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ON GAUSSIAN DECAY ESTIMATES OF SOLUTIONS TO SOME LINEAR ELLIPTIC EQUATIONS AND ITS APPLICATIONS

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1. INTRODUCTION

In this paper we consider the following linear elliptic equations

$$(1.1) \quad (-\mathcal{L} - B \cdot \nabla - \lambda)u = f, \quad x \in \mathbb{R}^n,$$

where $n \geq 1$ and

$$(1.2) \quad \mathcal{L} = \Delta + \frac{x}{2} \cdot \nabla + \frac{n}{2},$$

with $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ and $\nabla = \nabla_x = (\partial_{x_1}, \dots, \partial_{x_n})^\top$. The given scalar and vector valued functions $\lambda = \lambda(x)$ and $B = B(x)$ are assumed to belong to $L^\infty(\mathbb{R}^n)$ and $(L^\infty(\mathbb{R}^n))^n$, respectively, and f is a given scalar valued function. If we do not mention explicitly, each function in this paper is assumed to be real valued. Eq. (1.1) is interpreted in the weak sense, i.e., u belongs to $W_{loc}^{1,2}(\mathbb{R}^n)$ and satisfies

$$(1.3) \quad \int_{\mathbb{R}^n} \nabla u(x) \cdot \nabla \varphi(x) - \left\{ \left(\frac{x}{2} + B(x) \right) \cdot \nabla u(x) + \left(\lambda(x) + \frac{n}{2} \right) u(x) \right\} \varphi(x) dx = \int_{\mathbb{R}^n} f(x) \varphi(x) dx,$$

for any smooth function φ with compact support.

The operator \mathcal{L} often appears in the analysis of self-similar solutions to linear or nonlinear heat equations, and its properties are well understood by now. For example, \mathcal{L} is realized as a self-adjoint operator in the Gaussian weighted L^2 space:

$$(1.4) \quad L_G^2 = \left\{ f \in L^2(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{G(x)} \right\}, \quad G(x) = (4\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4}}.$$

In L_G^2 the spectrum $\sigma(\mathcal{L})$ consists of semisimple eigenvalues and is given by $\sigma(\mathcal{L}) = \left\{ -\frac{k}{2} \mid k = 0, 1, 2, \dots \right\}$; see Escobedo-Kavian [4]. Especially, for each $k \in \mathbb{N} \cup \{0\}$ the eigenspace of the eigenvalue $-\frac{k}{2}$ is spanned by the Hermite functions $\{\partial_x^\beta G\}_{|\beta|=k}$ where $\beta = (\beta_1, \dots, \beta_n)$ is a multi-index. If the space L_G^2 is replaced by the polynomial weighted spaces, then \mathcal{L} possesses the essential

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spectrum, and moreover, it is not even a sectorial operator. More precisely, for each $m \geq 0$ let L_m^2 be the Hilbert space defined by

$$(1.5) \quad L_m^2 = \{f \in L^2(\mathbb{R}^n) \mid \|f\|_{L_m^2}^2 = \int_{\mathbb{R}^n} (1 + |x|^2)^m |f(x)|^2 dx < \infty\}.$$

Then Galloway-Wayne [10] showed that \mathcal{L} is realized as a closed operator in L_m^2 with its spectrum

$$(1.6) \quad \sigma(\mathcal{L}) = \{\mu \in \mathbb{C} \mid \operatorname{Re}(\mu) \leq \frac{n}{4} - \frac{m}{2}\} \cup \{-\frac{k}{2} \mid k = 0, 1, 2, \dots\}.$$

Moreover, if $k \in \mathbb{N}$ satisfies $m > \frac{n}{2} + k$, then $-\frac{k}{2}$ is a semisimple eigenvalue with multiplicity $\frac{(n+k-1)!}{k!(n-1)!}$, and the set $\{\mu \in \sigma(\mathcal{L}) \mid \operatorname{Re}(\mu) \leq \frac{n}{4} - \frac{m}{2}\}$ consists of eigenvalues whose eigenfunctions decay at spatial infinity in some polynomial order. This implies by functional analytic considerations that in the case $\mu > \frac{n}{4} - \frac{m}{2}$, if f belongs to L_G^2 , then a solution $u \in L_m^2$ to $-\mathcal{L}u + \mu u = f$ actually belongs to L_G^2 . We can expect this property also for (1.1). Set

$$(1.7) \quad \lambda_* = \lim_{R \rightarrow \infty} \operatorname{esssup}_{|x| \geq R} \lambda(x), \quad B_* = \lim_{R \rightarrow \infty} \operatorname{esssup}_{|x| \geq R} |B(x)|.$$

Note that λ_* can be negative. Then we have

Proposition 1.1. *Let $f \in L_G^2$. Assume that $\frac{m}{2} > \frac{n}{4} + \lambda_* + \frac{B_*^2}{2}$. Let $u \in L_m^2 \cap W_{loc}^{1,2}(\mathbb{R}^n)$ be a solution to (1.1). Then $u \in L_G^2$.*

The proof of this proposition will be given in the appendix.

In this paper we are interested in pointwise estimates of solutions to (1.1) when f decays exponentially. Our main results are three lemmas stated below, in which we assume that $u \in L_G^2 \cap W_{loc}^{1,2}(\mathbb{R}^n)$. We can always replace this assumption by $u \in L_m^2 \cap W_{loc}^{1,2}(\mathbb{R}^n)$ with $\frac{m}{2} > \frac{n}{4} + \lambda_* + \frac{B_*^2}{2}$ by Proposition 1.1. Set

$$(1.8) \quad H_G^s = \{f \in L_G^2 \mid \partial_x^\beta f \in L_G^2, |\beta| \leq s\}.$$

Our first main result is

Lemma 1.1. *Assume that f satisfies*

$$(1.9) \quad |f(x)| \leq C_\epsilon e^{-\frac{1-\epsilon}{4}|x|^2}, \quad x \in \mathbb{R}^n,$$

for all $\epsilon > 0$. Then any solution $u \in L_G^2 \cap W_{loc}^{1,2}(\mathbb{R}^n)$ to (1.1) belongs to $u \in H_G^2 \cap W^{2,p}(\mathbb{R}^n)$ with $1 \leq p < \infty$ and satisfies

$$(1.10) \quad |u(x)| + |\nabla u(x)| \leq C'_\epsilon e^{-\frac{1-\epsilon}{4}|x|^2}, \quad x \in \mathbb{R}^n,$$

for all $\epsilon > 0$. Moreover, if f is nontrivial and if there is an $R_0 \geq 0$ such that both $f(x)$ and $u(x)$ are nonnegative for $|x| \geq R_0$, then we have

$$(1.11) \quad u(x) \geq C''_\epsilon e^{-\frac{1+\epsilon}{4}|x|^2}, \quad |x| \gg R_0,$$

for all $\epsilon > 0$. Here C'_ϵ and C''_ϵ are positive constants independent of $x \in \mathbb{R}^n$ with $|x| \gg R_0$.

Remark 1.1. If there is an $R_0 \geq 0$ such that both $f(x)$ and $u(x)$ are nonpositive for $|x| \geq R_0$, then the lower bound (1.11) follows for $-u(x)$. The same remark holds for Lemma 1.2 and Lemma 1.3 below.

Under the additional assumption on B we can show more precise pointwise estimates as follows.

Lemma 1.2. *Let $B \in (L^\infty(\mathbb{R}^n))^n$ satisfy*

$$(1.12) \quad \lim_{R \rightarrow \infty} \sup_{|x| \geq R} |x \cdot B(x)| = 0.$$

Assume that f satisfies

$$(1.13) \quad |f(x)| \leq C(1 + |x|^2)^{\lambda_* + \mu} e^{-\frac{|x|^2}{4}}, \quad x \in \mathbb{R}^n,$$

for some $\mu > 0$. Then any solution $u \in L_G^2 \cap W_{loc}^{1,2}(\mathbb{R}^n)$ to (1.1) satisfies

$$(1.14) \quad |u(x)| \leq C'(1 + |x|^2)^{\lambda_* + \mu} e^{-\frac{|x|^2}{4}}, \quad x \in \mathbb{R}^n.$$

If f satisfies (1.13) with $\mu = 0$ and if there is an $\epsilon_0 > 0$ such that

$$(1.15) \quad \lambda(x) + \left| \frac{x}{2} \cdot B(x) \right| \leq \lambda_* + \frac{1 - \epsilon_0}{\log(e + |x|^2)}, \quad \text{for } |x| \gg 1,$$

then (1.14) is replaced by

$$(1.16) \quad |u(x)| \leq C' \{ (1 + |x|^2)^{\lambda_*} \log(1 + |x|^2) \} e^{-\frac{|x|^2}{4}}, \quad x \in \mathbb{R}^n.$$

Moreover, if f is nontrivial and if there is an $R_0 \geq 0$ such that both $f(x)$ and $u(x)$ are nonnegative for $|x| \geq R_0$, then we have

$$(1.17) \quad u(x) \geq C'' (1 + |x|^2)^{\lambda_* - \nu} e^{-\frac{|x|^2}{4}}, \quad |x| \gg R_0,$$

for any $\nu > 0$. Here C' and C'' are positive constants independent of $x \in \mathbb{R}^n$ with $|x| \gg R_0$.

The logarithmic term in (1.16) is optimal. For example, if we set $f_{\mu_0} = \{\log(e + |x|^2)\}^{\mu_0} \partial_x^\beta e^{-\frac{|x|^2}{4}}$, then the direct calculations yield

$$(1.18) \quad \left| \left(\mathcal{L} + \frac{|\beta|}{2} \right) f_{\mu_0}(x) \right| = O(\{\log(e + |x|^2)\}^{\mu_0 - 1} \partial_x^\beta e^{-\frac{|x|^2}{4}}), \quad |x| \rightarrow \infty.$$

This implies the optimality of (1.16) when $B \equiv 0$, $\lambda = \frac{|\beta|}{2}$, and $\mu = 0$ in (1.13). In order to obtain the estimates without the logarithmic term we need the additional assumption on the decay of f . Let Δ_S be the Laplace-Beltrami operator on the unit sphere S^{n-1} . For simplicity, we assume that $B \equiv 0$ and λ is a constant in the next lemma. Especially, $\lambda_* = \lambda$ in this case.

Lemma 1.3. *Assume that λ is a given number and $B \equiv 0$. Let $\psi(r)$ be a given positive decreasing function on $[0, \infty)$ such that $\frac{\psi(r)}{r}$ is integrable over $[1, \infty)$. Assume that $f \in H_G^2$ satisfies*

$$(1.19) \quad |f(x)| \leq C\psi(|x|^2)(1 + |x|^2)^\lambda e^{-\frac{|x|^2}{4}}, \quad x \in \mathbb{R}^n,$$

$$(1.20) \quad |(\Delta_S f)(x)| \leq C(1 + |x|^2)^{\lambda + \mu} e^{-\frac{|x|^2}{4}}, \quad x \in \mathbb{R}^n,$$

for some $0 < \mu < 1$. Then any solution $u \in L_G^2 \cap W_{loc}^{1,2}(\mathbb{R}^n)$ to (1.1) satisfies

$$(1.21) \quad \left| u(x) - A\left(\frac{x}{|x|}\right) |x|^{2\lambda} e^{-\frac{|x|^2}{4}} \right| \leq C' |x|^{2\lambda} \left(|x|^{2(\mu-1)} + \int_{|x|^2}^\infty \frac{\psi(r)}{r} dr \right) e^{-\frac{|x|^2}{4}}, \quad |x| \gg 1.$$

Here A is a continuous functions on S^{n-1} . Moreover, if u satisfies (1.17) and if we can take $\psi(r) = (1+r)^{-\mu'}$ for some $\mu' > 0$ in (1.19), then $\min_{\sigma \in S^{n-1}} A(\sigma) > 0$. Especially, in this case we have

$$(1.22) \quad u(x) \geq \frac{C'}{2} |x|^{2\lambda} e^{-\frac{|x|^2}{4}}, \quad |x| \gg 1.$$

Remark 1.2. If f satisfies (1.20) for $\mu = 0$, then (1.21) can be replaced by

$$(1.23) \quad \left| u(x) - A\left(\frac{x}{|x|}\right) |x|^{2\lambda} e^{-\frac{|x|^2}{4}} \right| \leq C' |x|^{2\lambda} (|x|^{-2} \log(e+|x|^2)) + \int_{|x|^2}^{\infty} \frac{\psi(r)}{r} dr e^{-\frac{|x|^2}{4}}, \quad |x| \gg 1.$$

Remark 1.3. In view of (1.18) the assumption (1.19) is essential in Lemma 1.3. However, the author does not know so far whether the additional assumption (1.20) can be removed or not in order to verify the estimate $|u(x)| \leq C(1+|x|^2)^\lambda e^{-\frac{|x|^2}{4}}$.

Lemma 1.1 and Lemma 1.2 are proved by estimating the L^∞ norm of $\frac{u}{w}$ or $\frac{w}{u}$ with a suitable weight function w . We essentially use the Nash-Moser iteration arguments in this step. However, it seems to be difficult to prove Lemma 1.3 only by this iteration arguments, so we rewrite (1.1) in polar coordinates and use the representation of solutions obtained through the related ordinary differential equations. In this step we are forced to estimate $\Delta_S u$, which is the reason why (1.20) is required in our arguments. The positivity of $A(\sigma)$ in (1.22) is proved by a contradiction argument combining with Lemma 1.2 and the representation of solutions in polar coordinates.

Our lemmas are useful to study pointwise estimates of solutions to some nonlinear elliptic equations. Let us consider the equation

$$(1.24) \quad \begin{cases} -\mathcal{L}u = a \cdot \nabla(|u|^{\frac{1}{n}}u), & x \in \mathbb{R}^n, \\ \int_{\mathbb{R}^n} u(x) dx = \alpha, \end{cases}$$

where $n \geq 1$, $a \in \mathbb{R}^n$ is a given constant vector, $\alpha \in \mathbb{R}$ is a given number. Eq. (1.24) is related with the following convection-diffusion equation

$$(1.25) \quad \begin{cases} \partial_t v - \Delta v = a \cdot \nabla(|v|^{\frac{1}{n}}v), & t > 0, \quad x \in \mathbb{R}^n, \\ \int_{\mathbb{R}^n} v(x, t) dx = \alpha, & t \geq 0. \end{cases}$$

Indeed, if u_α is a solution to (1.24) then $t^{-\frac{n}{2}} u_\alpha\left(\frac{x}{\sqrt{t}}\right)$ is a solution to (1.25), which is called a self-similar solution to (1.25). In [1] Aquirre, Escobedo, and Zuazua studied (1.24) in L_G^2 and showed that

Theorem 1.1 ([1]). *For any $\alpha \in \mathbb{R}$ there is a unique solution u_α to (1.24) in $H_G^2 \cap W^{2,p}(\mathbb{R}^n)$ with $1 \leq p < \infty$ such that u_α is positive (negative) when $\alpha > 0$ (when $\alpha < 0$) and satisfies the estimate*

$$(1.26) \quad |u_\alpha(x)| \leq C(\alpha) e^{-\gamma|x|^2}, \quad x \in \mathbb{R}^n,$$

for $\gamma = \frac{1}{8}$ when $n = 1, 2, 3$, and $0 \leq \gamma < \frac{1}{2n}$ when $n \geq 4$.

Remark 1.4. Since the sign of u_α is constant in \mathbb{R}^n we can differentiate the nonlinear term $a \cdot \nabla(|u_\alpha|^{\frac{1}{n}}u_\alpha)$ pointwisely. Especially, u_α is shown to be smooth in \mathbb{R}^n by the usual elliptic regularity.

Remark 1.5. The self-similar solution $t^{-\frac{n}{2}}u_\alpha(\frac{x}{\sqrt{t}})$ is closely related with the large time behaviors of solutions to (1.25). Indeed, it is proved in Escobedo-Zuazua [6] that $t^{-\frac{n}{2}}u_\alpha(\frac{x}{\sqrt{t}})$ gives the large time asymptotic profile of solutions to (1.25).

When $n = 1$ the solution to (1.24) is explicitly written as

$$\begin{aligned} u_\alpha(x) &= -\frac{1}{2a} \frac{(e^{-a\alpha} - 1)e^{-\frac{x^2}{4}}}{\sqrt{\pi} + (e^{-a\alpha} - 1) \int_{\frac{x}{2}}^{\infty} e^{-y^2} dy}, & \alpha > 0, \\ u_\alpha(x) &= \frac{1}{2a} \frac{(e^{a\alpha} - 1)e^{-\frac{x^2}{4}}}{\sqrt{\pi} + (e^{a\alpha} - 1) \int_{\frac{x}{2}}^{\infty} e^{-y^2} dy}, & \alpha < 0. \end{aligned}$$

Especially, when $n = 1$ the solution satisfies the exact pointwise estimate

$$(1.27) \quad Ce^{-\frac{x^2}{4}} \leq |u_\alpha(x)| \leq C'e^{-\frac{x^2}{4}}, \quad x \in \mathbb{R}.$$

Motivated by (1.27) and the results of [1], Kawashima [17] studied (1.24) further in details, and improved (1.26) by

$$(1.28) \quad |u_\alpha(x)| + |\nabla_x u_\alpha(x)| \leq C(\alpha, \gamma)|\alpha|e^{-\gamma|x|^2}, \quad x \in \mathbb{R}^n,$$

for any $\gamma \in [0, \frac{1}{4}]$ and $n \geq 2$. However, it has been still open whether or not we can take $\gamma = \frac{1}{4}$ in (1.26). Moreover, it seems that no pointwise estimates have been established so far for higher order derivatives of u_α . The difficulty is that the nonlinear term $a \cdot \nabla(|u|^{\frac{1}{n}}u)$ is not smooth for $n \geq 2$, and we only know that the sign of u_α is constant in \mathbb{R}^n . In order to overcome this difficulty the precise pointwise lower bounds of u_α are also required.

Using Lemma 1.1-Lemma 1.3, we can obtain sharp pointwise estimates of solutions to (1.24) from above and below, together with the pointwise upper bounds for their derivatives.

Theorem 1.2. *Let u_α be the solution to (1.24) obtained in Theorem 1.1. Let β be any multi-index. Then u_α satisfies*

$$(1.29) \quad \left| u_\alpha(x) - A\left(\frac{x}{|x|}\right)e^{-\frac{|x|^2}{4}} \right| \leq C|x|^{-2}e^{-\frac{|x|^2}{4}}, \quad |x| \gg 1,$$

$$(1.30) \quad \left| \partial_x^\beta u_\alpha(x) - A_\beta\left(\frac{x}{|x|}\right)|x|^{|\beta|}e^{-\frac{|x|^2}{4}} \right| \leq C|x|^{|\beta|-2}e^{-\frac{|x|^2}{4}} \quad |x| \gg 1.$$

Here A is a positive (negative) continuous function on S^{n-1} when α is positive (negative), and each A_β is a continuous function on S^{n-1} . Especially, we have from (1.29),

$$(1.31) \quad C_1e^{-\frac{|x|^2}{4}} \leq |u(x)| \leq C_2e^{-\frac{|x|^2}{4}}, \quad x \in \mathbb{R}^n.$$

Remark 1.6. When $a = 0$ the solution to (1.24) is given by αG . Hence (1.30) is also considered to be optimal.

Theorem 1.2 is easily obtained from our lemmas. Especially, the "rough" lower bound (1.11) is important to ensure that each derivative of $a \cdot \nabla(|u_\alpha|^{\frac{1}{n}}u_\alpha)$ belongs to L_G^2 .

Next we consider the nonlinear elliptic problems of the Haraux-Weissler [11] type:

$$(1.32) \quad -\Delta u - \frac{x}{2} \cdot \nabla u - \frac{1}{p-1} u = |u|^{p-1} u.$$

There is much literature on (1.32), but here we just focus on the results about rapidly decreasing solutions. It is well-known that if a solution u to (1.32) is radially symmetric and decays in the order

$$(1.33) \quad u(x) = o(|x|^{-\frac{2}{p-1}}), \quad |x| \rightarrow \infty,$$

then u has actually the asymptotics of $A|x|^{\frac{2}{p-1}-n} e^{-\frac{|x|^2}{4}}$ with a constant $A \neq 0$ at $|x| \rightarrow \infty$; see Peletier-Terman-Weissler [21]. Moreover, for radially symmetric solutions, detailed structures of solutions have been achieved based on the number of the points where $u(x) = 0$; see Weissler [23, 24], Yanagida [25], Dohmen-Hirose [5], Hirose [12], and Hirose-Yanagida [15]. As for nonradially symmetric solutions, Escobedo and Kavian showed in [4] that there exist infinitely many solutions to (1.32) in H_G^1 if $1 < p < (\frac{n+2}{n-2})^+$. Here $(\frac{n+2}{n-2})^+ = \infty$ when $n = 1, 2$ and $(\frac{n+2}{n-2})^+ = \frac{n+2}{n-2}$ when $n \geq 3$. On the other hand, in [4] they also proved that there are no solutions to (1.32) in $W^{1,2}(\mathbb{R}^n)$ if $n \geq 3$ and $p \geq \frac{n+2}{n-2}$. Naito-Suzuki [18] proved that when $n \geq 2$ if a solution u is positive and satisfies (1.33) then it must be radially symmetric. As for the case of $n = 1$, any positive and rapidly decreasing solution must be even symmetric with respect to the origin by [15].

Our interest here is the decay estimates of solutions obtained in [4], in which Escobedo and Kavian proved that if u is a solution to (1.32) in H_G^1 then u satisfies

$$(1.34) \quad u(x) = O(e^{-\frac{|x|^2}{8}}), \quad |x| \gg 1.$$

Fukuizumi-Ozawa [8] discussed complex valued solutions to (1.32) and derived a condition for solutions in $W^{1,2}(\mathbb{R}^n)$ to belong to H_G^1 when $1 + \frac{4}{n} < p < (\frac{n+2}{n-2})^+$; see Remark 1.8 below. Note that the critical number $p - 1 = \frac{4}{n}$ is related with the bound of the essential spectrum of \mathcal{L} in $L^2(\mathbb{R})$ by (1.6). In [18] more general equations of the type $\Delta u + \frac{x}{2} \cdot \nabla u + ku + f(u) = 0$ are also studied, and it is proved that if $f \in C^1([0, \infty))$ satisfies $f(s) = O(s^\sigma)$ as $s \rightarrow 0$ for some $\sigma > 1$ and if a positive solution u satisfies $u = o(|x|^{-l})$ as $|x| \rightarrow \infty$ for some $l > 2k$, then $u = o(|x|^{-m})$ as $|x| \rightarrow \infty$ for every $m > 0$.

As a consequence of Proposition 1.1 and Lemma 1.2, we can improve (1.34) as follows.

Theorem 1.3. *Let $1 < p < (\frac{n+2}{n-2})^+$ and let u be a solution to (1.32) in $L_m^2 \cap W^{1,2}(\mathbb{R}^n)$ for some $\frac{m}{2} > \frac{1}{p-1} - \frac{n}{4}$. Then u satisfies*

$$(1.35) \quad |u(x)| \leq C \left\{ (1 + |x|^2)^{\frac{1}{p-1} - \frac{n}{2}} \log(e + |x|^2) \right\} e^{-\frac{|x|^2}{4}}, \quad x \in \mathbb{R}^n,$$

$$(1.36) \quad |\nabla u(x)| \leq C \left\{ (1 + |x|^2)^{\frac{1}{p-1} - \frac{n-1}{2}} \log(e + |x|^2) \right\} e^{-\frac{|x|^2}{4}}, \quad x \in \mathbb{R}^n.$$

If we apply Lemma 1.3 to (1.32) in order to establish the optimal bound without logarithmic term, we need a restriction on p due to the lack of the smoothness of the nonlinear term $|u|^{p-1}u$.

Theorem 1.4. *Let $n \leq 5$ and $2 \leq p < (\frac{n+2}{n-2})^+$. Let u be a solution to (1.32) in $L_m^2 \cap W^{1,2}(\mathbb{R}^n)$ for some $\frac{m}{2} > \frac{1}{p-1} - \frac{n}{4}$. Then u satisfies*

$$(1.37) \quad \left| u(x) - A\left(\frac{x}{|x|}\right) |x|^{\frac{2}{p-1}-n} e^{-\frac{|x|^2}{4}} \right| \leq C \{ |x|^{\frac{2}{p-1}-n-2} \log(e+|x|^2) \} e^{-\frac{|x|^2}{4}}, \quad |x| \gg 1.$$

Here A is a continuous functions on S^{n-1} .

Remark 1.7. When we apply our lemmas to (1.32) we regard $|u|^{p-1}$ or $|u|^{p-1}u$ as a given function. Especially, it is not difficult to check that Theorem 1.3 and Theorem 1.4 hold also for complex valued solutions. In this case the function A in Theorem 1.4 is in general complex valued.

Remark 1.8. In [8] it is proved that if $1 + \frac{4}{n} < p < (\frac{n+2}{n-2})^+$ and if a solution $u \in W^{1,2}(\mathbb{R}^n)$ to (1.32) satisfies $\|u\|_{L^\infty(\{|x| \geq R\})} \leq (\frac{n}{4} - \frac{1}{p-1})^{\frac{1}{p-1}}$ for some $R > 0$, then $u \in H_G^1$. In the proof of Theorem 1.3 we will show that any solution u to (1.32) in $W^{1,2}(\mathbb{R}^n)$ satisfies $u \in L^\infty(\mathbb{R}^n)$ and $\lim_{R \rightarrow \infty} \|u\|_{L^\infty(\{|x| \geq R\})} = 0$, if $1 < p < (\frac{n+2}{n-2})^+$. Especially, the smallness assumption on u stated in [8] is shown to be always satisfied.

Remark 1.9. The restriction $p \geq 2$ in Theorem 1.4 will not be optimal. The difficulty is that we need the estimate for $\Delta_S |u|^{p-1}u$ in order to apply Lemma 1.3. But since Δ_S includes the second order derivatives, the condition $p \geq 2$ is required in our arguments. The condition $n \leq 5$ comes from $2 \leq p < (\frac{n+2}{n-2})^+$. We note that if u is positive, then we can verify (1.37) for all $\sigma \in S^{n-1}$ since $|u|^{p-1}u$ becomes smooth and is estimated as

$$|\partial_x^\beta (|u(x)|^{p-1}u(x))| \leq C_\beta e^{-\frac{1+\epsilon_0}{4}|x|^2}, \quad x \in \mathbb{R}^n,$$

for any multi-index β by Lemma 1.1. However, when the solution $u \in L_G^2$ is positive it is already known by [18] that u must be radially symmetric, and thus, the asymptotics (1.37) is already established by [21].

Finally we consider the nonlinear elliptic equations of the form

$$(1.38) \quad -\Delta v + k|x|^2 v + \omega v = |v|^{p-1}v, \quad x \in \mathbb{R}^n,$$

where v is a complex valued function on \mathbb{R}^n , $k > 0$, $\omega \in \mathbb{R}$, $p > 1$, and $n \geq 1$. Eq. (1.38) is related with standing wave solutions of nonlinear Schrödinger equations. We are interested in the pointwise estimates of solutions to (1.38) which belong to the complex Hilbert space $X = \{v \in W^{1,2}(\mathbb{R}^n) \mid |x|v \in L^2(\mathbb{R}^n)\}$ with scalar product

$$\langle u, v \rangle_X = \int_{\mathbb{R}^n} \nabla u(x) \cdot \nabla \bar{v}(x) dx + \int_{\mathbb{R}^n} u(x) \bar{v}(x) dx + \int_{\mathbb{R}^n} |x|^2 u(x) \bar{v}(x) dx.$$

In Kavian-Weissler [16] the existence of infinitely many real valued solutions to (1.38) in X is proved for $1 < p < (\frac{n+2}{n-2})^+$ and for all $\omega \in \mathbb{R}$ by variational methods. But we do not go into the details on the existence or the stability of solutions to (1.38) here; for details, see the results and references in Fukuizumi [7]. As for the estimates of solutions, in [16] it is proved that when $k = \frac{1}{4}$ and $1 < p \leq (\frac{n+2}{n-2})^+$, any real valued solution $v \in X$ belongs to $C^2(\mathbb{R}^n)$ and satisfies

$$(1.39) \quad |v(x)| + |\nabla v(x)| + |\nabla^2 v(x)| \leq C_\epsilon e^{-\frac{1-\epsilon}{4}|x|^2}, \quad x \in \mathbb{R}^n,$$

for all $\epsilon \in (0, 1)$. Moreover, if $\omega > -\frac{n}{2}$ then (1.39) is valid also for $\epsilon = 0$. Under the setting of $k = 1$, $\omega > -n$, and $1 < p < (\frac{n+2}{n-2})^+$, Hirose-Ohta [13, 14] showed that any positive radially symmetric solution v to (1.38) in X satisfies the estimate

$$(1.40) \quad v(x) = A|x|^{-\frac{n+\omega}{2}} e^{-\frac{|x|^2}{2}} (1 + o(1)), \quad |x| \gg 1,$$

where A is a positive number. In [16] or [13] the maximum principle or the ODE methods are essentially used. On the other hand, Fukuizumi-Ozawa [9] discussed complex valued solutions when $k = 1$ and $1 < p < (\frac{n+2}{n-2})^+$, and established the estimate

$$(1.41) \quad |v(x)| \leq C_\epsilon e^{-\frac{1-\epsilon}{n+2}|x|^2}, \quad x \in \mathbb{R}^n,$$

for all $\epsilon > 0$. We remark that in [9] more general potentials other than the harmonic potential $|x|^2$ in (1.38) are treated. In Pankov [20] nonlinear elliptic equations of the form $-\Delta u + V(x)u = f(x, u)$, which includes (1.38), are discussed and some exponential upper bounds of solutions are obtained. In Shirai [22] the lower bound in the sense of $L^2(\mathbb{R}^n)$ for solutions to nonlinear Schrödinger equations with magnetic field are established.

By regarding the term $|v|^{p-1}$ as a given term and by using suitable transformations we can reduce (1.38) to (1.1) with a real valued solution. Then our lemmas lead to

Theorem 1.5. *Let $k > 0$, $\omega \in \mathbb{R}$, and $p > 1$. Let v be a complex valued solution to (1.38) in $X \cap L^\infty(\mathbb{R}^n)$. Then v satisfies*

$$(1.42) \quad |v(x)| \leq C \left\{ (1 + |x|^2)^{-\frac{n}{4} - \frac{\omega}{4\sqrt{k}}} \log(e + |x|^2) \right\} e^{-\frac{\sqrt{k}}{2}|x|^2}, \quad x \in \mathbb{R}^n,$$

$$(1.43) \quad |\nabla v(x)| \leq C \left\{ (1 + |x|^2)^{-\frac{n-2}{4} - \frac{\omega}{4\sqrt{k}}} \log(e + |x|^2) \right\} e^{-\frac{\sqrt{k}}{2}|x|^2}, \quad x \in \mathbb{R}^n.$$

As in the case of (1.32), in order to drop the logarithmic term in (1.42), so far we need the restriction of $p \geq 2$ due to the lack of the smoothness of the nonlinear term.

Theorem 1.6. *Let $k > 0$, $\omega \in \mathbb{R}$, and $p \geq 2$. Let v be a complex valued solution to (1.38) in $X \cap L^\infty(\mathbb{R}^n)$. Then v satisfies*

$$(1.44) \quad \left| v(x) - A \left(\frac{x}{|x|} \right) |x|^{-\frac{n}{2} - \frac{\omega}{2\sqrt{k}}} e^{-\frac{\sqrt{k}}{2}|x|^2} \right| \leq C \left\{ |x|^{-\frac{n}{2} - \frac{\omega}{2\sqrt{k}} - 2} \log(e + |x|^2) \right\} e^{-\frac{\sqrt{k}}{2}|x|^2},$$

for $|x| \gg 1$. Here A is a complex valued continuous function on S^{n-1} .

Remark 1.10. When $1 < p < (\frac{n+2}{n-2})^+$ we do not need to assume v to be in $L^\infty(\mathbb{R}^n)$ in the above theorems. Indeed, in this case if a solution v to (1.38) belongs to X then it must belong to $L^\infty(\mathbb{R}^n)$; see [9] or [20].

This paper is organized as follows. Section 2 is the main contribution of this paper. In this section we establish the pointwise estimates of solutions to the linear equation (1.1), and prove Lemma 1.1, Lemma 1.2, and Lemma 1.3. In Section 3 we consider (1.24) and prove Theorem 1.2. In Section 4 we discuss (1.32) and show Theorem 1.3 and Theorem 1.4. In Section 5 we deal with (1.38) and prove Theorem 1.5 and Theorem 1.6. The proof of Proposition 1.1 is given in the appendix.

2. POINTWISE ESTIMATES OF SOLUTIONS TO LINEAR EQUATIONS

In this section we consider pointwise estimates of solutions to (1.1). Section 2.1 is devoted to establish the upper bound of solutions stated in Lemma 1.1, Lemma 1.2, and Lemma 1.3. The lower bounds in these lemmas are proved in Section 2.2.

2.1. Pointwise upper bounds. To prove the upper bounds of solutions to (1.1) we prepare some fundamental inequalities. Due to the presence of the term $\frac{x}{2} \cdot \nabla$ we need to be careful to verify calculations at spatial infinity.

Let $u \in W_{loc}^{1,2}(\mathbb{R}^n)$ be the solution to (1.1) and let w be a positive function such that $\frac{1}{w}$ is smooth and $\frac{f}{w} \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then by direct calculations we observe that $U = \frac{u}{w}$ satisfies the equation

$$(2.1) \quad -\Delta U - \left(\frac{x}{2} + B + \frac{2\nabla w}{w}\right) \cdot \nabla U = \left(\lambda + \frac{n}{2} + \left(\frac{x}{2} + B\right) \cdot \frac{\nabla w}{w} + \frac{\Delta w}{w}\right)U + \frac{f}{w}.$$

That is, for any compactly supported smooth function φ , we have

$$(2.2) \quad \int_{\mathbb{R}^n} \nabla U \cdot \nabla \varphi dx = \int_{\mathbb{R}^n} \left(\frac{x}{2} + B + \frac{2\nabla w}{w}\right) \cdot \nabla U \varphi dx + \int_{\mathbb{R}^n} \frac{f}{w} \varphi dx \\ + \int_{\mathbb{R}^n} \left(\lambda + \frac{n}{2} + \left(\frac{x}{2} + B\right) \cdot \frac{\nabla w}{w} + \frac{\Delta w}{w}\right)U \varphi dx.$$

Clearly (2.2) holds for all $\varphi \in W^{1,2}(\mathbb{R}^n)$ with a compact support. Note that, since $\frac{f}{w} \in L^\infty(\mathbb{R}^n)$, by the elliptic regularity we always have $U \in L^\infty(\{|x| \leq R\})$ for each $R > 0$. Let χ_R be a smooth cut off function such that $\chi_R = 1$ if $|x| \leq R$ and $\chi_R = 0$ if $|x| \geq 2R$. We can take χ_R as satisfying $|x \cdot \nabla \chi_R(x)| \leq C$ where the constant C is independent of R . Then, since $|U|^{2p-2}U\chi_R \in W^{1,2}(\mathbb{R}^n)$ for all $1 \leq p < \infty$, we get

$$(2p-1) \int_{\mathbb{R}^n} \chi_R |U|^{2p-2} |\nabla U|^2 dx + \int_{\mathbb{R}^n} |U|^{2p-2} U \nabla U \cdot \nabla \chi_R dx \\ = \int_{\mathbb{R}^n} \chi_R \left(\frac{x}{2} + B + \frac{2\nabla w}{w}\right) \cdot \nabla U |U|^{2p-2} U dx \\ + \int_{\mathbb{R}^n} \chi_R \left\{ \lambda + \frac{n}{2} + \left(\frac{x}{2} + B\right) \cdot \frac{\nabla w}{w} + \frac{\Delta w}{w} \right\} |U|^{2p} dx + \int_{\mathbb{R}^n} \chi_R \frac{f}{w} |U|^{2p-2} U dx.$$

Thus, by the Young inequality $ab \leq \frac{1}{q}a^q + \frac{1}{q'}b^{q'}$ with $\frac{1}{q} + \frac{1}{q'} = 1$, $1 \leq q, q' \leq \infty$, and by the integration by parts, we have

$$(2.3) \quad \frac{2p-1}{2p^2} \int_{\mathbb{R}^n} \chi_R |\nabla(|U|^p)|^2 dx \\ \leq \int_{\mathbb{R}^n} \chi_R \left\{ \Phi_p(w) + B \cdot \frac{\nabla w}{w} + \frac{1}{2p} |B|^2 + \eta \right\} |U|^{2p} dx \\ + \frac{1}{2p} \int_{\mathbb{R}^n} \chi_R \left| \frac{f}{w\eta} \right|^{2p} dx + \frac{1}{2p} \int_{\mathbb{R}^n} \left\{ \Delta \chi_R - \nabla \chi_R \cdot \left(\frac{x}{2} + \frac{2\nabla w}{w}\right) \right\} |U|^{2p} dx,$$

where $\eta \leq 1$ is a given positive number and

$$(2.4) \quad \Phi_p(w) = -\frac{n}{4p} + \frac{1}{p} \left| \frac{\nabla w}{w} \right|^2 + \lambda + \frac{n}{2} + \frac{x}{2} \cdot \frac{\nabla w}{w} + \left(1 - \frac{1}{p}\right) \frac{\Delta w}{w}.$$

By taking $w \equiv 1$ we first claim that if $U = u \in L^2(\mathbb{R}^n)$ then $u \in L^p(\mathbb{R}^n)$ for all $2 \leq p < \infty$. Indeed, by taking $p = 1$ in (2.3), since $U \in L^2(\mathbb{R}^n)$ and

$|x \cdot \nabla \chi_R(x)| \leq C$, we can pass R to ∞ and get the bound of $\|\nabla U\|_{L^2}$ by the Fatou lemma. Hence $U \in W^{1,2}(\mathbb{R}^n)$ and the claim holds for $n = 1, 2$ by the Sobolev imbedding theorem. Let $n \geq 3$. From the Sobolev inequality

$$(2.5) \quad \|h\|_{L^{\frac{2n}{n-2}}} \leq C_s \|\nabla h\|_{L^2},$$

we have $u \in L^{\frac{2n}{n-2}}(\mathbb{R}^n)$. Set $p = \frac{n}{n-2}$ in (2.3). Since $u \in L^{\frac{2n}{n-2}}(\mathbb{R}^n)$, taking the limit $R \rightarrow \infty$, we observe again from the Fatou lemma that $\nabla(|u|^{\frac{n}{n-2}}) \in L^2(\mathbb{R}^n)$. Hence, $u \in L^{2(\frac{n}{n-2})^2}(\mathbb{R}^n)$ by (2.5). Repeating this arguments, we see that $u \in L^p(\mathbb{R}^n)$ with $p = 2(\frac{n}{n-2})^l$ for every $l \in \mathbb{N}$. This proves the claim.

Next we consider the case $u \in L^2(\mathbb{R}^n)$ and $\frac{1}{w}$ is smooth and bounded together with its derivatives. Then $U = \frac{u}{w} \in W^{1,2}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for all $2 \leq p < \infty$ by the above claim. Hence we can take the limit $R \rightarrow \infty$ in (2.3) and obtain

$$(2.6) \quad \frac{2p-1}{p^2} \int_{\mathbb{R}^n} |\nabla(|U|^p)|^2 dx \\ \leq \int_{\mathbb{R}^n} (\Phi_p(w) + |B \cdot \frac{\nabla w}{w}| + \frac{1}{2p}|B|^2 + \eta)|U|^{2p} dx + \frac{1}{2p} \int_{\mathbb{R}^n} |\frac{f}{w\eta}|^{2p} dx.$$

Recalling the Nash inequality

$$(2.7) \quad \|h\|_{L^2} \leq C_n \|\nabla h\|_{L^2}^{\frac{n}{n+2}} \|h\|_{L^1}^{\frac{2}{n+2}},$$

we get

$$(2.8) \quad C_n^{-\frac{2(n+2)}{n}} \frac{\|U\|_{L^{2p}}^{\frac{2p(n+2)}{n}}}{\|U\|_{L^p}^{\frac{4p}{n}}} \leq 2p \int_{\mathbb{R}^2} (\Phi_p(w) + |B \cdot \frac{\nabla w}{w}| + \frac{1}{2p}|B|^2 + \eta)|U|^{2p} dx \\ + \int_{\mathbb{R}^2} |\frac{f}{w\eta}|^{2p} dx.$$

The inequality (2.8) is a key tool of the Nash-Moser type iteration arguments below.

2.1.1. Gaussian upper bounds in Lemma 1.1. In this section we prove the upper bounds (1.10) in Lemma 1.1 by using (2.8) with a suitable weight function w .

For $l > 0$ let w_l be a positive and smooth function satisfying

$$w_l = 1 \quad \text{if } |x| \leq 1, \quad w_l = |x|^{-2l} \quad \text{if } |x| \geq 2,$$

and set

$$U_l = \frac{u}{w_l}.$$

We first prove

Proposition 2.1. *Let $l \geq 0$ and assume that $U_l \in L^2(\mathbb{R}^n)$. Then $U_l \in L^q(\mathbb{R}^n)$ for all $q \in [2, \infty]$.*

Proof. It suffices to show $U_l \in L^\infty(\mathbb{R}^n)$. For any $\delta > 0$ we also set $w_{l,\delta} = w_l + \delta$ and consider $U_{l,\delta} = \frac{u}{w_{l,\delta}}$. Note that by the above preparations we

already have $U_{l,\delta} \in W^{1,2}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for all $2 \leq p < \infty$ and (2.8). By direct calculations we see

$$\begin{aligned}\Phi_p(w_{l,\delta}) &= -\frac{n}{4p} + \lambda + \frac{n}{2}, & \text{if } |x| \leq 1, \\ \Phi_p(w_{l,\delta}) &= -\frac{n}{4p} + \frac{4l^2}{p|x|^2(1+\delta|x|^{2k})^2} + \lambda + \frac{n}{2} - \frac{l}{1+\delta|x|^{2l}} + (1-\frac{1}{p})\frac{2l\{2(l+1)-n\}}{|x|^2(1+\delta|x|^{2l})} \\ &\leq -\frac{n}{4p} + \lambda + \frac{n}{2} - \frac{l}{1+\delta|x|^{2l}} + \frac{4l(l+1)}{|x|^2}, & \text{if } |x| \geq 2.\end{aligned}$$

Especially, $\Phi_p(w_{l,\delta})$ is bounded uniformly in $0 \leq \delta < 1$ and $2 \leq p \leq \infty$. Let $\eta = 1$. Since $|B \cdot \frac{\nabla w}{w}| + \frac{1}{2p}|B|^2 \leq C$ uniformly in $0 \leq \delta < 1$ and $2 \leq p \leq \infty$, we have from (2.8),

$$C_n^{-\frac{2(n+2)}{n}} \frac{\|U_{l,\delta}\|_{L^{2p}(\mathbb{R}^n)}^{\frac{2p(n+2)}{n}}}{\|U_{l,\delta}\|_{L^p(\mathbb{R}^n)}^{\frac{4p}{n}}} \leq 2Cp \int_{\mathbb{R}^n} |U_{l,\delta}|^{2p} dx + \int_{\mathbb{R}^n} \frac{f}{w_{l,\delta}} |^{2p} dx,$$

that is,

$$(2.9) \quad \|U_{l,\delta}\|_{L^{2p}} \leq (2Cp)^{\frac{n}{2p(n+2)}} \|U_{l,\delta}\|_{L^p}^{\frac{2}{n+2}} \left(\|U_{l,\delta}\|_{L^{2p}}^{2p} + \left\| \frac{f}{w_{l,\delta}} \right\|_{L^{2p}}^{2p} \right)^{\frac{n}{2p(n+2)}}.$$

Here $C \geq 1$ depends only on $n, l, \|\lambda\|_{L^\infty}$, and $\|B\|_{L^\infty}$. We set $p_k = 2^k$ with $k \in \mathbb{N}$ and set $\xi_k = \|U_{l,\delta}\|_{L^{p_k}}$, $d = \sup_{2 \leq p < \infty} \left\| \frac{f}{w_{l,\delta}} \right\|_{L^p}$. Then (2.9) implies

$$(2.10) \quad \xi_{k+1} \leq (Cp_{k+1})^{\frac{n}{p_{k+1}(n+2)}} \xi_k^{\frac{2}{n+2}} (\xi_{k+1}^{p_{k+1}} + d^{p_{k+1}})^{\frac{n}{p_{k+1}(n+2)}}.$$

We may assume $d \geq 1$. Then we claim that

$$(2.11) \quad \xi_k \leq \prod_{j=1}^k (2Cp_j)^{\frac{n}{2p_j}} (\xi_1 + d).$$

Indeed, it is clear that the case $k = 1$ is true. Suppose that (2.11) is true for ξ_k but $\xi_{k+1} \geq \prod_{j=1}^{k+1} (2Cp_j)^{\frac{n}{2p_j}} (\xi_1 + d)$ holds. Especially, $\xi_{k+1} > d$ and thus from (2.10) we have

$$\begin{aligned}\xi_{k+1} &\leq (Cp_{k+1})^{\frac{n}{p_{k+1}(n+2)}} \xi_k^{\frac{2}{n+2}} \xi_{k+1}^{\frac{n}{n+2}} \left(1 + \left(\frac{d}{\xi_{k+1}} \right)^{p_{k+1}} \right)^{\frac{n}{p_{k+1}(n+2)}} \\ &< (2Cp_{k+1})^{\frac{n}{p_{k+1}(n+2)}} \xi_k^{\frac{2}{n+2}} \xi_{k+1}^{\frac{n}{n+2}},\end{aligned}$$

i.e.,

$$\xi_{k+1} < (2Cp_{k+1})^{\frac{n}{2p_{k+1}}} \xi_k \leq (2Cp_{k+1})^{\frac{n}{2p_{k+1}}} \prod_{j=1}^k (2Cp_j)^{\frac{n}{2p_j}} (\xi_1 + d).$$

This is a contradiction, and (2.11) holds. Hence, we see ξ_k is bounded uniformly in k and obtain

$$(2.12) \quad \|U_{l,\delta}\|_{L^\infty} \leq \limsup_{k \rightarrow \infty} \|U_{l,\delta}\|_{L^{p_k}} \leq C(\xi_1 + d),$$

where C is independent of δ . Since $d \leq \sup_{2 \leq p < \infty} \left\| \frac{f}{w_l} \right\|_{L^p}$ and $\xi_1 = \|U_{l,\delta}\|_{L^2} \leq \|U_l\|_{L^2}$, we have the uniform bound for $\|U_{l,\delta}\|_{L^\infty}$. This implies $U_l \in L^\infty(\mathbb{R}^n)$. Now the proof is completed.

The main goal of this section is to establish the following moment bounds for u .

Proposition 2.2. *Suppose that $f \in L_G^2$ satisfies the conditions in Lemma 1.1. Let $u \in L_G^2 \cap W_{loc}^{1,2}(\mathbb{R}^n)$ be the solution to (1.1). Then for any $\epsilon > 0$ there is a positive constant C_ϵ such that for any $k \in \mathbb{N}$ the following estimate holds.*

$$(2.13) \quad |x|^{2k}|u(x)| \leq C_\epsilon \{4(1+\epsilon)\}^k k!, \quad x \in \mathbb{R}^n.$$

Proof. To simplify calculations we take $\tilde{w}_k = |x|^{-2k}$ and set $\tilde{U}_k = \frac{u}{\tilde{w}_k}$. By Proposition 2.1 we see that \tilde{U}_k belongs to $L^\infty(\mathbb{R}^n)$ for each $k \in \mathbb{N}$ and satisfies (2.8). It suffices to show (2.13) for large k and $|x|$. We recall that

$$\Phi_p(\tilde{w}_k) \leq -\frac{n}{4p} + \lambda + \frac{n}{2} - k + \frac{4k(k+1)}{|x|^2},$$

for all $x \in \mathbb{R}^n$. We take $\eta = 1$ in (2.8). Then we have for any $\epsilon > 0$,

$$\Phi_p(\tilde{w}_k) + |B \cdot \frac{\nabla \tilde{w}_k}{\tilde{w}_k}| + \frac{1}{2p}|B|^2 \leq \lambda + \frac{n}{2} + (\frac{1}{2\epsilon} + 2p)\|B\|_{L^\infty}^2 - k + \frac{k(4k + \epsilon k + 4)}{|x|^2}.$$

Let $\alpha_{k,p,\epsilon} > 0$ be the number given by

$$(2.14) \quad \alpha_{k,p,\epsilon}^{\frac{2p}{2p-1}} = \{(4+2\epsilon)k\}^{-1}$$

Then we have

$$\begin{aligned} \frac{k(4k + \epsilon k + 4)}{|x|^2} \tilde{U}_k^{2p} &= k(4k + \epsilon k + 4) \tilde{U}_k^{2p-1} \tilde{U}_{k-1} \\ &\leq \frac{4k + \epsilon k + 4}{4 + 2\epsilon} \tilde{U}_k^{2p} + \frac{k(4k + \epsilon k + 4)}{2p} \alpha_{k,p,\epsilon}^{-2p} \tilde{U}_{k-1}^{2p}. \end{aligned}$$

Hence, there is a $k_0 \in \mathbb{N}$ such that if $k \geq k_0$ then

$$\begin{aligned} &\int_{\mathbb{R}^n} (\Phi_p(\tilde{w}_k) + |B \cdot \frac{\nabla \tilde{w}_k}{\tilde{w}_k}| + \frac{1}{2p}|B|^2 + 1) |\tilde{U}_k|^{2p} dx \\ &\leq \frac{k(4k + \epsilon k + 4)}{2p} \alpha_{k,p,\epsilon}^{-2p} \|\tilde{U}_{k-1}\|_{L^{2p}}^{2p}. \end{aligned}$$

Note that k_0 does not depend on $p \geq 2$. Then we have from (2.8) that

$$\|\tilde{U}_k\|_{L^{2p}}^{\frac{n+2}{2p}} \leq (Cp)^{\frac{1}{2p}} \|\tilde{U}_k\|_{L^p}^{\frac{2}{n}} \left\{ \left(\frac{k(4k + \epsilon k + 4)}{2p} \right)^{\frac{1}{2p}} \alpha_{k,p,\epsilon}^{-1} \|\tilde{U}_{k-1}\|_{L^{2p}} + \left\| \frac{f}{\tilde{w}_k} \right\|_{L^{2p}} \right\}.$$

Letting $p \rightarrow \infty$, we finally get

$$(2.15) \quad \|\tilde{U}_k\|_{L^\infty} \leq (4+2\epsilon)k \|\tilde{U}_{k-1}\|_{L^\infty} + \left\| \frac{f}{\tilde{w}_k} \right\|_{L^\infty}.$$

Putting $a_k = \|\tilde{U}_k\|_{L^\infty}$ and $b_k = \left\| \frac{f}{\tilde{w}_k} \right\|_{L^\infty}$, we have from (2.15),

$$(2.16) \quad \begin{aligned} a_{k+1} &\leq (4+2\epsilon)(k+1)a_k + b_{k+1} \\ &\leq (4+2\epsilon)^{k-k_0} \frac{(k+1)!}{k_0!} a_{k_0} + \sum_{j=0}^{k-k_0} (4+2\epsilon)^j b_{k+1-j} \frac{(k+1)!}{(k+1-j)!}. \end{aligned}$$

From the assumption of f we have

$$b_k \leq C \sup_{r>0} r^k (1+r)^\mu e^{-\frac{r}{4}} \leq C \left\{ \frac{4(k+\mu)}{e} \right\}^{k+\mu},$$

which gives

$$\begin{aligned}
& \sum_{j=0}^{k-k_0} (4+2\epsilon)^j b_{k+1-j} \frac{(k+1)!}{(k+1-j)!} \\
& \leq C(4+2\epsilon)^k \sum_{j=0}^{k-k_0} (4+2\epsilon)^{j-k} \left\{ \frac{4(k+1+\mu-j)}{e} \right\}^{k+1+\mu-j} \frac{(k+1+\mu)!}{(k+1+\mu-j)!} \\
& \leq C(k+1+\mu)^\mu (4+2\epsilon)^k (k+1)! \sum_{j=0}^k \left(\frac{4}{4+2\epsilon} \right)^{k-j} \frac{\{(k+1+\mu-j)\}^{k+1+\mu-j}}{e^{k+1+\mu-j} (k+1+\mu-j)!}
\end{aligned}$$

From the Stirling formula $\lim_{l \rightarrow \infty} \frac{l! \sqrt{2\pi l}}{e^l l!} = 1$, the term $\frac{\{(k+1+\mu-j)\}^{k+1+\mu-j}}{e^{k+1+\mu-j} (k+1+\mu-j)!}$ is bounded uniformly in μ and $k \geq j$. Thus we have

$$\sum_{j=0}^{k-k_0} (4+2\epsilon)^j b_{k+1-j} \frac{(k+1)!}{(k+1-j)!} \leq C_\epsilon \{4(1+\epsilon)\}^k (k+1)!.$$

Substituting this to (2.16), we get (2.13). This completes the proof.

Proposition 2.2 immediately leads to

Corollary 2.1. *Suppose that $f \in L_G^2$ satisfies the conditions in Lemma 1.1. Let $u \in L_G^2 \cap W_{loc}^{1,2}(\mathbb{R}^n)$ be the solution to (1.1). Then for any $\epsilon > 0$ there is a positive constant C_ϵ such that*

$$(2.17) \quad |u(x)| \leq C_\epsilon e^{-\frac{1-\epsilon}{4}|x|^2}, \quad x \in \mathbb{R}^n.$$

Proof of the upper bounds (1.10) in Lemma 1.1. It remains to prove the estimate for derivatives of u . We return to (2.1) and set $w = w_\epsilon = e^{-\frac{1-\epsilon}{4}|x|^2}$ in this case. From Corollary 2.1 the function $U_\epsilon = \frac{u}{w_\epsilon}$ belongs to $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for all $\epsilon > 0$. We rewrite (2.1) as

$$\begin{aligned}
-\Delta U_\epsilon &= \nabla \cdot \left\{ \left(\frac{x}{2} + B + 2\nabla \log w_\epsilon \right) U_\epsilon \right\} \\
&\quad + \left\{ \lambda - \nabla \cdot B + \left(\frac{x}{2} + B \right) \cdot \nabla \log w_\epsilon + \frac{\Delta w_\epsilon}{w_\epsilon} - \Delta \log w_\epsilon \right\} U_\epsilon + \frac{f}{w_\epsilon}.
\end{aligned}$$

Then the Calderón-Zygmund inequality and the Hardy-Littlewood-Sobolev inequality yield that $U_\epsilon \in W^{1,p}(\mathbb{R}^n)$ for sufficiently large $p < \infty$. Since $\partial_x U_\epsilon = -\frac{\partial_x w_\epsilon}{w_\epsilon^2} u + \frac{\partial_x u}{w_\epsilon}$ and $\frac{\partial_x w_\epsilon}{w_\epsilon^2} u \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ by Corollary 2.1, we see that $\frac{\partial_x u}{w_\epsilon} \in L^p(\mathbb{R}^n)$ for sufficiently large p for all $\epsilon > 0$. Hence $\left(\frac{x}{2} + B + 2\nabla \log w_\epsilon \right) \cdot \nabla U_\epsilon$ and the right-hand side of (2.1) belongs to $L^p(\mathbb{R}^n)$ for some $p > n$. Then by the Calderón-Zygmund inequality we get $U_\epsilon \in W^{2,p}(\mathbb{R}^n)$ for some $p > n$, which gives $\partial_x U_\epsilon \in L^\infty(\mathbb{R}^n)$ for all $\epsilon > 0$. It is easy to see $u \in H_G^2 \cap W^{2,p}(\mathbb{R}^n)$ with $1 \leq p < \infty$ from the above arguments. Now the proof of upper bounds in Lemma 1.1 is completed.

2.1.2. Gaussian upper bounds in Lemma 1.2. In this section we prove (1.14) and (1.16) in Lemma 1.2. First let us consider the case f satisfies (1.13) with $\mu > 0$. For any $\epsilon > 0$ let us take $w = w_\epsilon = (1 + |x|^2)^{\lambda_* + \mu} e^{-\frac{1-\epsilon}{4}|x|^2}$ in (2.1). Then by Lemma 1.1 the inequality (2.8) is verified for $U = U_\epsilon = \frac{u}{w_\epsilon}$ with all $1 \leq p < \infty$.

From direct calculations we observe that

$$\begin{aligned}\frac{\nabla w_\epsilon}{w_\epsilon} &= 2x\left(\frac{\lambda_* + \mu}{1 + |x|^2} - \frac{1 - \epsilon}{4}\right), \\ \frac{\Delta w_\epsilon}{w_\epsilon} &= \frac{(1 - \epsilon)^2}{4}|x|^2 - (1 - \epsilon)\left\{\frac{n}{2} + 2(\lambda_* + \mu)\right\} \\ &\quad + \frac{2(\lambda_* + \mu)}{1 + |x|^2}\left\{n - 1 - \epsilon + 2(\lambda_* + \mu) - \frac{2(\lambda_* + \mu - 1)}{1 + |x|^2}\right\}.\end{aligned}$$

Hence we have

$$\begin{aligned}\Phi_p(w_\epsilon) &= -\frac{\epsilon(1 - \epsilon)}{4}|x|^2 + \left\{\frac{1}{2p} + \left(1 - \frac{1}{p}\right)\epsilon\right\}\frac{n}{2} + 2\epsilon(\lambda_* + \mu) \\ &\quad + \lambda(x) - \lambda_* - \mu + O\left(\frac{1}{1 + |x|^2}\right), \quad |x| \gg 1.\end{aligned}$$

Then, by the assumptions on λ and B there are constants $R >$ and $\eta > 0$ such that if $\epsilon > 0$ is sufficiently small and p is sufficiently large then it follows that

$$(2.18) \quad \int_{\mathbb{R}^n} \left(\Phi_p(w_\epsilon) + |B \cdot \frac{\nabla w_\epsilon}{w_\epsilon}| + \frac{1}{2p}|B|^2 + \eta\right) |U_\epsilon|^{2p} dx \leq C \|U_\epsilon \chi_R\|_{L^{2p}}^{2p}.$$

Here the constant C is independent of $0 < \epsilon \ll 1$ and $p \gg 1$.

Then by (2.8) and (2.18) we have

$$\|U_\epsilon\|_{L^{2p}}^{\frac{n+2}{n}} \leq (2CC_s^2 p)^{\frac{1}{2p}} \|U_\epsilon\|_{L^p}^{\frac{2}{n}} (\|U_\epsilon \chi_R\|_{L^{2p}} + \|\frac{f}{w_\epsilon \eta}\|_{L^{2p}}),$$

for all $p \gg 1$. This implies that

$$\|U_\epsilon\|_{L^\infty} \leq \|U_\epsilon \chi_R\|_{L^\infty} + \|\frac{f}{w_\epsilon \eta}\|_{L^\infty},$$

i.e., we have by taking $\epsilon \rightarrow 0$,

$$(2.19) \quad \|\frac{u}{w_0}\|_{L^\infty} \leq \|\frac{u}{w_0} \chi_R\|_{L^\infty} + \|\frac{f}{w_0 \eta}\|_{L^\infty},$$

where $w_0(x) = (1 + |x|^2)^{\lambda_* + \mu} e^{-\frac{|x|^2}{4}}$. This gives (1.14).

Next we consider the case f satisfies (1.13) with $\mu = 0$. We recall that it is also assumed that $\lambda(x) + |\frac{x}{2} \cdot B(x)| \leq \lambda_* + \frac{1 - \epsilon_0}{\log(e + |x|^2)}$ for $|x| \gg 1$. Let us take $w = w_\epsilon = \{(1 + |x|^2)^{\lambda_*} \log(e + |x|^2)\} e^{-\frac{1 - \epsilon}{4}|x|^2}$ and $\eta = \eta(x) = \frac{\epsilon_0}{2 \log(e + |x|^2)}$ in (2.1). Then we have

$$\begin{aligned}&\Phi_p(w_\epsilon) + |B \cdot \frac{\nabla w_\epsilon}{w_\epsilon}| + \frac{1}{2p}|B|^2 + \eta \\ &\leq -\frac{\epsilon(1 - \epsilon)}{4}|x|^2 + \left\{\frac{1}{2p} + \left(1 - \frac{1}{p}\right)\epsilon\right\}\frac{n}{2} + 2\epsilon\lambda_* + \frac{1}{2p}\|B\|_{L^\infty}^2 \\ &\quad + \lambda(x) - \lambda_* - \frac{1 - 2\epsilon}{\log(e + |x|^2)} + \left|\frac{x}{2} \cdot B(x)\right| + \frac{\epsilon_0}{2 \log(e + |x|^2)} + \frac{C}{1 + |x|^2} \\ &\leq -\frac{\epsilon(1 - \epsilon)}{4}|x|^2 + \left\{\frac{1}{2p} + \left(1 - \frac{1}{p}\right)\epsilon\right\}\frac{n}{2} + 2\epsilon\lambda_* \\ &\quad + \frac{1}{2p}\|B\|_{L^\infty}^2 - \frac{\epsilon_0 - 4\epsilon}{2 \log(e + |x|^2)} + \frac{C}{1 + |x|^2}, \quad |x| \gg 1.\end{aligned}$$

Then we have the estimate (2.18) with $R \gg 1$ which is independent of $0 < \epsilon \ll 1$ and $p \geq p(\epsilon) \gg 1$. This is enough to conclude (2.19) with $w_0 = \{(1 + |x|^2)^{\lambda^*} \log(e + |x|^2)\} e^{-\frac{|x|^2}{4}}$. This completes the proof.

2.1.3. *Gaussian upper bounds in Lemma 1.3.* To prove (1.21) in Lemma 1.3 we rewrite (2.1) into the equation in polar coordinates $x = r\sigma$.

$$(2.20) \quad \partial_r^2 u + \left(\frac{n-1}{r} + \frac{r}{2}\right) \partial_r u + \left(\frac{n}{2} + \lambda\right) u + \frac{1}{r^2} \Delta_S u = -f,$$

where Δ_S is the Laplace-Beltrami operator on the unit sphere S^{n-1} . The next proposition gives the estimate for $\Delta_S u$.

Proposition 2.3. *Under the assumptions in Lemma 1.3, $(\Delta_S u)(x)$ belongs to $H_G^2 \cap W^{2,p}(\mathbb{R}^n)$ with $1 \leq p < \infty$ and satisfies the estimate*

$$(2.21) \quad |(\Delta_S u)(x)| \leq C(1 + |x|^2)^{\lambda + \mu} e^{-\frac{|x|^2}{4}}.$$

Here μ is the number in Lemma 1.3. Moreover, if f satisfies (1.20) with $\mu = 0$, then (2.21) is replaced by

$$(2.22) \quad |(\Delta_S u)(x)| \leq C\{(1 + |x|^2)^\lambda \log(e + |x|^2)\} e^{-\frac{|x|^2}{4}}.$$

Proof. Since $f \in H_G^2$ we have $u \in W_{loc}^{4,2}(\mathbb{R}^n)$ by the elliptic regularity. We first assert that $\Delta_S u \in L_G^2$. Indeed, since $|(\Delta_S u)(x)| \leq C \sum_{|\beta| \leq 2} (1 + |x|^2) |\partial_x^\beta u(x)|$, the assertion follows from the fact $|x|^2 u \in H_G^2$, which is already proved since we showed in the proof of (1.10) that for any $\epsilon > 0$ the function $e^{\frac{1-\epsilon}{4}|x|^2} u$ belongs to $W^{2,p}(\mathbb{R}^n)$ for sufficiently large $p > n$. In order to prove $\Delta_S u \in H_G^2 \cap W^{2,p}(\mathbb{R}^n)$ and (2.21), we note that $\Delta_S \mathcal{L}u = \mathcal{L} \Delta_S u$ and thus $\Delta_S u$ satisfies the equation

$$(2.23) \quad -(\mathcal{L} + \lambda) \Delta_S u = \Delta_S f.$$

Then the claim follows from the assumption of $\Delta_S f$ and the results of Lemma 1.1 and Lemma 1.2. This completes the proof.

Set $u(r\sigma) = \omega(\frac{r^2}{4}\sigma)$, $f(r\sigma) = g(\frac{r^2}{4}\sigma)$, and $\tau = \frac{r^2}{4}$. Then ω satisfies

$$(2.24) \quad \partial_\tau^2 \omega + \left(1 + \frac{n}{2\tau}\right) \partial_\tau \omega + \left(\frac{n}{2\tau} + \frac{\lambda}{\tau}\right) \omega = -\frac{1}{4\tau^2} \Delta_S \omega - \frac{1}{\tau} g =: b(\tau, \sigma), \quad \tau > 0.$$

By Proposition 2.3 and the assumption on f it is not difficult to see that $b(\tau, \sigma)$ is continuous with respect to $\sigma \in S^{n-1}$ for a.e. $\tau > 0$. Regarding the term $b(\tau, \sigma)$ as the inhomogeneous term, let us consider the linear ordinary differential equation

$$(2.25) \quad \frac{d^2}{d\tau^2} h + \left(1 + \frac{n}{2\tau}\right) \frac{d}{d\tau} h + \left(\frac{n}{2\tau} + \frac{\lambda}{\tau}\right) h = 0.$$

Then by [3, Chapter 3, Theorem 8.1] there are two linearly independent solutions $\varphi_1(\tau)$, $\varphi_2(\tau)$ to (2.25) such that

$$(2.26) \quad \lim_{\tau \rightarrow \infty} \tau^{-\lambda} e^\tau (\varphi_1(\tau), \varphi_1'(\tau)) = (1, -1), \quad \lim_{\tau \rightarrow \infty} \tau^{\frac{n}{2} + \lambda} (\varphi_2(\tau), \varphi_2'(\tau)) = (1, 0).$$

Moreover, we can show the asymptotics of φ_1 such that

$$(2.27) \quad \varphi_1(\tau) = \tau^\lambda e^{-\tau} (1 + O(\tau^{-1})), \quad \tau \gg 1.$$

Although (2.27) seems to be well-known, we give the proof of it in the appendix for convenience to the reader.

Noting that the Wronskian determinant of φ_1 and φ_2 is given by $\tau^{-\frac{n}{2}}e^{-\tau}$, we see that ω is represented as

$$(2.28) \quad \omega(\tau\sigma) = C(\tau, \sigma)\varphi_1(\tau) + D(\tau, \sigma)\varphi_2(\tau),$$

where

$$\begin{aligned} C(\tau, \sigma) &= C(\sigma) - \int_1^\tau s^{\frac{n}{2}} e^s \varphi_2(s) b(s, \sigma) ds, \\ D(\tau, \sigma) &= D(\sigma) + \int_1^\tau s^{\frac{n}{2}} e^s \varphi_1(s) b(s, \sigma) ds. \end{aligned}$$

Now we claim that

$$(2.29) \quad \sup_{\tau \geq 1, \sigma \in S^{n-1}} |C(\tau, \sigma)| < \infty, \quad \lim_{\tau \rightarrow \infty} C(\tau, \sigma) \text{ exists,}$$

$$(2.30) \quad \lim_{\tau \rightarrow \infty} \sup_{\sigma \in S^{n-1}} |D(\tau, \sigma)| = 0.$$

Let $\mu \in [0, 1)$ and ψ be the number and the function stated in Lemma 1.3, respectively. By the assumption on f and Proposition 2.3 we have

$$(2.31) \quad |b(\tau, \sigma)| \leq \frac{C\tilde{\psi}(\tau)\tau^\lambda e^{-\tau}}{\tau}, \quad \tau > 1,$$

where $\tilde{\psi}(\tau) = \tau^{\mu-1} + \psi(4\tau)$ is a positive decreasing function such that $\frac{\tilde{\psi}(\tau)}{\tau}$ is integrable over $[1, \infty)$, and C is a constant which does not depend on $\sigma \in S^{n-1}$. Thus, by (2.26) each function $s^{\frac{n}{2}} e^{-s} \varphi_i(s) b(s, \sigma)$ is integrable over $[1, \infty)$. Since we already know that ω decays exponentially at $\tau \rightarrow \infty$, from the fact $\varphi_2(\tau) \approx \tau^{-\frac{n}{2}-\lambda}$ at $\tau \rightarrow \infty$, we must have $\lim_{\tau \rightarrow \infty} D(\tau, \sigma) = 0$ for each $\sigma \in S^{n-1}$. Hence we get

$$(2.32) \quad D(\tau, \sigma) = D(\sigma) + \int_1^\tau s^{\frac{n}{2}} e^s \varphi_1(s) b(s, \sigma) ds = - \int_\tau^\infty s^{\frac{n}{2}} e^s \varphi_1(s) b(s, \sigma) ds,$$

which implies (2.30) by (2.26) and (2.31). To show (2.29) it suffices to prove $C(\sigma)$ is continuous in $\sigma \in S^{n-1}$. Let $\tau_0 > 1$ be the number such that $\varphi_1(\tau_0) \neq 0$. Then we have

$$C(\sigma) = \int_1^{\tau_0} s^{\frac{n}{2}} e^s \varphi_2(s) b(s, \sigma) ds + \varphi_1(\tau_0)^{-1} (\omega(\tau_0\sigma) - D(\tau_0, \sigma)\varphi_2(\tau_0)),$$

from which we can easily get the continuity of $C(\sigma)$.

Proof of (1.21) in Lemma 1.3. From the above preparations let us establish the asymptotics (1.21). From (2.28), (2.29), and (2.32), we can write

$$(2.33) \quad \omega(\tau\sigma) = C_\infty(\sigma)\varphi_1(\tau) + \int_\tau^\infty s^{\frac{n}{2}} e^s (\varphi_1(\tau)\varphi_2(s) - \varphi_1(s)\varphi_2(\tau)) b(s, \sigma) ds,$$

where $C_\infty(\sigma)$ is a continuous function on S^{n-1} given by

$$C_\infty(\sigma) = \lim_{\tau \rightarrow \infty} C(\tau, \sigma) = C(\sigma) - \int_1^\infty s^{\frac{n}{2}} e^s \varphi_2(s) b(s, \sigma) ds.$$

Then from (2.26) and (2.31) we observe that

$$\begin{aligned}
& \left| \int_{\tau}^{\infty} s^{\frac{n}{2}} e^s (\varphi_1(\tau)\varphi_2(s) - \varphi_1(s)\varphi_2(\tau)) b(s, \sigma) ds \right| \\
& \leq C \int_{\tau}^{\infty} (\tau^{-\frac{n}{2}-\lambda} s^{\frac{n}{2}+2\lambda} e^{-s} + e^{-\tau} \tau^{\lambda}) \frac{\tilde{\psi}(s)}{s} ds \\
(2.34) \quad & \leq C \tau^{\lambda} e^{-\tau} \int_{\tau}^{\infty} \frac{\tilde{\psi}(s)}{s} ds, \quad \tau \gg 1.
\end{aligned}$$

In the last line we used the fact that $s^{\frac{n}{2}+2\lambda} e^{-s}$ is decreasing at $s \gg 1$. Collecting (2.26), (2.27), (2.33), and (2.34), we finally get (1.21). This completes the proof.

Remark 2.1. If f satisfies (1.20) with $\mu = 0$, then by Proposition 2.3 the function $b(\tau, \sigma)$ is estimated as

$$|b(\tau, \sigma)| \leq C(\tau^{-1} \log \tau + \psi(4\tau)) \tau^{\lambda-1} e^{-\tau}, \quad \tau > 1.$$

This leads to (1.23) in Remark 1.2 from (2.27) and (2.33).

Remark 2.2. If we can take $\psi(r) = (1+r)^{-1}$ and $\mu = 0$ in the assumptions of Lemma 1.3 and if f satisfies in addition that $f \in H_G^4$ and

$$(2.35) \quad |(\Delta_S^2 f)(x)| \leq C(1 + |x|^2)^{\lambda+\mu} e^{-\frac{|x|^2}{4}}, \quad x \in \mathbb{R}^n,$$

for some $0 < \mu < 1$, then (1.21) is replaced by

$$(2.36) \quad \left| u(x) - A\left(\frac{x}{|x|}\right) |x|^{2\lambda} e^{-\frac{|x|^2}{4}} \right| \leq C|x|^{2\lambda-2} e^{-\frac{|x|^2}{4}}, \quad |x| \gg 1.$$

Indeed, under the above assumptions we can verify the estimate $|(\Delta_S u)(x)| \leq C|x|^{2\lambda} e^{-\frac{|x|^2}{4}}$ at $|x| \gg 1$ by applying Lemma 1.3 to the equation $-(\mathcal{L} + \lambda)(\Delta_S u) = \Delta_S f$. Hence (2.31) is replaced by $|b(\tau, \sigma)| \leq C\tau^{\lambda-2} e^{-\tau}$ in this case, which leads to (2.36) from (2.27) and (2.33).

2.2. Pointwise lower bounds. In this section we establish the Gaussian lower bounds for solutions to (1.1) and complete the proofs of Lemma 1.1, Lemma 1.2, and Lemma 1.3.

2.2.1. Gaussian lower bounds in Lemma 1.1. In this section we prove the estimate (1.11) when $f(x)$, $u(x) \geq 0$ for $|x| \geq R_0$. We first note that we may assume $u(x) \geq 0$ for all $x \in \mathbb{R}^n$. Indeed, if $u(x) \geq 0$ for $|x| \geq R_0$ then it suffices to consider $\tilde{u} = u + 2\|u\|_{L^\infty} \chi_{R_0}$ instead of u , where χ_{R_0} , $0 \leq \chi_{R_0} \leq 1$, is a smooth cut-off function with $\chi_{R_0} = 1$ if $|x| \leq R_0$ and $\chi_{R_0} = 0$ if $|x| \geq 2R_0$.

Let w be a positive and smooth function such that $w^{-1} \in W^{2,p}(\mathbb{R}^n)$ for all $p \in [n, \infty]$. Set $M = 2\|u\|_{L^\infty}$ and let $0 < \delta \ll 1$. We consider the function

$$(2.37) \quad U_\delta = w^{-1} \log\left(\frac{u}{M} + \delta\right).$$

Then by (1.10) and the fact $u \in H_G^2$, the function U_δ belongs to $W_{loc}^{2,2}(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$ for all $p \in [n, \infty]$. From the choice of M and δ , the function $U_\delta(x)$

is strictly negative in \mathbb{R}^n . The direct calculation leads to

$$\begin{aligned}\nabla U_\delta &= \frac{\nabla u}{w(u+M\delta)} - \frac{\nabla w}{w}U_\delta, \\ \Delta U_\delta &= \frac{\Delta u}{w(u+M\delta)} - \frac{2\nabla w}{w} \cdot \nabla U_\delta - \frac{\Delta w}{w}U_\delta - \frac{|\nabla u|^2}{w(u+M\delta)^2}.\end{aligned}$$

Hence U_δ satisfies the equation

$$\begin{aligned}-\Delta U_\delta &= \left(\frac{x}{2} + \frac{2\nabla w}{w} + B\right) \cdot \nabla U_\delta + \left(\frac{\Delta w}{w} + \frac{x}{2} \cdot \frac{\nabla w}{w} + B \cdot \frac{\nabla w}{w}\right)U_\delta \\ &\quad + \frac{u(\lambda + \frac{n}{2}) + f}{w(u+M\delta)} + \frac{|\nabla u|^2}{w(u+M\delta)^2}.\end{aligned}$$

By replacing $\frac{\nabla u}{u+M\delta} = w\nabla U_\delta + U_\delta\nabla w$, we have

$$(2.38) \quad \begin{aligned}-\Delta U_\delta &= \left(\frac{x}{2} + \frac{2\nabla w}{w} + 2U_\delta\nabla w + B\right) \cdot \nabla U_\delta \\ &\quad + \left(\frac{\Delta w}{w} + \frac{x}{2} \cdot \frac{\nabla w}{w} + \frac{|\nabla w|^2}{w}U_\delta + B \cdot \frac{\nabla w}{w}\right)U_\delta \\ &\quad + \frac{u(\lambda + \frac{n}{2}) + f}{w(u+M\delta)} + w|\nabla U_\delta|^2.\end{aligned}$$

Let $p \in \mathbb{N}$ with $p \geq n$. Multiplying both sides of (2.38) by U_δ^{2p-1} and integrating over \mathbb{R}^n , we get

$$\begin{aligned}&(2p-1) \int_{\mathbb{R}^n} U_\delta^{2p-2} |\nabla U_\delta|^2 dx \\ &= \int_{\mathbb{R}^n} \left(-\frac{n}{4p} + \frac{x}{2} \cdot \frac{\nabla w}{w} + \frac{|\nabla w|^2}{w}U_\delta + \left(1 - \frac{1}{p}\right)\frac{\Delta w}{w}\right)U_\delta^{2p} dx \\ &\quad + \int_{\mathbb{R}^n} \left(\frac{1}{p}\left|\frac{\nabla w}{w}\right|^2 - \frac{1}{p}\nabla \cdot (U_\delta\nabla w) + B \cdot \frac{\nabla w}{w}\right)U_\delta^{2p} dx + \int_{\mathbb{R}^n} B \cdot \nabla U_\delta U_\delta^{2p-1} dx \\ &\quad + \int_{\mathbb{R}^n} \frac{\{u(\lambda + \frac{n}{2}) + f\}U_\delta^{2p-1}}{w(u+M\delta)} dx + \int_{\mathbb{R}^n} w|\nabla U_\delta|^2 U_\delta^{2p-1} dx.\end{aligned}$$

By the integration by parts we see

$$\int_{\mathbb{R}^n} \nabla \cdot (U_\delta\nabla w)U_\delta^{2p-1} dx = \frac{2p}{2p+1} \int_{\mathbb{R}^n} \Delta w U_\delta^{2p+1} dx.$$

From $U_\delta < 0$, $u \geq 0$ for $x \in \mathbb{R}^n$ and $f(x) \geq 0$ for $|x| \geq R_0$, we have

$$\begin{aligned}&\frac{(2p-1)}{2} \int_{\mathbb{R}^n} U_\delta^{2p-2} |\nabla U_\delta|^2 dx \\ &\leq \int_{\mathbb{R}^n} \left(-\frac{n}{4p} + \frac{x}{2} \cdot \frac{\nabla w}{w} + \frac{|\nabla w|^2}{w}U_\delta + \left(1 - \frac{1}{p}\right)\frac{\Delta w}{w} + \frac{|\lambda|}{w}\right)U_\delta^{2p} dx \\ &\quad + \int_{\mathbb{R}^n} \left(\frac{1}{p}\left|\frac{\nabla w}{w}\right|^2 - \frac{2\Delta w}{2p+1}U_\delta + B \cdot \frac{\nabla w}{w} + \frac{|B|^2}{2p}\right)U_\delta^{2p} dx + \int_{|x| \leq R_0} \frac{|f||U_\delta|^{2p-1}}{wM\delta} dx.\end{aligned}$$

Set $w = (K + |x|^2)$ with $K > 1$, which is determined later. Then we have for any $\epsilon > 0$,

$$\begin{aligned}
& \frac{(2p-1)}{2} \int_{\mathbb{R}^n} U_\delta^{2p-2} |\nabla U_\delta|^2 dx \\
\leq & \int_{U_\delta < -\frac{1}{4}-\epsilon} \left(\frac{x}{2} \cdot \nabla w - \frac{(1+\epsilon)|\nabla w|^2}{4} + \Delta w + B \cdot \nabla w + |\lambda| \right) \frac{U_\delta^{2p}}{w} dx \\
& + \int_{U_\delta < -\frac{1}{4}-\epsilon} \left(\frac{1}{p} \left| \frac{\nabla w}{w} \right|^2 - \frac{2\Delta w}{2p+1} U_\delta + \frac{|B|^2}{2p} \right) U_\delta^{2p} dx \\
& + C \int_{U_\delta \geq -\frac{1}{4}-\epsilon} U_\delta^{2p} dx + \int_{|x| \leq R_0} \frac{|f|}{wM\delta} |U_\delta|^{2p-1} dx \\
\leq & \int_{U_\delta < -\frac{1}{4}-\epsilon} \left(\frac{x}{2} \cdot \nabla w - \frac{(1+\epsilon)|\nabla w|^2}{4} + \Delta w + B \cdot \nabla w + \frac{Cw}{p} + |\lambda| \right) \frac{U_\delta^{2p}}{w} dx \\
& + C \int_{U_\delta \geq -\frac{1}{4}-\epsilon} U_\delta^{2p} dx + \frac{1}{M\delta} \left\| \frac{f}{wU_\delta} \right\|_{L^\infty} \int_{|x| \leq R_0} |U_\delta|^{2p} dx,
\end{aligned}$$

where the constant C depends only on n , $\|\lambda\|_{L^\infty}$, $\|B\|_{L^\infty}$, and $\|U_\delta\|_{L^\infty}$. We take $p > n$ so large that $\frac{CK}{p} \leq \frac{\epsilon}{2}$. Then a direct calculation shows that

$$\begin{aligned}
& \frac{x}{2} \cdot \nabla w - \frac{(1+\epsilon)|\nabla w|^2}{4} + \Delta w + B \cdot \nabla w + \frac{Cw}{p} + |\lambda| \\
\leq & -\epsilon|x|^2 + 2n + \|B\|_{L^\infty}|x| + \frac{C(K+|x|^2)}{p} + \|\lambda\|_{L^\infty} \\
\leq & -\frac{\epsilon}{2}|x|^2 + 2n + 2 + \|B\|_{L^\infty}|x| + \|\lambda\|_{L^\infty} \leq -\frac{\epsilon}{4}|x|^2, \quad \text{if } |x| \geq R,
\end{aligned}$$

where $R \geq 1$ does not depend on $\|U_\delta\|_{L^\infty}$, K , $p \gg 1$, and δ . We may assume that $R \geq R_0$. Thus we obtain

$$\begin{aligned}
& \frac{(2p-1)}{2p^2} \int_{\mathbb{R}^n} |\nabla U_\delta^p|^2 dx \\
= & \frac{(2p-1)}{2} \int_{\mathbb{R}^n} U_\delta^{2p-2} |\nabla U_\delta|^2 dx \\
\leq & C \int_{U_\delta < -\frac{1}{4}-\epsilon, |x| \leq R} U_\delta^{2p} dx + C \int_{U_\delta \geq -\frac{1}{4}-\epsilon} U_\delta^{2p} dx + \frac{1}{M\delta} \left\| \frac{f}{wU_\delta} \right\|_{L^\infty} \int_{|x| \leq R} |U_\delta|^{2p} dx,
\end{aligned}$$

where C depends only on n , $\|\lambda\|_{L^\infty}$, $\|B\|_{L^\infty}$, and $\|U_\delta\|_{L^\infty}$. By using the Nash inequality (2.7) we have

$$C_n^{-\frac{2(n+2)}{n}} \|U_\delta\|_{L^{2p}}^{\frac{2p(n+2)}{n}} \leq Cp \|U_\delta\|_{L^p}^{\frac{4p}{n}} \left(\|U_\delta \chi_{\{|x| \leq R\}}\|_{L^{2p}}^{2p} + \|U_\delta \chi_{\{|U_\delta| \leq \frac{1}{4} + \epsilon\}}\|_{L^{2p}}^{2p} \right),$$

i.e.,

$$\|U_\delta\|_{L^{2p}}^{\frac{n+2}{n}} \leq \|U_\delta\|_{L^p}^{\frac{2}{n}} \left(C_n^{-\frac{2(n+2)}{n}} Cp \right)^{\frac{1}{2p}} \left(\|U_\delta \chi_{\{|x| \leq R\}}\|_{L^{2p}} + \|U_\delta \chi_{\{|U_\delta| \leq \frac{1}{4} + \epsilon\}}\|_{L^{2p}} \right),$$

where χ_A is the characteristic function on the measurable set A . Taking the limit $p \rightarrow \infty$, we have

$$\|U_\delta\|_{L^\infty} \leq \|U_\delta \chi_{\{|x| \leq R\}}\|_{L^\infty} + \|U_\delta \chi_{\{|U_\delta| \leq \frac{1}{4} + \epsilon\}}\|_{L^\infty} \leq \|U_\delta \chi_{\{|x| \leq R\}}\|_{L^\infty} + \frac{1}{4} + \epsilon.$$

We recall that $R \geq R_0$ does not depend on K and δ , so if $|x| \leq R$ then

$$\begin{aligned} |U_\delta(x)| &= -(K + |x|^2)^{-1} \log\left(\frac{u(x)}{M} + \delta\right) \leq -K^{-1} \log \frac{u(x)}{M} \\ &\leq -K^{-1} \log \frac{\inf_{|x| \leq R} u(x)}{M} \leq \epsilon, \end{aligned}$$

if K is sufficiently large depending only on R , M , and ϵ . Thus we have

$$|U_\delta(x)| = -(K + |x|^2)^{-1} \log\left(\frac{u(x)}{M} + \delta\right) \leq \frac{1}{4} + 2\epsilon,$$

for any $\epsilon > 0$, which implies

$$\log\left(\frac{u(x)}{M} + \delta\right) \geq -\left(\frac{1}{4} + 2\epsilon\right)(K + |x|^2), \quad x \in \mathbb{R}^n,$$

that is,

$$(2.39) \quad u(x) + M\delta \geq Me^{-(\frac{1}{4} + 2\epsilon)(K + |x|^2)}, \quad x \in \mathbb{R}^n.$$

Taking the limit $\delta \rightarrow 0$, we get (1.11). This completes the proof of Lemma 1.1.

2.2.2. Gaussian lower bounds in Lemma 1.2. In this section we establish the estimate (1.17) under the assumption

$$\lim_{R \rightarrow \infty} \sup_{|x| \geq R} |x \cdot B(x)| = 0.$$

By considering $\tilde{u} = u + 2\|u\|_{L^\infty} \chi_{R_0}$ if necessary as in the previous section, we may assume $u(x) > 0$ in \mathbb{R}^n . Fix $\nu > 0$. For any $\epsilon > 0$ we set

$$\begin{aligned} w_\epsilon(x) &= \rho_\epsilon(|x|^2) := (1 + |x|^2)^{\lambda_* - \nu} e^{-\frac{1+\epsilon}{4}|x|^2}, \\ U_\epsilon(x) &= \frac{w_\epsilon(x)}{u(x)}. \end{aligned}$$

By Lemma 1.1 we have $U_\epsilon \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap W^{2,2}(\mathbb{R}^n)$. By direct calculations we have

$$\begin{aligned} \nabla U_\epsilon &= -U_\epsilon \frac{\nabla u}{u} + \frac{\nabla w_\epsilon}{u}, \\ \Delta U_\epsilon &= -\frac{\Delta u}{u} U_\epsilon - 2 \frac{\nabla u}{u} \cdot \nabla U_\epsilon + \frac{\Delta w_\epsilon}{u}, \end{aligned}$$

and hence,

$$\begin{aligned} -\Delta U_\epsilon &= \left(\frac{x}{2} + \frac{2\nabla u}{u}\right) \cdot \nabla U_\epsilon - \left(\lambda + \frac{n}{2} + \frac{f}{u}\right) U_\epsilon - B \cdot \nabla u U_\epsilon \\ &\quad - \frac{1}{u} (\Delta w_\epsilon + \frac{x}{2} \cdot \nabla w_\epsilon). \end{aligned}$$

Rewriting $\frac{\nabla u}{u} = -\frac{\nabla U_\epsilon}{U_\epsilon} + \frac{\nabla w_\epsilon}{w_\epsilon}$, we get

$$\begin{aligned} -\Delta U_\epsilon &= \left(\frac{x}{2} + \frac{2\nabla w_\epsilon}{w_\epsilon}\right) \cdot \nabla U_\epsilon - \frac{2}{U_\epsilon} |\nabla U_\epsilon|^2 - \left(\lambda + \frac{n}{2} + \frac{f}{u}\right) U_\epsilon + B \cdot \nabla U_\epsilon \\ &\quad - B \cdot \frac{\nabla w_\epsilon}{w_\epsilon} U_\epsilon - \left(\frac{\Delta w_\epsilon}{w_\epsilon} + \frac{x}{2} \cdot \frac{\nabla w_\epsilon}{w_\epsilon}\right) U_\epsilon. \end{aligned}$$

Multiplying U_ϵ^{2p-1} both sides above and integrating over \mathbb{R}^n , we have

$$\begin{aligned}
(2.40) \quad & (2p-1) \int_{\mathbb{R}^n} |\nabla U_\epsilon|^2 U_\epsilon^{2p-2} dx \\
&= -\frac{1}{p} \int_{\mathbb{R}^n} \left(\frac{n}{4} + \frac{\Delta w_\epsilon}{w_\epsilon} - \left| \frac{\nabla w_\epsilon}{w_\epsilon} \right|^2 \right) U_\epsilon^{2p} dx - 2 \int_{\mathbb{R}^n} |\nabla U_\epsilon|^2 U_\epsilon^{2p-2} dx \\
&\quad - \int_{\mathbb{R}^n} \left(\lambda + \frac{n}{2} + \frac{f}{u} + \frac{\Delta w_\epsilon}{w_\epsilon} + \frac{x}{2} \cdot \frac{\nabla w_\epsilon}{w_\epsilon} \right) U_\epsilon^{2p} dx \\
&\quad - \int_{\mathbb{R}^n} B \cdot \frac{\nabla w_\epsilon}{w_\epsilon} U_\epsilon^{2p} dx + \int_{\mathbb{R}^n} U_\epsilon^{2p-1} B \cdot \nabla U_\epsilon dx.
\end{aligned}$$

Since

$$\begin{aligned}
\frac{\nabla w_\epsilon}{w_\epsilon} &= -2x \left(\frac{1+\epsilon}{4} + \frac{\nu - \lambda_*}{1+|x|^2} \right), \\
\frac{\Delta w_\epsilon}{w_\epsilon} &= \frac{(1+\epsilon)^2}{4} |x|^2 - \left(\frac{n}{2} + 2\lambda_* - 2\nu \right) (1+\epsilon) \\
&\quad + \frac{2(\lambda_* - \nu)(n-1+2\lambda_* - 2\nu + \epsilon)}{1+|x|^2} + \frac{4(\lambda_* - \nu)(1 - \lambda_* + \nu)}{(1+|x|^2)^2},
\end{aligned}$$

we obtain from (2.40),

$$\begin{aligned}
& \frac{2p-1}{2p^2} \int_{\mathbb{R}^n} |\nabla U_\epsilon^p|^2 dx \\
&\leq -\frac{1}{p} \int_{\mathbb{R}^n} \left(\frac{n}{4} + \frac{\Delta w_\epsilon}{w_\epsilon} - \left| \frac{\nabla w_\epsilon}{w_\epsilon} \right|^2 \right) U_\epsilon^{2p} dx - \int_{\mathbb{R}^n} \left(\lambda + \frac{n}{2} + \frac{\Delta w_\epsilon}{w_\epsilon} + \frac{x}{2} \cdot \frac{\nabla w_\epsilon}{w_\epsilon} \right) U_\epsilon^{2p} dx \\
&\quad + \int_{\mathbb{R}^n} \left(C|x \cdot B| + \frac{\|B\|_{L^\infty}^2}{2p} \right) U_\epsilon^{2p} dx + \left\| \frac{f}{u} \right\|_{L^\infty} \int_{|x| \leq R_0} U_\epsilon^{2p} dx \\
&\leq \int_{\mathbb{R}^n} \left\{ \frac{n(1+2\epsilon)}{4p} + \epsilon \left(\frac{n}{2} + 2\lambda_* \right) - \nu(1+2\epsilon) + \frac{C|\lambda_* - \nu|}{1+|x|^2} + C|x \cdot B| + \frac{\|B\|_{L^\infty}^2}{2p} \right\} U_\epsilon^{2p} dx \\
&\quad + \int_{\mathbb{R}^n} |\lambda - \lambda_*| U_\epsilon^{2p} dx + \left\| \frac{f}{u} \right\|_{L^\infty} \int_{|x| \leq R_0} U_\epsilon^{2p} dx \\
&\leq \int_{\mathbb{R}^n} \left\{ -\frac{\nu}{2} + |\lambda - \lambda_*| + \frac{C|\lambda_* - \nu|}{1+|x|^2} + C|x \cdot B| \right\} U_\epsilon^{2p} dx + \left\| \frac{f}{u} \right\|_{L^\infty} \int_{|x| \leq R_0} U_\epsilon^{2p} dx,
\end{aligned}$$

if $\epsilon > 0$ is sufficiently small and p is sufficiently large. From the assumption on B , there is an $R = R(\nu) \geq R_0$ such that

$$\frac{2p-1}{2p^2} \int_{\mathbb{R}^n} |\nabla U_\epsilon^p|^2 dx \leq C \|U_\epsilon \chi_{\{|x| \leq R\}}\|_{L^{2p}}^{2p}.$$

Here the constants C and R do not depend on $p \gg 1$ and $\epsilon > 0$. Hence from the Nash inequality (2.7), we get

$$\|U_\epsilon\|_{L^{2p}}^{\frac{n+2}{n}} \leq \|U_\epsilon\|_{L^p}^{\frac{2}{n}} (C_n)^{\frac{2(n+2)}{n}} (Cp)^{\frac{1}{2p}} \|U_\epsilon \chi_{\{|x| \leq R\}}\|_{L^{2p}}.$$

By taking $p \rightarrow \infty$ we have

$$\|U_\epsilon\|_{L^\infty} \leq \|U_\epsilon \chi_{\{|x| \leq R\}}\|_{L^\infty}.$$

Since the choice of R does not depend on $\epsilon > 0$, we can pass the limit $\epsilon \rightarrow 0$ and obtain the L^∞ bound of $\frac{(1+|x|^2)^{\lambda_* - \nu} e^{-\frac{|x|^2}{4}}}{u(x)}$, which gives (1.17). This completes the proof of Lemma 1.2.

2.2.3. *Gaussian lower bounds in Lemma 1.3.* In this section we prove the sharp lower bound for solutions to (1.1) and complete the proof of Lemma 1.3. Note that f is assumed to satisfy

$$(2.41) \quad |f(x)| \leq C(1 + |x|^2)^{\lambda - \mu'} e^{-\frac{|x|^2}{4}},$$

for some $\mu' > 0$. We use the representation of the solution in polar coordinates stated in Section 2.1.3: $u(x) = u(r\sigma) = \omega(\frac{r^2}{4}\sigma)$, $\tau = \frac{r^2}{4}$. Let us recall that by (2.33) ω is written as

$$(2.42) \quad \omega(\tau\sigma) = C_\infty(\sigma)\varphi_1(\tau) + \int_\tau^\infty s^{\frac{n}{2}} e^s (\varphi_1(\tau)\varphi_2(s) - \varphi_2(\tau)\varphi_1(s)) b(s, \sigma) ds,$$

where

$$(2.43) \quad \lim_{\tau \rightarrow \infty} \tau^{-\lambda} e^\tau \varphi_1(\tau) = 1, \quad \lim_{\tau \rightarrow \infty} \tau^{\frac{n}{2} + \lambda} \varphi_2(\tau) = 1,$$

$$(2.44) \quad b(\tau, \sigma) = -\frac{1}{4\tau^2} \Delta_S \omega - \frac{\tilde{f}}{\tau}, \quad \tilde{f}(\tau\sigma) = f\left(\frac{\sqrt{\tau}}{2}\sigma\right),$$

and $C_\infty(\sigma)$ is a continuous function on S^{n-1} .

By Proposition 2.3 we have

$$|(\Delta_S \omega)(\tau\sigma)| = |(\Delta_S u)\left(\frac{\sqrt{\tau}}{2}\sigma\right)| \leq C_\mu (1 + \tau)^{\lambda + \mu} e^{-\tau},$$

where $\mu \in (0, 1)$ is the number in Lemma 1.3. Set

$$\mu^* = \min\{1 - \mu, \mu'\} > 0.$$

Then, $b(\tau, \sigma)$ is estimated as

$$(2.45) \quad |b(\tau, \sigma)| \leq C \tau^{\lambda - \mu^* - 1} e^{-\tau}, \quad \tau > 1.$$

Combining this with (2.43), we see

$$(2.46) \quad \int_\tau^\infty s^{\frac{n}{2}} e^s |\varphi_1(\tau)\varphi_2(s) - \varphi_2(\tau)\varphi_1(s)| |b(s, \sigma)| ds \leq C \tau^{\lambda - \mu^*} e^{-\tau}, \quad \tau \gg 1,$$

We claim that

$$(2.47) \quad C_\infty := \inf_{\sigma \in S^{n-1}} C_\infty(\sigma) > 0.$$

Indeed, since ω is positive for $\tau \gg 1$, we have from (2.42), (2.43), and (2.46),

$$\begin{aligned} 0 &\leq \lim_{\tau \rightarrow \infty} \tau^{-\lambda} e^\tau \omega(\tau\sigma) \\ &\leq \lim_{\tau \rightarrow \infty} \tau^{-\lambda} e^\tau \varphi_1(\tau) C_\infty(\sigma) + \lim_{\tau \rightarrow \infty} \tau^{-\lambda} e^\tau \int_\tau^\infty s^{\frac{n}{2}} e^s |\varphi_1(\tau)\varphi_2(s) - \varphi_1(s)\varphi_2(\tau)| |b(s, \sigma)| ds \\ &\leq C_\infty(\sigma) + C \lim_{\tau \rightarrow \infty} \tau^{-\mu^*} = C_\infty(\sigma). \end{aligned}$$

Suppose that $C_\infty(\sigma) = 0$ for some $\sigma \in S^{n-1}$. Then from (2.42) we have

$$0 < \omega(\tau\sigma) = \int_\tau^\infty s^{\frac{n}{2}} e^s (\varphi_1(\tau)\varphi_2(s) - \varphi_1(s)\varphi_2(\tau)) b(s, \sigma) ds \leq C \tau^{\lambda - \mu^*} e^{-\tau}, \quad \tau \gg 1.$$

On the other hand, we have already proved in Lemma 1.2 that $u(x) \geq C_\nu (1 + |x|^2)^{\lambda - \nu} e^{-\frac{|x|^2}{4}}$ for any $\nu > 0$, which is a contradiction. Hence $C_\infty(\sigma) > 0$ for each $\sigma \in S^{n-1}$. Since $C_\infty(\sigma)$ is continuous on S^{n-1} , we conclude that

$\inf_{\sigma \in S^{n-1}} C_\infty(\sigma) > 0$. The claim is proved. From (2.43), (2.46), and (2.47), we achieve the estimate

$$\begin{aligned} \omega(\tau\sigma) &= C_\infty(\sigma)\varphi_1(\tau) + \int_\tau^\infty s^{\frac{n}{2}} e^s (\varphi_1(\tau)\varphi_2(s) - \varphi_1(s)\varphi_2(\tau)) b(s, \sigma) ds \\ &\geq C_\infty \tau^\lambda e^{-\tau} (\tau^{-\lambda} e^\tau \varphi_1(\tau) - C\tau^{-\mu^*}), \end{aligned}$$

which implies $\omega \geq \frac{C_\infty}{2} \tau^\lambda e^{-\tau}$ for $\tau \gg 1$. The proof of Lemma 1.3 is completed.

3. POINTWISE ESTIMATES FOR SOLUTIONS TO (1.24)

We are now in position to prove Theorem 1.2. Let u_α be the solution to (1.24), i.e., u_α solves

$$(3.1) \quad -\mathcal{L}u_\alpha - (1 + \frac{1}{n})|u_\alpha|^{\frac{1}{n}} a \cdot \nabla u_\alpha = 0, \quad x \in \mathbb{R}^n.$$

Without loss of generality we may assume that $\alpha > 0$ and thus u_α is strictly positive. Although we can assume $u_\alpha \in H_G^2 \cap W^{2,p}(\mathbb{R}^n)$ with $1 \leq p < \infty$ and both u_α and $\partial_x u_\alpha$ decay exponentially by Theorem 1.1 and (1.28), below we start from the regularity condition $u_\alpha \in L_G^2 \cap W_{loc}^{1,2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, which is enough to apply the results obtained in the previous section. We first take

$$B = a(1 + \frac{1}{n})u_\alpha^{\frac{1}{n}} \in (L^\infty(\mathbb{R}^n))^n \text{ and obtain by Lemma 1.1 that } u_\alpha \in H_G^2 \text{ and}$$

$$(3.2) \quad C'_\epsilon e^{-\frac{1+\epsilon}{4}|x|^2} \leq u_\alpha(x) \leq C_\epsilon e^{-\frac{1-\epsilon}{4}|x|^2}, \quad |\nabla u_\alpha(x)| \leq C_\epsilon e^{-\frac{1-\epsilon}{4}|x|^2},$$

for any $\epsilon > 0$. Then differentiating both sides of (3.1), we see $\partial_{x_j} u_\alpha$ satisfies

$$(3.3) \quad -\mathcal{L}\partial_{x_j} u_\alpha - (1 + \frac{1}{n})u_\alpha^{\frac{1}{n}} a \cdot \nabla \partial_{x_j} u_\alpha - \frac{\partial_{x_j} u_\alpha}{2} = \frac{n+1}{n^2} u_\alpha^{\frac{1}{n}-1} \partial_{x_j} u_\alpha a \cdot \nabla u_\alpha, \quad x \in \mathbb{R}^n.$$

From (3.2) we have

$$|u_\alpha^{\frac{1}{n}-1} \partial_{x_j} u_\alpha a \cdot \nabla u_\alpha(x)| \leq C e^{-\frac{1+\epsilon'}{4}|x|^2},$$

for some $\epsilon' > 0$. Thus from Lemma 1.1 with $\lambda = \frac{1}{2}$ and $B = a(1 + \frac{1}{n})u_\alpha^{\frac{1}{n}}$ we conclude $\partial_{x_j} u_\alpha \in H_G^2$ and satisfies

$$|\nabla \partial_{x_j} u_\alpha(x)| \leq C e^{-\frac{1-\epsilon}{4}|x|^2},$$

for any $\epsilon > 0$. Repeating this arguments, we obtain the estimates for derivatives of the nonlinear term in (1.25) as follows.

Proposition 3.1. *Let β be any multi-index. Then $\partial_x^\beta u_\alpha \in H_G^2$ and there is an $\epsilon_0 > 0$ such that the following estimate holds.*

$$(3.4) \quad |\partial_x^\beta (|u_\alpha|^{\frac{1}{n}} u_\alpha)(x)| \leq C_\beta e^{-\frac{1+\epsilon_0}{4}|x|^2}, \quad x \in \mathbb{R}^n.$$

Theorem 1.2 follows from Proposition 3.1 and Lemma 1.3. Indeed, differentiating both sides of (1.24), we observe that $\partial_x^\beta u_\alpha$ satisfies

$$(3.5) \quad (-\mathcal{L} - \frac{|\beta|}{2} I) \partial_x^\beta u_\alpha = a \cdot \partial_x^\beta \nabla (|u_\alpha|^{\frac{1}{n}} u_\alpha).$$

Then, noting the pointwise inequality $|(\Delta_S h)(x)| \leq C \sum_{1 \leq |\beta'| \leq 2} (1 + |x|^2) |\partial_x^{\beta'} h(x)|$, from Proposition 3.1 we can check that the term $f = a \cdot \partial_x^\beta \nabla (|u_\alpha|^{\frac{1}{n}} u_\alpha)$ satisfies the assumptions stated in Remark 2.2. Hence (1.29) and (1.30) hold

by Lemma 1.3 and Remark 2.2 with $\lambda = \frac{|\beta|}{2}$. This completes the proof of Theorem 1.2.

4. POINTWISE ESTIMATES FOR SOLUTIONS TO (1.32)

In this section we apply the results in Section 2 to Eq. (1.32) and prove Theorem 1.3 and Theorem 1.4.

We first show that if $1 < p < \frac{n+2}{n-2}$ then a solution u to (1.32) in $W^{1,2}(\mathbb{R}^n)$ belongs to $L^l(\mathbb{R}^n)$ for all $2 \leq l < \infty$. It suffices to consider the case $n \geq 3$. From the Sobolev imbedding theorem we have $u \in L^{\frac{2n}{n-2}}(\mathbb{R}^n)$. As in the calculations of (2.6), we can obtain

$$(4.1) \quad \frac{2q-1}{q^2} \int_{\mathbb{R}^n} |\nabla(|u|^q)|^2 dx \leq \int_{\mathbb{R}^n} \left(\frac{1}{p-1} - \frac{n}{4q} \right) |u|^{2q} dx + \int_{\mathbb{R}^n} |u|^{2q+p-1} dx,$$

for $q = q_1$ with $2q_1 + p - 1 = \frac{2n}{n-2}$ (by the assumption of $p < \frac{n+2}{n-2}$, we have $q_1 > 1$). Then by the Sobolev inequality (2.5) we have $u \in L^{\frac{2nq_1}{n-2}}(\mathbb{R}^n)$. Thus we have (4.1) with $q = q_2$ where $2q_2 + p - 1 = \frac{2nq_1}{n-2}$, which gives $u \in L^{\frac{2nq_2}{n-2}}(\mathbb{R}^n)$. Repeating this argument, we have $u \in L^{\frac{2nq_{k+1}}{n-2}}(\mathbb{R}^n)$ where $2q_{k+1} + p - 1 = \frac{2nq_k}{n-2}$. We observe that

$$(4.2) \quad q_k = \left\{ 1 - \frac{(n-2)(p-1)}{4} \right\} \left(\frac{n}{n-2} \right)^k + \frac{(n-2)(p-1)}{4}.$$

Since k is arbitrary and $1 - \frac{(n-2)(p-1)}{4} > 0$, we have $u \in L^l(\mathbb{R}^n)$ for all $2 \leq l < \infty$.

Next we will show that

$$(4.3) \quad u \in L^\infty(\mathbb{R}^n), \quad \lim_{R \rightarrow \infty} \|u\|_{L^\infty(\{|x| \geq R\})} = 0.$$

Indeed, we first note that $|u(x)|^{p-1}u \in L^2(\mathbb{R}^n)$. Then, since $\frac{n}{2}$ is in the resolvent set of \mathcal{L} in $L^2(\mathbb{R}^n)$ by (1.6), $(-\mathcal{L} + \frac{n}{2})^{-1}(\frac{u}{p-1} + |u|^{p-1}u)$ makes sense and belongs to the domain of \mathcal{L} in $L^2(\mathbb{R}^n)$ which includes $W^{1,2}(\mathbb{R}^n)$. On the other hand, as in the calculations of (2.6) it is easy to see that the uniqueness in $W^{1,2}(\mathbb{R}^n)$ holds for solutions to $-\mathcal{L}u + \frac{n}{2}u = f \in L^2(\mathbb{R}^n)$ (especially, we do not need the condition on $\frac{x}{2} \cdot \nabla u$ for the uniqueness). Thus we have $u = (-\mathcal{L} + \frac{n}{2})^{-1}(gu)$, where $g = \frac{1}{p-1} + |u|^{p-1}$.

Then by the Laplace formula we can write

$$(4.4) \quad u = (-\mathcal{L} + \frac{n}{2})^{-1}(gu) = \int_0^\infty e^{-\frac{n}{2}t} e^{t\mathcal{L}}(gu) dt,$$

where $e^{t\mathcal{L}}$ is the semigroup generated by \mathcal{L} which is explicitly represented as

$$(4.5) \quad (e^{t\mathcal{L}}f)(x) = \frac{e^{\frac{n}{2}t}}{(4\pi a(t))^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4a(t)}} f(ye^{\frac{t}{2}}) dy.$$

Here $a(t) = 1 - e^{-t}$; see [10] for details. Using (4.5), we can show for $1 \leq q \leq \infty$ and $R > 0$,

$$(4.6) \quad \|e^{t\mathcal{L}}f\|_{L^\infty(\{|x| \geq 2R\})} \leq \frac{C e^{\frac{n}{2}(1-\frac{1}{q})t} e^{-\frac{R^2}{8}}}{a(t)^{\frac{n}{2q}}} \|f\|_{L^q} + \frac{C e^{\frac{n}{2}(1-\frac{1}{q})t}}{a(t)^{\frac{n}{2q}}} \|f\|_{L^q(\{|x| \geq R\})}.$$

Here the constant C does not depend on $R > 0$ and f . Indeed, when $|x| \geq 2R$ we have

$$\begin{aligned}
& \frac{e^{\frac{n}{2}t}}{(4\pi a(t))^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4a(t)}} f(ye^{\frac{t}{2}}) dy \\
&= \frac{e^{\frac{n}{2}t}}{(4\pi a(t))^{\frac{n}{2}}} \int_{|y| \leq R} e^{-\frac{|x-y|^2}{4a(t)}} f(ye^{\frac{t}{2}}) dy + \frac{e^{\frac{n}{2}t}}{(4\pi a(t))^{\frac{n}{2}}} \int_{|y| \geq R} e^{-\frac{|x-y|^2}{4a(t)}} f(ye^{\frac{t}{2}}) dy \\
&\leq \frac{C e^{\frac{n}{2}t} e^{-\frac{R^2}{8}}}{a(t)^{\frac{n}{2}}} \int_{|y| \leq R} e^{-\frac{|x-y|^2}{8a(t)}} |f(ye^{\frac{t}{2}})| dy + \frac{C e^{\frac{n}{2}t}}{a(t)^{\frac{n}{2q}}} \|f(\cdot e^{\frac{t}{2}})\|_{L^q(\{|x| \geq R\})} \\
&\leq \frac{C e^{\frac{n}{2}(1-\frac{1}{q})t} e^{-\frac{R^2}{8}}}{a(t)^{\frac{n}{2q}}} \|f\|_{L^q} + \frac{C e^{\frac{n}{2}(1-\frac{1}{q})t}}{a(t)^{\frac{n}{2q}}} \|f\|_{L^q(\{|x| \geq R\})}.
\end{aligned}$$

Here we used the Young inequality. Let $\frac{n}{2} < q < \infty$. Then, recalling that $u \in L^l(\mathbb{R}^n)$ for all $2 \leq l < \infty$, we have from (4.5) and (4.6),

$$\begin{aligned}
& \|u\|_{L^\infty(\{|x| \geq 2R\})} \\
&\leq \int_0^1 \|e^{t\mathcal{L}} g u\|_{L^\infty(\{|x| \geq 2R\})} dt + \int_1^\infty e^{-\frac{n}{2}t} \|e^{t\mathcal{L}} g u\|_{L^\infty(\{|x| \geq 2R\})} dt \\
&\leq \int_0^1 \left(\frac{C e^{-\frac{R^2}{8}}}{a(t)^{\frac{n}{2q}}} \|g u\|_{L^q} + \frac{C}{a(t)^{\frac{n}{2q}}} \|g u\|_{L^q(\{|x| \geq R\})} \right) dt \\
&\quad + C \int_1^\infty e^{-\frac{n}{4}t} (e^{-\frac{R^2}{8}} \|g u\|_{L^2} + \|g u\|_{L^2(\{|x| \geq R\})}) dt \\
&\leq C \left\{ e^{-\frac{R^2}{8}} (\|g u\|_{L^q} + \|g u\|_{L^2}) + \|g u\|_{L^q(\{|x| \geq R\})} + \|g u\|_{L^2(\{|x| \geq R\})} \right\}.
\end{aligned}$$

In the above calculations the facts $\frac{n}{2} < q < \infty$ is essentially used. It is not difficult to see $u \in L^\infty(\mathbb{R}^n)$. Now (4.3) has been proved.

Proof of Theorem 1.3. Let u be a solution to (1.32) in $L_m^2 \cap W^{1,2}(\mathbb{R}^n)$ with $\frac{m}{2} > \frac{1}{p-1} - \frac{n}{4}$. Set $\lambda(x) = \frac{1}{p-1} - \frac{n}{2} + |u(x)|^{p-1}$. Since $\lambda \in L^\infty(\mathbb{R}^n)$ and $\lambda_* = \frac{1}{p-1} - \frac{n}{2}$ by (4.3), from Proposition 1.1 (or we can apply [8] when $m = 0$) we have $u \in L_G^2$. Then by Lemma 1.1 with $\lambda = \frac{1}{p-1} - \frac{n}{2} + |u|^{p-1}$, $B = 0$, and $f = 0$, we have

$$(4.7) \quad |u(x)| + |\nabla u(x)| \leq C_\epsilon e^{-\frac{1-\epsilon}{4}|x|^2}, \quad x \in \mathbb{R}^n,$$

for all $\epsilon > 0$. Thus by Lemma 1.2 with $\lambda = \frac{1}{p-1} - \frac{n}{2}$, $B = 0$, and $f = |u|^{p-1}u$, we get (1.35). The estimate for derivatives (1.36) is proved obtained similarly by differentiating both sides of (1.32). This completes the proof of Theorem 1.3.

Proof of Theorem 1.4. By Theorem 1.3 and Lemma 1.1 with $\lambda(x) = \frac{1}{p-1} - \frac{n}{2} + |u(x)|^{p-1}$ we already have $u \in H_G^2 \cap W^{2,p}(\mathbb{R}^n)$ with $1 \leq p < \infty$, and (4.7) holds for all $\epsilon > 0$. Since $\partial_{x_j} u$ satisfies $-(\mathcal{L} + \frac{1-n}{2} + \frac{1}{p-1})\partial_{x_j} u = (p-1)|u|^{p-1}\partial_{x_j} u$, we have by Lemma 1.1 with $\lambda(x) = \frac{1-n}{2} + \frac{1}{p-1} - (p-1)|u|^{p-1}$ that

$$|\nabla \partial_x u(x)| \leq C_\epsilon e^{-\frac{1-\epsilon}{4}|x|^2},$$

for all $\epsilon > 0$. Hence we see that $f = |u|^{p-1}u$ satisfies the assumptions in Lemma 1.3 with $\mu = 0$ and $\psi(r) = e^{-\delta_0 r}$ for some $\delta_0 > 0$ if $p \geq 2$. Thus by

Lemma 1.3 and Remark 1.2 with $\lambda = \frac{1}{p-1} - \frac{n}{2}$ and $f = |u|^{p-1}u$, we obtain the estimate

$$(4.8) \quad \left| u(x) - A\left(\frac{x}{|x|}\right) |x|^{\frac{2}{p-1}-n-2} e^{-\frac{|x|^2}{4}} \right| \leq C \{ |x|^{\frac{2}{p-1}-n-2} \log(e+|x|^2) \} e^{-\frac{|x|^2}{4}}, \quad |x| \gg 1.$$

This completes the proof of Theorem 1.4.

Remark 4.1. The author does not know if there is a $\sigma_0 \in S^{n-1}$ such that $A(\sigma_0) \neq 0$ in general. At least when the solution $u \in H_G^1$ is radially symmetric, it is already known by [21] that $A(\sigma) = A \neq 0$ if u is not trivial. If $A(\sigma_0) \neq 0$ then u is smooth in the set

$$\{x \in \mathbb{R}^n \mid |x| \gg 1, \frac{x}{|x|} \in S^{n-1} \text{ is near } \sigma_0\},$$

since $u(x) \neq 0$ in this set and so the nonlinear term $|u|^{p-1}u$ can be shown to be smooth without any restrictions on p in this set. From this point of view it seems to be important to find any structures of the set $\{\sigma \in S^{n-1} \mid A(\sigma) = 0\}$. This problem will be also related with the problem whether we can relax the condition $p \geq 2$ in Theorem 1.4 or not, since the information on the points where $u(x) = 0$ is important to estimate the second order derivatives of $|u|^{p-1}u$ (recall that the condition $p \geq 2$ comes from the difficulty of the possible singularity of the second order derivatives of $|u|^{p-1}u$).

5. POINTWISE ESTIMATES OF SOLUTIONS TO (1.38)

In this section we establish pointwise estimates of solutions to (1.38). To apply our lemmas we first consider the rescaling

$$\tilde{v}(x) = v\left(\frac{x}{2k^{\frac{1}{4}}}\right).$$

Then \tilde{v} satisfies

$$(5.1) \quad L\tilde{v} + \frac{1}{4}\left(n + \frac{\omega}{\sqrt{k}}\right)\tilde{v} = \frac{1}{4\sqrt{k}}|\tilde{v}|^{p-1}\tilde{v},$$

where $L = -\Delta + \frac{|x|^2}{16} - \frac{n}{4}$. Since $L = G^{-\frac{1}{2}}(-\mathcal{L})G^{\frac{1}{2}}$, by setting $u = G^{\frac{1}{2}}\tilde{v}$, we have from (5.1),

$$(5.2) \quad -\mathcal{L}u + \frac{1}{4}\left(n + \frac{\omega}{\sqrt{k}}\right)u = \frac{1}{4\sqrt{k}}|\tilde{v}|^{p-1}u,$$

with $\tilde{v} = G^{-\frac{1}{2}}u$.

Proof of Theorem 1.5. Let $u = u_1 + \sqrt{-1}u_2$ where each u_i is real valued. Since $v \in L^2(\mathbb{R})$ we see $u_i \in L_G^2$. Then under the assumptions of Theorem 1.5 we have from Lemma 1.1 with $\lambda = -\frac{1}{4}\left(n + \frac{\omega}{\sqrt{k}}\right) + \frac{1}{4\sqrt{k}}|\tilde{v}|^{p-1} \in L^\infty(\mathbb{R}^n)$,

$$(5.3) \quad |u_i(x)| + |\nabla u_i(x)| \leq C_\epsilon e^{-\frac{1-\epsilon}{4}|x|^2}, \quad x \in \mathbb{R}^n,$$

for all $\epsilon > 0$. This leads to $|\tilde{v}(x)| \leq C e^{-\frac{|x|^2}{16}}$ and so $\lambda_* = -\frac{1}{4}\left(n + \frac{\omega}{\sqrt{k}}\right)$. Hence by Lemma 1.2 with $\lambda = -\frac{1}{4}\left(n + \frac{\omega}{\sqrt{k}}\right) + \frac{1}{4\sqrt{k}}|\tilde{v}|^{p-1}$, we get

$$|u(x)| \leq C \{(1 + |x|^2)^{-\frac{n}{4} - \frac{\omega}{4\sqrt{k}}} \log(e + |x|^2)\} e^{-\frac{|x|^2}{4}},$$

which gives (1.42) by the relation $v(x) = (G^{-\frac{1}{2}}u)(2k^{\frac{1}{4}}x)$. By differentiating both sides of (5.2) we also get the estimate

$$|\nabla u(x)| \leq C\{(1 + |x|^2)^{-\frac{n-2}{4} - \frac{\omega}{4\sqrt{k}}} \log(e + |x|^2)\}e^{-\frac{|x|^2}{4}}.$$

This yields (1.43). The proof of Theorem 1.5 is completed.

Proof of Theorem 1.6. By differentiating both sides of (5.2) and by applying Lemma 1.1 we get

$$|\nabla \partial_x u_i(x)| \leq C_\epsilon e^{-\frac{1-\epsilon}{4}|x|^2}, \quad x \in \mathbb{R}^n,$$

for all $\epsilon > 0$. Hence if $p \geq 2$ then the term $f = |\tilde{v}|^{p-1}u_i = G^{-\frac{p-1}{2}}(u_1^2 + u_2^2)^{\frac{p-1}{2}}u_i$ satisfies the assumption of Lemma 1.3. Thus from Lemma 1.3 with $\lambda = -\frac{1}{4}(n + \frac{\omega}{\sqrt{k}}) + \frac{1}{4\sqrt{k}}$ and $f = |\tilde{v}|^{p-1}u_i = G^{-\frac{p-1}{2}}(u_1^2 + u_2^2)^{\frac{p-1}{2}}u_i$, we have

$$(5.4) \quad |u_i(x) - A_i(\frac{x}{|x|})| |x|^{-\frac{n}{2} - \frac{\omega}{2\sqrt{k}}} e^{-\frac{|x|^2}{4}} \leq C\{|x|^{-\frac{n}{2} - \frac{\omega}{2\sqrt{k}} - 2} \log(e + |x|^2)\} e^{-\frac{|x|^2}{4}},$$

for $|x| \gg 1$. This implies (1.44) again by $v(x) = (G^{-\frac{1}{2}}u)(2k^{\frac{1}{4}}x)$, which completes the proof of Theorem 1.6.

Remark 5.1. As in the case of (1.32) the author does not know whether there is a $\sigma_0 \in S^{n-1}$ such that $A(\sigma_0) \neq 0$ or not, in general. When a solution $u \in X$ is positive (in this case u must be radially symmetric by Li-Ni [19]) it is already known that $A > 0$ by [13, 14].

6. APPENDIX

6.1. Proof of Proposition 1.1. We prove Proposition 1.1 by using the idea of [8]. Let $\frac{m}{2} > \frac{n}{4} + \lambda_* + \frac{1}{2}B_*^2$ and let $u \in L_m^2 \cap W_{loc}^{1,2}(\mathbb{R}^n)$ satisfy (1.3). It is not difficult to check that we can take $R \rightarrow \infty$ in (2.3) with $p = 1$ and $w = (1 + |x|^2)^{-\frac{m}{2}}$ when $f \in L_G^2$. Thus we may assume that $\partial_{x_j} u \in L_m^2$ for each j . We first show that $e^{-\frac{1-\epsilon}{8}|x|^2} u(x) \in L^2(\mathbb{R}^n)$ for all $\epsilon > 0$. Motivated by [8], for $k \geq 1$, $\epsilon > 0$, $l \geq 1$, and $\theta \in [0, m]$, we set

$$(6.1) \quad \rho_{k,\epsilon}(x) = e^{\frac{(1-\epsilon)k|x|^2}{4k+|x|^2}}, \quad \zeta_{l,\theta}(x) = \frac{l}{l+|x|^2}(1+|x|^2)^\theta.$$

Then we can take $\varphi = \zeta_{l,\theta} \rho_{k,\epsilon} u$ in (1.3) and we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \zeta_{l,\theta} \rho_{k,\epsilon} |\nabla u|^2 dx + \int_{\mathbb{R}^n} u \nabla u \cdot \nabla(\zeta_{l,\theta} \rho_{k,\epsilon}) dx + \frac{1}{4} \int_{\mathbb{R}^n} |u|^2 x \cdot \nabla(\zeta_{l,\theta} \rho_{k,\epsilon}) dx \\ &= \int_{\mathbb{R}^n} \zeta_{l,\theta} \rho_{k,\epsilon} u B \cdot \nabla u dx + \int_{\mathbb{R}^n} \zeta_{l,\theta} \rho_{k,\epsilon} (\lambda + \frac{n}{4}) |u|^2 dx + \int_{\mathbb{R}^n} \zeta_{l,\theta} \rho_{k,\epsilon} f u dx. \end{aligned}$$

Since

$$\nabla \zeta_{l,\theta} = 2x \left(\frac{\theta}{1+|x|^2} - \frac{1}{l+|x|^2} \right) \zeta_{l,\theta}, \quad \nabla \rho_{k,\epsilon} = \frac{8(1-\epsilon)k^2 \rho_{k,\epsilon} x}{(4k+|x|^2)^2},$$

we have for $\eta_1 > 0$,

$$\begin{aligned}
& \int_{\mathbb{R}^n} u \nabla u \cdot \nabla (\zeta_{l,\theta} \rho_{k,\epsilon}) dx \\
&= \int_{\mathbb{R}^n} |u|^2 x \cdot \nabla \left\{ \left(\frac{\theta}{1+|x|^2} - \frac{1}{l+|x|^2} \right) \zeta_{l,\theta} \rho_{k,\epsilon} \right\} dx + \int_{\mathbb{R}^n} \frac{8(1-\epsilon)k^2 \zeta_{l,\theta} \rho_{k,\epsilon}}{(4k+|x|^2)^2} ux \cdot \nabla u dx \\
&\geq \int_{\mathbb{R}^n} |u|^2 x \cdot \nabla \left\{ \left(\frac{\theta}{1+|x|^2} - \frac{1}{l+|x|^2} \right) \zeta_{l,\theta} \rho_{k,\epsilon} \right\} dx - \int_{\mathbb{R}^n} \frac{2(1-\epsilon)k \zeta_{l,\theta} \rho_{k,\epsilon}}{4k+|x|^2} |ux| |\nabla u| dx \\
&\geq -8(1-\epsilon)\theta \int_{\mathbb{R}^n} \frac{k^2 \zeta_{l,\theta} \rho_{k,\epsilon} |xu|^2}{(4k+|x|^2)^2 (1+|x|^2)} dx - \int_{\mathbb{R}^n} \left(\frac{C \zeta_{l,\theta} \rho_{k,\epsilon}}{1+|x|^2} + x \cdot \nabla \left(\frac{\zeta_{l,\theta} \rho_{k,\epsilon}}{l+|x|^2} \right) \right) |u|^2 dx \\
&\quad - (1-\eta_1) \int_{\mathbb{R}^n} \zeta_{l,\theta} \rho_{k,\epsilon} |\nabla u|^2 dx - \frac{(1-\epsilon)^2}{1-\eta_1} \int_{\mathbb{R}^n} \frac{k^2 \zeta_{l,\theta} \rho_{k,\epsilon} |ux|^2}{(4k+|x|^2)^2} dx,
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{4} \int_{\mathbb{R}^n} |u|^2 x \cdot \nabla (\zeta_{l,\theta} \rho_{k,\epsilon}) dx \\
&= \frac{1}{2} \int_{\mathbb{R}^n} \zeta_{l,\theta} \rho_{k,\epsilon} |xu|^2 \left(\frac{\theta}{1+|x|^2} - \frac{1}{l+|x|^2} \right) dx + 2(1-\epsilon) \int_{\mathbb{R}^n} \frac{k^2 \zeta_{l,\theta} \rho_{k,\epsilon} |xu|^2}{(4k+|x|^2)^2} dx.
\end{aligned}$$

Here the constant $C > 0$ does not depend on l , k , and ϵ . Set $\zeta_\theta = (1+|x|^2)^\theta$. We observe that we can take the limit $l \rightarrow \infty$ in each term above by the Lebesgue convergence theorem, and obtain for $\eta_2, \eta_3 > 0$,

$$\begin{aligned}
& (6.2) \\
& (\eta_1 - \eta_2) \int_{\mathbb{R}^n} \zeta_\theta \rho_{k,\epsilon} |\nabla u|^2 dx + \int_{\mathbb{R}^n} \frac{(1-\epsilon)k^2 \zeta_\theta \rho_{k,\epsilon} |xu|^2}{(4k+|x|^2)^2} \left(\frac{1-2\eta_1+\epsilon}{1-\eta_1} - \frac{8\theta}{1+|x|^2} \right) dx \\
&\leq \int_{\mathbb{R}^n} \zeta_\theta \rho_{k,\epsilon} \left(\frac{C}{1+|x|^2} + \lambda + \frac{n}{4} + \frac{|B|^2}{4\eta_2} + \eta_3 - \frac{\theta}{2} \right) |u|^2 dx + \frac{1}{4\eta_3} \int_{\mathbb{R}^n} \zeta_\theta \rho_{k,\epsilon} |f|^2 dx.
\end{aligned}$$

Now we take $\eta_1 = \eta_2 = \frac{1}{2}$ and $\theta = m$ in (6.2). Then, since $\frac{m}{2} > \frac{n}{4} + \lambda_* + \frac{1}{2}B_*^2$ there is an $R > 0$ independent of $k \geq 1$ such that if $\eta_3 > 0$ is sufficiently small, then we have

$$(1-\epsilon)\epsilon \int_{\mathbb{R}^n} \frac{k^2 \zeta_\theta \rho_{k,\epsilon} |xu|^2}{(4k+|x|^2)^2} dx \leq C \int_{|x| \leq R} \zeta_\theta \rho_{k,\epsilon} |u|^2 dx + \frac{1}{4\eta_3} \int_{\mathbb{R}^n} \zeta_\theta \rho_{k,\epsilon} |f|^2 dx,$$

where C is independent of $k \geq 1$. Hence by the Fatou lemma we get

$$\begin{aligned}
& (1-\epsilon)\epsilon \int_{\mathbb{R}^n} (1+|x|^2)^m e^{\frac{1-\epsilon}{4}|x|^2} |xu|^2 dx \\
&\leq C(R) \int_{|x| \leq R} |u|^2 dx + \frac{1}{4\eta_3} \int_{\mathbb{R}^n} (1+|x|^2)^m e^{\frac{1-\epsilon}{4}|x|^2} |f|^2 dx,
\end{aligned}$$

which gives $e^{\frac{1-\epsilon}{8}|x|^2} u(x) \in L^2(\mathbb{R}^n)$ for all $\epsilon > 0$. Next we take $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{1}{8}$, $\eta_3 = 1$, and $\theta = 0$ in (6.2). Then by the Lebesgue convergence theorem we have

$$\begin{aligned}
& \frac{1}{8} \int_{\mathbb{R}^n} e^{\frac{1-\epsilon}{4}|x|^2} |\nabla u|^2 dx + \frac{1-\epsilon}{24} \int_{\mathbb{R}^n} e^{\frac{1-\epsilon}{4}|x|^2} |xu|^2 dx \\
&\leq C \int_{\mathbb{R}^n} e^{\frac{1-\epsilon}{4}|x|^2} |u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^n} e^{\frac{1-\epsilon}{4}|x|^2} |f|^2 dx,
\end{aligned}$$

where C does not depend on $\epsilon > 0$. This inequality yields that

$$\begin{aligned} & \frac{1}{8} \int_{\mathbb{R}^n} e^{\frac{1-\epsilon}{4}|x|^2} |\nabla u|^2 dx + \frac{1-\epsilon}{48} \int_{\mathbb{R}^n} e^{\frac{1-\epsilon}{4}|x|^2} |xu|^2 dx \\ & \leq C \int_{|x| \leq R'} e^{\frac{1-\epsilon}{4}|x|^2} |u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^n} e^{\frac{1-\epsilon}{4}|x|^2} |f|^2 dx, \end{aligned}$$

for some $R' > 0$ independent of $\epsilon > 0$. Taking the limit $\epsilon \rightarrow 0$, we obtain $u \in H_G^1$. The proof is now completed.

6.2. Asymptotic behavior of φ_1 . Here we give the proof of (2.27), which is based on the arguments of Brezis-Peletier-Terman [2]. For simplicity of notations we write $h'(\tau) = \frac{d}{d\tau} h(\tau)$ and $h''(\tau) = \frac{d^2}{d\tau^2} h(\tau)$. Note that by (2.26) we already have

$$\lim_{\tau \rightarrow \infty} \frac{\varphi_1'(\tau)}{\varphi_1(\tau)} = -1.$$

Set

$$E(\tau) = \tau(\varphi_1'(\tau) + \varphi_1(\tau)).$$

Then by the l'Hôpital rule, we have

$$\begin{aligned} \lim_{\tau \rightarrow \infty} \frac{E(\tau)}{\varphi_1(\tau)} &= \lim_{\tau \rightarrow \infty} \frac{E'(\tau)}{\varphi_1'(\tau)} = \lim_{\tau \rightarrow \infty} \frac{(1 - \frac{n}{2})(\varphi_1'(\tau) + \varphi_1(\tau)) - \lambda\varphi_1(\tau)}{\varphi_1'(\tau)} \\ &= \lambda. \end{aligned}$$

Next set

$$F(\tau) = \tau(E(\tau) + \lambda\varphi_1(\tau)).$$

Then again by the l'Hôpital rule, we see

$$\begin{aligned} \lim_{\tau \rightarrow \infty} \frac{F(\tau)}{\varphi_1(\tau)} &= \lim_{\tau \rightarrow \infty} \frac{F'(\tau)}{\varphi_1'(\tau)} \\ &= \lim_{\tau \rightarrow \infty} \frac{E(\tau) + \lambda\varphi_1(\tau)}{\varphi_1'(\tau)} + \lim_{\tau \rightarrow \infty} \frac{\tau(E'(\tau) + \lambda\varphi_1'(\tau))}{\varphi_1'(\tau)} \\ &= \lim_{\tau \rightarrow \infty} \frac{E(\tau) + \lambda\varphi_1(\tau)}{\varphi_1(\tau)} \frac{\varphi_1(\tau)}{\varphi_1'(\tau)} + (1 - \frac{n}{2} - \lambda) \lim_{\tau \rightarrow \infty} \frac{\tau(\varphi_1'(\tau) + \varphi_1(\tau))}{\varphi_1(\tau)} \frac{\varphi_1(\tau)}{\varphi_1'(\tau)} \\ &= -2\lambda - (1 - \frac{n}{2} - \lambda) \lim_{\tau \rightarrow \infty} \frac{(1 - \frac{n}{2})(\varphi_1'(\tau) + \varphi_1(\tau)) - \lambda\varphi_1(\tau)}{\varphi_1'(\tau)} \\ &= -2\lambda - \lambda(1 - \frac{n}{2} - \lambda) = \lambda(\frac{n}{2} - 3 + \lambda). \end{aligned}$$

By the definitions of E and F we get

$$\frac{\varphi_1'(\tau)}{\varphi_1(\tau)} = -1 - \frac{\lambda}{\tau} + \frac{\lambda(\frac{n}{2} - 3 + \lambda) + o(1)}{\tau^2}, \quad \tau \gg 1,$$

which yields for $\tau_0 > \tau \gg 1$,

$$\tau_0^{-\lambda} e^{\tau_0} \varphi_1(\tau_0) = \tau^{-\lambda} e^{\tau} \varphi_1(\tau) e^{\{\lambda(\frac{n}{2} - 3 + \lambda) + o(1)\}(\frac{1}{\tau} - \frac{1}{\tau_0})}.$$

Tending $\tau_0 \rightarrow \infty$, since the left-hand side of the equality converges to 1, we finally obtain

$$\varphi_1(\tau) = \tau^\lambda e^{-\lambda} e^{-\frac{\lambda}{\tau} \{\frac{n}{2} - 3 + \lambda + o(1)\}}, \quad \tau \gg 1.$$

This completes the proof.

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