satisfy the relation  $sts^{-1} = t^{q_v}$ . Let  $c_1$  be the 1-cocycle corresponding to  $\rho_1$ . The space  $L_v$  is spanned by the class of  $c_1$ . Thus we have  $\dim_{\mathbf{k}} L_v = 1$ .

(ii) The conditions (P1), (P2), (P3), (P6) and (P7) follow from the definition of  $(\mathcal{C}_v, L_v)$ .

First we prove the condition (P4). Suppose that we have rings  $(R_1, \mathfrak{m}_{R_1}), (R_2, \mathfrak{m}_{R_2}) \in \mathcal{A}$ , lifts  $\rho_i \in \mathcal{C}_v(R_i)$ , ideals  $I_i \subset R_i$ , and an identification  $\phi : R_1/I_1 \xrightarrow{\sim} R_2/I_2$  under which  $\rho_1 \pmod{I_1} = \rho_2 \pmod{I_2}$ . Take  $\alpha_i \in \text{Hom}_{\mathcal{A}}(W(\mathbf{k})[[X]], R_i)$  and  $M_i \in 1 + \text{M}_2(\mathfrak{m}_{R_i})$  such that  $\rho_i = M_i(\alpha_i \circ \rho_v)M_i^{-1}$ ,  $i = 1, 2$ . We claim that there exist  $\alpha \in \text{Hom}_{\mathcal{A}}(W(\mathbf{k})[[X]], R_3)$  and  $M \in 1 + \text{M}_2(\mathfrak{m}_{R_3})$  such that  $M(\alpha \circ \rho_v)M^{-1} = \rho_1 \oplus \rho_2$ . By conjugating  $\rho_1$  by some lift of  $M_2 \pmod{I_2}$  to  $R_1$ , we may assume that  $M_2 = 1$ . Since  $\alpha_1 \circ \rho_v(s) = \alpha_2 \circ \rho_v(s)$ , the matrix  $M_1 \pmod{I_1}$  commutes with  $(\alpha_1 \pmod{I_1}) \circ \rho_v(s)$ . Let  $\begin{pmatrix} 1+m_1 & m_2 \\ 0 & 1+m_3 \end{pmatrix} \in 1 + \text{M}_2(\mathfrak{m}_{R_1})$  be a lift of  $M_1 \pmod{I_1}$ . Put  $M'_1 := \begin{pmatrix} 1+m_1 & m_2 \\ 0 & 1+m_3-x \end{pmatrix}$ ,

where  $x := (q_v - 1)m_2 - m_1 + m_3$ . Note that  $x \in I_1$ . Then  $M'_1 \in 1 + M_2(\mathfrak{m}_{R_1})$  commutes with  $\alpha_1 \circ \rho_v(s)$ . We now replace  $M_1$  by  $\widetilde{M}_1 := M_1 M_1'^{-1}$  and  $\alpha_1$  by some  $\widetilde{\alpha}_1 : W(\mathbf{k})[[X]] \rightarrow R_1$  such that  $\widetilde{M}_1(\widetilde{\alpha}_1 \circ \rho_v)\widetilde{M}_1^{-1} = M_1(\alpha_1 \circ \rho_v)M_1^{-1}$ . Defining  $M := (\widetilde{M}_1, 1) \in 1 + M_2(\mathfrak{m}_{R_3})$  and  $\alpha := (\widetilde{\alpha}_1, \alpha_2) : W(\mathbf{k})[[X]] \rightarrow R_3$ , the condition (P4) is verified.

Next we prove the condition (P5). Suppose that we have a ring  $R \in \mathcal{A}$  and a  $\delta_v$ -lift  $\rho$  of  $\bar{\rho}_v$  to  $R$  such that each  $\rho \pmod{\mathfrak{m}_R^n} \in \mathcal{C}_v(R/\mathfrak{m}_R^n)$ . Put  $\rho_n := \rho \pmod{\mathfrak{m}_R^n}$ . Take  $\alpha_n \in \text{Hom}_{\mathcal{A}}(W(\mathbf{k})[[X]], R/\mathfrak{m}_R^n)$  and  $M_n \in 1 + M_2(\mathfrak{m}_R/\mathfrak{m}_R^n)$  such that  $\rho_n = M_n(\alpha_n \circ \rho_v)M_n^{-1}$ . We claim that there exist  $\alpha \in \text{Hom}_{\mathcal{A}}(R_v, R)$  and  $M \in 1 + M_2(\mathfrak{m}_R)$  such that  $M(\alpha \circ \rho_v)M^{-1} = \rho$ . Put  $S_n := \{(\alpha'_n, M'_n) \mid \rho_n = M'_n(\alpha'_n \circ \rho_v)M_n'^{-1}\}$ . Since  $\mathcal{C}_v(R/\mathfrak{m}_R^n)$  is finite,  $S_n$  is finite. For each  $n$ ,  $S_n$  is not empty set. Thus  $\varinjlim_n S_n$  is not empty set, the condition (P5) is verified.  $\square$

### 3.2 Case II

Suppose  $\bar{\rho}_v$  is ramified and  $v \nmid p$ . In addition, suppose  $\bar{\rho}_v(I_v)$  is of order prime to  $p$ . Define the functor  $\mathcal{C}_v : \mathcal{A} \rightarrow \mathbf{Sets}$  by

$$\mathcal{C}_v(R) := \{\rho : G_v \rightarrow \text{GL}_2(R) \mid \rho \pmod{\mathfrak{m}_R} = \bar{\rho}_v, \rho(I_v) \xrightarrow{\sim} \bar{\rho}_v(I_v), \det \rho = \delta_v\}.$$

Moreover, if  $\rho_0 : G_v \rightarrow \text{GL}_2(\mathbf{k}[X]/(X^2))$  denotes the trivial lift of  $\bar{\rho}_v$ , we define a subspace  $L_v \subset H^1(G_v, \text{Ad}^0 \bar{\rho}_v)$  to be the set

$$\{[c] \in H^1(G_v, \text{Ad}^0 \bar{\rho}_v) \mid (1 + Xc)\rho_0 \in \mathcal{C}_v(\mathbf{k}[X]/(X^2))\}.$$

**Lemma 2.** *We have*

- (i)  $\dim_{\mathbf{k}} L_v = \dim_{\mathbf{k}} H^0(G_v, \text{Ad}^0 \bar{\rho}_v)$ .
- (ii) *The pair  $(\mathcal{C}_v, L_v)$  satisfies the conditions (P1)-(P7) of Definition 1.*

*Proof.* This lemma follows from the definitions and the Schur-Zassenhaus theorem.  $\square$

### 3.3 Case III

Suppose  $\bar{\rho}_v$  is ramified and  $v \nmid p$ . In addition, suppose the order of  $\bar{\rho}_v(I_v)$  is divisible by  $p$ . By Lemma 3.1 of [G], since  $p \geq 7$ , we may assume that  $\bar{\rho}_v$  is given by the form

$$\bar{\rho}_v = \begin{pmatrix} \varphi \bar{\chi}_p & \gamma \\ 0 & \varphi \end{pmatrix},$$

for a character  $\varphi : G_v \rightarrow \mathbf{k}^\times$  and a nonzero continuous function  $\gamma : G_v \rightarrow \mathbf{k}$ . The functor  $\mathcal{C}_v : \mathcal{A} \rightarrow \mathbf{Sets}$  is given by

$$\mathcal{C}_v(R) := \{\rho : G_v \rightarrow \text{GL}_2(R) \mid \text{there are } \tilde{\gamma} \in \text{Map}(G_v, R) \text{ and } M \in 1 + M_2(\mathfrak{m}_R) \text{ such that } \rho = M \begin{pmatrix} \widehat{\varphi} \chi_p & \tilde{\gamma} \\ 0 & \widehat{\varphi} \end{pmatrix} M^{-1}, \tilde{\gamma} \pmod{\mathfrak{m}_R} = \gamma\}.$$

Moreover, if  $\rho_0 : G_v \rightarrow \text{GL}_2(\mathbf{k}[X]/(X^2))$  denotes the trivial lift of  $\bar{\rho}_v$ , we define a subspace  $L_v \subset H^1(G_v, \text{Ad}^0 \bar{\rho}_v)$  to be the set

$$\{[c] \in H^1(G_v, \text{Ad}^0 \bar{\rho}_v) \mid (1 + Xc)\rho_0 \in \mathcal{C}_v(\mathbf{k}[X]/(X^2))\}.$$

**Lemma 3.** *We have*

(i)  $\dim_{\mathbf{k}} L_v = \dim_{\mathbf{k}} H^0(G_v, \text{Ad}^0 \bar{\rho}_v)$ .

(ii) *The pair  $(\mathcal{C}_v, L_v)$  satisfies the conditions (P1)-(P7) of Definition 1.*

*Proof.* The proof of this lemma is almost identical argument as in [T, Section 1(E3)].  $\square$

## 4 Lifting theorem over arbitrary number fields

In this section, we give a generalization of Theorem 1 of [R1] to arbitrary number fields.

We define  $\delta : G_{K,S} \rightarrow W(\mathbf{k})^\times$  by  $\widehat{\det \bar{\rho}} \widehat{\chi}_p^{-1} \chi_p$ . Throughout this section, we consider lifts of a fixed determinant  $\delta$  and we always assume the following:

- The order of the image of  $\bar{\rho}$  is divisible by  $p$ .

By the Schur-Zassenhaus theorem, if the order of the image of  $\bar{\rho}$  is prime to  $p$ , we can find a lift to  $W(\mathbf{k})$  of  $\bar{\rho}$ . Since  $p \geq 7$  and the order of the image of  $\bar{\rho}$  is divisible by  $p$ , we see from Section 260 of [D] that the image of  $\bar{\rho}$  is contained in the Borel subgroup of  $\text{GL}_2(\mathbf{k})$  or the projective image of  $\bar{\rho}$  is conjugate to either  $\text{PGL}_2(\mathbb{F}_{p^r})$  or  $\text{PSL}_2(\mathbb{F}_{p^r})$  for some  $r \in \mathbb{Z}_{>0}$ . In the Borel case, by Theorem 2 of [K] we have a lift of  $\bar{\rho}$  to  $W(\mathbf{k})$ . Thus we may assume that the projective image of  $\bar{\rho}$  is equal to  $\text{PSL}_2(\mathbb{F}_{p^r})$  or  $\text{PGL}_2(\mathbb{F}_{p^r})$ . Then, by Lemma 17 of [R1],  $\text{Ad}^0 \bar{\rho}$  is an irreducible  $G_{K,S}$ -module. (Note that one may replace the assumption that the image of  $\bar{\rho}$  contains  $\text{SL}_2(\mathbf{k})$  in [R1] with the assumption that the projective image of  $\bar{\rho}$  contains  $\text{PSL}_2(\mathbb{F}_p)$  without affecting the proof.) The irreducibility of  $\text{Ad}^0 \bar{\rho}$  implies that of  $\text{Ad}^0 \bar{\rho}(1)$ .

Let  $K(\text{Ad}^0 \bar{\rho})$  be the fixed field of  $\text{Ker}(\text{Ad}^0 \bar{\rho})$ . Put  $E = K(\text{Ad}^0 \bar{\rho})K(\mu_p)$  and  $D = K(\text{Ad}^0 \bar{\rho}) \cap K(\mu_p)$ .

**Lemma 4.** *We have*

$$H^1(\text{Gal}(E/K), \text{Ad}^0 \bar{\rho}) = H^1(\text{Gal}(E/K), \text{Ad}^0 \bar{\rho}(1)) = 0.$$

*Proof.* First we prove that  $H^1(\text{Gal}(E/K), \text{Ad}^0 \bar{\rho}) = 0$ . It suffices to show that  $H^1(\text{SL}_2(\mathbb{F}_{p^r}), \text{Ad}^0 \bar{\rho}) = 0$  and  $H^1(\text{GL}_2(\mathbb{F}_{p^r}), \text{Ad}^0 \bar{\rho}) = 0$ , where  $\text{GL}_2(\mathbb{F}_{p^r})$  and  $\text{SL}_2(\mathbb{F}_{p^r})$  act on  $\text{Ad}^0 \bar{\rho}$  by conjugation. By Lemma 2.48 of [DDT], we see  $H^1(\text{SL}_2(\mathbb{F}_{p^r}), \text{Ad}^0 \bar{\rho}) = 0$ . Since the index of  $\text{SL}_2(\mathbb{F}_{p^r})$  in  $\text{GL}_2(\mathbb{F}_{p^r})$  is prime to  $p$ , we have  $H^1(\text{GL}_2(\mathbb{F}_{p^r}), \text{Ad}^0 \bar{\rho}) = 0$ .

Next we prove that  $H^1(\text{Gal}(E/K), \text{Ad}^0 \bar{\rho}(1)) = 0$ . As  $D \subset K(\mu_p)$ , we see  $\text{Gal}(K(\text{Ad}^0 \bar{\rho})/D)$  contains the commutator subgroup of  $\text{Gal}(K(\text{Ad}^0 \bar{\rho})/K)$ . Since the projective image of  $\bar{\rho}$  is equal to  $\text{PSL}_2(\mathbb{F}_{p^r})$  or  $\text{PGL}_2(\mathbb{F}_{p^r})$ , we see this commutator subgroup is just  $\text{PSL}_2(\mathbb{F}_{p^r})$ . Thus  $\text{Gal}(K(\text{Ad}^0 \bar{\rho})/K)/\text{PSL}_2(\mathbb{F}_{p^r}) \rightarrow \text{Gal}(D/K)$  is surjective, and so  $[D : K] = 1$  or  $2$ . Assume that  $[K(\mu_p) : K] = 1$ , then  $H^1(\text{Gal}(E/K), \text{Ad}^0 \bar{\rho}(1))$  is isomorphic to  $H^1(\text{Gal}(E/K), \text{Ad}^0 \bar{\rho})$ . Consequently  $H^1(\text{Gal}(E/K), \text{Ad}^0 \bar{\rho}(1)) = 0$ .

Assume that  $[K(\mu_p) : K] \geq 3$ , or  $[K(\mu_p) : K] = 2$  and  $[D : K] = 1$ . We apply the inflation-restriction sequence to  $\text{Gal}(E/K)$  and its normal subgroup  $\text{Gal}(E/K(\text{Ad}^0 \bar{\rho}))$ . Since  $\text{Gal}(K_S/E)$  fixes  $\text{Ad}^0 \bar{\rho}(1)$  we see  $\text{Ad}^0 \bar{\rho}(1)^{\text{Gal}(E/K(\text{Ad}^0 \bar{\rho}))} = \text{Ad}^0 \bar{\rho}(1)^{\text{Gal}(K_S/K(\text{Ad}^0 \bar{\rho}))}$ . We get the exact sequence

$$\begin{aligned} 0 \rightarrow H^1(\text{Gal}(K(\text{Ad}^0 \bar{\rho})/K), \text{Ad}^0 \bar{\rho}(1)^{\text{Gal}(K_S/K(\text{Ad}^0 \bar{\rho}))}) &\rightarrow H^1(\text{Gal}(E/K), \text{Ad}^0 \bar{\rho}(1)) \\ &\rightarrow H^1(\text{Gal}(E/K(\text{Ad}^0 \bar{\rho})), \text{Ad}^0 \bar{\rho}(1)^{\text{Gal}(K(\text{Ad}^0 \bar{\rho})/K)}). \end{aligned}$$

The last term is trivial as  $\text{Gal}(E/K(\text{Ad}^0 \bar{\rho}))$  has order prime to  $p$ . As  $\text{Gal}(K_S/K(\text{Ad}^0 \bar{\rho}))$  acts trivially on  $\text{Ad}^0 \bar{\rho}$  we see the action of  $\text{Gal}(K_S/K(\text{Ad}^0 \bar{\rho}))$  is  $\chi_p|_{\text{Gal}(K_S/K(\text{Ad}^0 \bar{\rho}))}$ , which is nontrivial, so  $\text{Ad}^0 \bar{\rho}(1)^{\text{Gal}(K_S/K(\text{Ad}^0 \bar{\rho}))} = 0$ . Thus the left term in the sequence is trivial, so  $H^1(\text{Gal}(E/K), \text{Ad}^0 \bar{\rho}(1)) = 0$ .

Assume that  $[K(\mu_p) : K] = 2$  and  $[D : K] = 2$ , then we have  $K(\mu_p) = D$ . Note that  $\text{PSL}_2(\mathbb{F}_{p^r})$  has no non-trivial abelian quotients. If the projective image of  $\bar{\rho}$  is  $\text{PSL}_2(\mathbb{F}_{p^r})$  for some  $r \in \mathbb{Z}_{>0}$ , then  $\text{Gal}(E/K)$  has no non-trivial abelian quotients. This contradicts the assumption that  $[K(\mu_p) : K] = 2$ . Hence, we assume that the projective image of  $\bar{\rho}$  is  $\text{PGL}_2(\mathbb{F}_{p^r})$  for some  $r \in \mathbb{Z}_{>0}$ . Since the index of  $\text{PSL}_2(\mathbb{F}_{p^r})$  in  $\text{PGL}_2(\mathbb{F}_{p^r})$  is equal to the index of  $\text{Gal}(E/K(\mu_p))$  in  $\text{Gal}(E/K)$ ,  $\text{Gal}(E/K(\mu_p))$  is isomorphic to  $\text{PSL}_2(\mathbb{F}_{p^r})$ . We have

$$H^1(\text{Gal}(E/K), \text{Ad}^0 \bar{\rho}(1)) \hookrightarrow H^1(\text{Gal}(E/K(\mu_p)), \text{Ad}^0 \bar{\rho}(1)).$$

Since  $\text{Ad}^0 \bar{\rho}(1)$  is isomorphic to  $\text{Ad}^0 \bar{\rho}$  as a  $\text{Gal}(E/K(\mu_p))$ -module and the cohomology group  $H^1(\text{Gal}(E/K(\mu_p)), \text{Ad}^0 \bar{\rho})$  is zero, the proof is complete.  $\square$

**Lemma 5.** *If a pair  $(\mathcal{C}_v, L_v)$  which is locally admissible is given for each  $v \in S_f$  and each elements  $\phi \in H^1_{\{L_v^\perp\}}(G_{K,S}, \text{Ad}^0 \bar{\rho}(1))$  and  $\psi \in H^1_{\{L_v\}}(G_{K,S}, \text{Ad}^0 \bar{\rho})$  are not zero, then we can find a prime  $w \notin S$  and a locally admissible pair  $(\mathcal{C}_w, L_w)$  such that*

- (1)  $\dim_{\mathbf{k}} H^1(G_w/I_w, \text{Ad}^0 \bar{\rho}) = \dim_{\mathbf{k}} L_w = 1$ ,
- (2) *the image of  $\psi$  in  $H^1(G_w/I_w, \text{Ad}^0 \bar{\rho})$  is not zero,*
- (3) *the image of  $\phi$  in  $H^1(G_w, \text{Ad}^0 \bar{\rho}(1))/L_w^\perp$  is not zero.*

*Proof.* Note that Lemma 4 implies that the restrictions of the cocycles  $\psi$  and  $\phi$  are non-zero homomorphisms  $\phi : \text{Gal}(K_S/E) \rightarrow \text{Ad}^0 \bar{\rho}(1)$  and  $\psi : \text{Gal}(K_S/E) \rightarrow \text{Ad}^0 \bar{\rho}$ . Let  $E_\phi$  and  $E_\psi$  be the fixed fields of the respective kernels. Then,  $\text{Gal}(E_\phi/E) \rightarrow \text{Ad}^0 \bar{\rho}(1)$  and  $\text{Gal}(E_\psi/E) \rightarrow \text{Ad}^0 \bar{\rho}$  are injective homomorphisms of  $\mathbb{F}_p[G_{K,S}]$ -modules. Since  $\text{Ad}^0 \bar{\rho}$  is irreducible  $G_{K,S}$ -module, these morphisms are bijective, and we see  $E_\phi \cap E_\psi = E_\psi (= E_\phi)$  or  $E$ . If the intersection is  $E$ , then  $\text{Gal}(E_\phi E_\psi/E)$  is isomorphic to  $\text{Gal}(E_\phi/E) \times \text{Gal}(E_\psi/E)$ . If the intersection is  $E_\psi$ , then  $\text{Gal}(E_\phi E_\psi/E)$  is isomorphic to  $\text{Gal}(E_\psi/E)$  and  $\text{Gal}(E_\phi/E)$ . Therefore,  $\text{Gal}(E_\phi E_\psi/E)$  may be regarded as a  $\mathbf{k}[\text{Gal}(E/K)]$ -module, moreover, natural homomorphisms  $\text{Gal}(E_\phi E_\psi/E) \rightarrow \text{Ad}^0 \bar{\rho}(1)$  and  $\text{Gal}(E_\phi E_\psi/E) \rightarrow \text{Ad}^0 \bar{\rho}$  are surjective. Since  $\text{PSL}_2(\mathbb{F}_{p^r})$  has no non-trivial abelian quotients, the image of the morphism  $\tilde{\rho} \times \chi_p : G_{K,S} \rightarrow \text{PGL}_2(\mathbf{k}) \times \mathbf{k}^\times$  contains  $\text{PSL}_2(\mathbb{F}_{p^r}) \times 1$ , where  $\tilde{\rho}$  is the projective image of  $\bar{\rho}$  and  $\chi_p$  is the mod  $p$  cyclotomic character of  $G_{K,S}$ . Thus there is an element  $\sigma \in \text{Gal}(E/K)$  such that  $\chi_p(\sigma) = 1$  and  $\bar{\rho}(\sigma) = \begin{pmatrix} \lambda & \lambda \\ 0 & \lambda \end{pmatrix}$ , for some element  $\lambda \in \mathbf{k}^\times$ . We denote by  $\tilde{\sigma}$  a lift to  $\text{Gal}(E_\phi E_\psi/K)$  of  $\sigma$ . Let  $L$  be the subset of  $\text{Ad}^0 \bar{\rho}$  whose elements have the form  $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$  and let  $L'$  be the subset of  $\text{Ad}^0 \bar{\rho}(1)$  whose elements have the form  $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ . Since  $L$  and  $L'$  are two-dimensional, there exists  $\tau \in \text{Gal}(E_\phi E_\psi/E)$  such that  $\psi(\tau) \notin -\psi(\tilde{\sigma}) + L$  and  $\phi(\tau) \notin -\phi(\tilde{\sigma}) + L'$ .

By the Čebotarev density theorem, we can choose a place  $w \notin S$  which is unramified in  $E_\phi E_\psi/K$  such that  $\text{Frob}_w = \tau \tilde{\sigma}$ . Take  $\mathcal{C}_w$  and  $L_w$  as in Case I. By Lemma 1 of this paper and Lemma 4.8 of [BK], it follows that  $(w, \mathcal{C}_w, L_w)$

has the desired properties. (Note that one may replace function fields in [BK] with number fields without affecting the proof.)  $\square$

**Lemma 6.** *Suppose that one is given locally admissible pairs  $(\mathcal{C}_v, L_v)_{v \in S_f}$  such that*

$$\sum_{v \in S_f} \dim_{\mathbf{k}} L_v \geq \sum_{v \in S} \dim_{\mathbf{k}} H^0(G_v, \text{Ad}^0 \bar{\rho}).$$

*Then we can find a finite set of places  $T \supset S$  and locally admissible pairs  $(\mathcal{C}_v, L_v)_{v \in T \setminus S}$  such that*

$$H^1_{\{L_v^\perp\}}(G_{K,T}, \text{Ad}^0 \bar{\rho}(1)) = 0.$$

*Proof.* Suppose that  $0 \neq \phi \in H^1_{\{L_v^\perp\}}(G_{K,S}, \text{Ad}^0 \bar{\rho}(1))$ . By the assumption of the lemma and Theorem 4.50 of [H], we see that  $\dim_{\mathbf{k}} H^1_{\{L_v\}}(G_{K,S}, \text{Ad}^0 \bar{\rho}) \geq \dim_{\mathbf{k}} H^1_{\{L_v^\perp\}}(G_{K,S}, \text{Ad}^0 \bar{\rho}(1))$ . Then we can find  $0 \neq \psi \in H^1_{\{L_v\}}(G_{K,S}, \text{Ad}^0 \bar{\rho})$ . Thus we can find a place  $w \notin S$  and a locally admissible pair  $(\mathcal{C}_w, L_w)$  such that

- (1)  $\dim_{\mathbf{k}} H^1(G_w/I_w, \text{Ad}^0 \bar{\rho}) = \dim_{\mathbf{k}} L_w$ ,
- (2)  $H^1_{\{L_v\}}(G_{K,S}, \text{Ad}^0 \bar{\rho}) \rightarrow H^1(G_w/I_w, \text{Ad}^0 \bar{\rho})$  is surjective,
- (3) the image of  $\phi$  in  $H^1(G_w, \text{Ad}^0 \bar{\rho}(1))/L_w^\perp$  is not zero,

by Lemma 5. We have an injection

$$H^1_{\{L_v^\perp\}}(G_{K,S}, \text{Ad}^0 \bar{\rho}(1)) \hookrightarrow H^1_{\{L_v^\perp\} \cup \{H^1(G_w, \text{Ad}^0 \bar{\rho}(1))\}}(G_{K, S \cup \{w\}}, \text{Ad}^0 \bar{\rho}(1))$$

and we see that its cokernel has order equal to

$$\#\text{Coker}(H^1_{\{L_v\}}(G_{K,S}, \text{Ad}^0 \bar{\rho}) \rightarrow H^1(G_w/I_w, \text{Ad}^0 \bar{\rho})),$$

by applying Theorem 4.50 of [H] to

$$H^1_{\{L_v^\perp\}}(G_{K,S}, \text{Ad}^0 \bar{\rho}(1))$$

and

$$H^1_{\{L_v^\perp\} \cup \{H^1(G_w, \text{Ad}^0 \bar{\rho}(1))\}}(G_{K, S \cup \{w\}}, \text{Ad}^0 \bar{\rho}(1)).$$

Thus

$$H^1_{\{L_v^\perp\}}(G_{K,S}, \text{Ad}^0 \bar{\rho}(1)) = H^1_{\{L_v^\perp\} \cup \{H^1(G_w, \text{Ad}^0 \bar{\rho}(1))\}}(G_{K, S \cup \{w\}}, \text{Ad}^0 \bar{\rho}(1)),$$

and we obtain an exact sequence

$$\begin{aligned} 0 &\rightarrow H^1_{\{L_v^\perp\} \cup \{L_w^\perp\}}(G_{K, S \cup \{w\}}, \text{Ad}^0 \bar{\rho}(1)) \rightarrow H^1_{\{L_v^\perp\}}(G_{K,S}, \text{Ad}^0 \bar{\rho}(1)) \\ &\rightarrow H^1(G_w, \text{Ad}^0 \bar{\rho}(1))/L_w^\perp. \end{aligned}$$

Hence  $\phi \notin H^1_{\{L_v^\perp\} \cup \{L_w^\perp\}}(G_{K, S \cup \{w\}}, \text{Ad}^0 \bar{\rho}(1)) \subset H^1_{\{L_v^\perp\}}(G_{K,S}, \text{Ad}^0 \bar{\rho}(1))$ . The lemma will follow by repeating such a computation.  $\square$

Let  $S'$  denote the set of places of  $K$  consisting of the places above  $p$ , the infinite places and the places at which  $\bar{\rho}$  is ramified.

*Proof of Theorem.* This follows almost at once from Proposition 1 and Lemma 6. For each places  $v$  satisfying  $v \in S'_f$  and  $v \nmid p$ , take  $\mathcal{C}_v$  and  $L_v$  as in Case II or Case III. For places  $v \mid p$ , take  $\mathcal{C}_v$  and  $L_v$  as the collection of all  $\delta|_{G_v}$ -lifts of  $\bar{\rho}|_{G_v}$  and  $H^1(G_v, \text{Ad}^0 \bar{\rho})$ , respectively. By Theorem 4.52 of [H] and the assumption of Theorem, we have

$$\sum_{v|p} \dim_{\mathbf{k}} L_v = \sum_{v|p} \dim_{\mathbf{k}} H^0(G_v, \text{Ad}^0 \bar{\rho}) + \sum_{v|p} [K_v : \mathbb{Q}_p] \dim_{\mathbf{k}} \text{Ad}^0 \bar{\rho}$$

and thus we obtain

$$\sum_{v \in S'_f} \dim_{\mathbf{k}} L_v \geq \sum_{v \in S'} \dim_{\mathbf{k}} H^0(G_v, \text{Ad}^0 \bar{\rho}).$$

□

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