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A short introduction to shape analysis of apparent contours by "panorama views"

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Abstract. Shape analysis of simple closed plane curves is an important subject including many chances of application to industry, especially when the curves are apparent contours of objects projected into the plane. A notion of panorama view is introduced, with presentations of some ideas of its application to the subject.

Keywords. Hough transform, panorama view, projective dual

1. Introduction

Shape analysis of apparent contours of objects is a common interest in wide areas of industry. It can be used, for example, in automatic gradings of fruits and vegetables by shapes, assortment mechanisms for mixed industrial parts on a conveyor-line in a factory, *etc*. Much more applications will be brought, if a method of shape analysis is given that is feasible and robust enough in practical scenes. In this article, we give a mathematical idea of such a method.

The Hough transform (originally in [5], and in a modified style now more popular in [3]) is the standard tool in image analysis used to detect straight line segments, such as window frames of buildings, in a digital image. Since it is unveiled, a lot of attempts are made to modify it to detect other shapes such as circles, arcs, ellipses, *etc*, in an image. The general idea of these modifications is to consider higher dimensional parameter space, where the dimension varies according to the shape demanded to be detected. In contrast, we use the Hough transform in the style of [3] and [5], or a transform to the 2-dimensional parameter space, for the shape analysis of apparent contours. More precisely, we will apply the Hough transform to plane curves to know their shapes, or the geometric features such as inflexions and bi-tangent lines.

Our method using a *panorama view* is simple and requires no analytic descriptions such as defining equations or parametrisations of the curve, hence has a high feasibility. It can be applicable even to a digital curve, without following the points on the curve in order. These natures of the panorama view will expand the opportunity of its applications to various branches in industry. We can apply it for example, in various ways such as discrimination of two curves, estimation of "roundness" of a simple closed convex curve, finding the optimal container box for a closed curve, *etc*, as shown in the final section.

The use of Hough transform in the original styles for the shape analysis of apparent contours is previously discussed by M.Wright-A.Fitzgibbon-P.Giblin-R.Fisher in [6]. Their key concept is that Hough transform for a plane curve is basically so called the projective dual in mathematics. A panorama view is again another aspect of the Hough transform, but it is worth to consider independently, because of its simpleness and feasibility mentioned above.

In this article, we first introduce the notion of a panorama view, show the principle of our feature extractions, point out its equivalences to the projective dual and to the Hough transform, and finally give examples of simple applications.

Though our method and results can be applicable to the digital curves, all curves in the following description will be planar, smooth, regular, and closed, unless otherwise stated. We assume also that the curves do not pass through the origin of the plane.

2. Panorama view

We first give the definition of a panorama view in a slightly intrinsic manner, and after that give another simple formulation.

Let $C \subset \mathbb{R}^2$ be a curve. For $q \in [0, \pi]$, denote by $\gamma_q : \mathbb{R}^2 \to \mathbb{R}_q$ the linear orthogonal projection onto the tangent line $\mathbb{R}_q$ of the unit circle at q (Refer to Figure 1; note that there is no need of C being contained in the unit sphere, regardless of the picture). We give $\mathbb{R}_q$ the orientation by the direction of $(-\sin q, \cos q)$. For a function $\gamma_q|C : C \to \mathbb{R}_q$, denote by $\mathrm{Crit}(\gamma_q|C)$ the set of its critical values, where $p \in C$ is a critical point of $\gamma_q|C$ if the differential $\mathrm{d}(\gamma_q|C)$ vanishes at p. We denote by $\mathbb{R}P^1 = [0, \pi]/\sim$ the 1-dimensional projective space, or the angles from 0 to π but π is regarded to be the same angle as 0.

We regard the tangent bundle of the half circle $[0, \pi]$ as $\mathbb{R} \times [0, \pi]$, where a fibre $\mathbb{R}_q$ is identified with $\mathbb{R} \times \{q\}$. Then a line bundle M_b over $\mathbb{R}P^1$ is induced as $\mathbb{R} \times [0, \pi]/\sim$, where $(x, 0)$ and $(-x, \pi)$ are identified for each $x \in \mathbb{R}$.

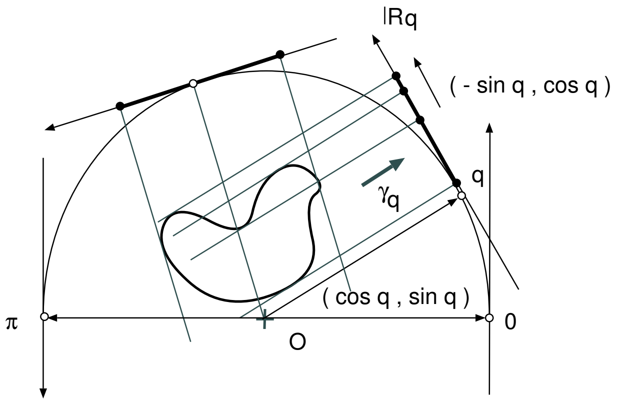

Figure 1: Projections of a curve to tangent lines of a half unit circle

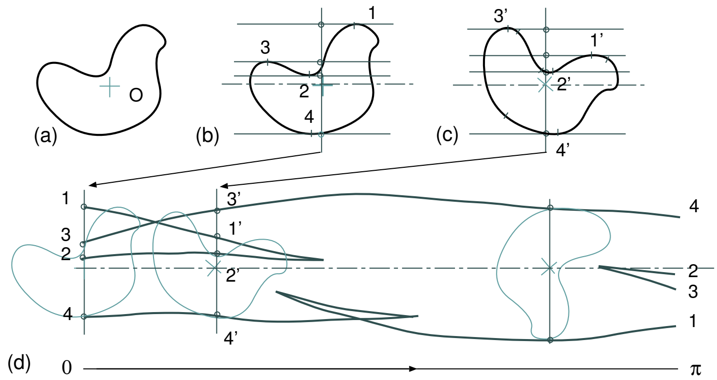

Figure 2: A panorama view taken by projection of a twisted tube

It is topologically the Möbius strip. To save notations, we write $(s,t) \in M_b$ for $s \in \mathbb{R}, t \in [0,\pi]$, to denote a point in M_b, with a little abuse of notation. Now note that $\gamma_0(C) = -\gamma_\pi(C)$ and $\mathrm{Crit}(\gamma_0|C) = -\mathrm{Crit}(\gamma_\pi|C)$. Hence the disjoint unions

$$\mathrm{PV}(C) = \coprod_q \gamma_q(C) \ ; \ q \in \mathbb{R}P^1$$

and

$$\mathrm{PV}_0(C) = \coprod_q \mathrm{Crit}(\gamma_q|C) \ ; \ q \in \mathbb{R}P^1$$

are well-defined compact subsets of M_b.

Definition 1. For a curve C, the pair of the compact sets

$$\mathcal{PV}(C) = (\mathrm{PV}(C), \mathrm{PV}_0(C))$$

is called the *panorama view* of C.

The first and the second components of $\mathcal{PV}(C)$ are referred to as the *image* and the *critical locus* of the panorama view, respectively.

We give another formulation of what has been mentioned above. Consider a twisted embedding $C \times [0,\pi] \to \mathbb{R}^2 \times [0,\pi]$ given by $(p,q) \mapsto (p\cdot q^{-1}, q)$, where multiplication by q^{-1} is defined as a rotation of angle $-q$ around the origin $O \in \mathbb{R}^2$, and succeedingly take the canonical projection of its image through the projection $\tilde{\gamma} : \mathbb{R}^2 \times [0,\pi] \to \mathbb{R} \times [0,\pi]$ onto the last two components: $(x,y,t) \mapsto (y,t)$. The composite naturally defines a smooth map $\gamma : C \times \mathbb{R}P^1 \to M_b$. It is easy to see that γ is given by $\gamma(p,q) = (\gamma_q(p), q)$. We define the singular points of γ to be the points where the differential $\mathrm{d}\gamma$ is not surjective, and denote by $\mathrm{Crit}(\gamma)$ the set of the images of the singular points of γ. The proposition below is now clear, which gives a formulation of a panorama view, and at the same time, offers a practical way to take a panorama view (refer to Example 1).

Proposition 1. *Let $\gamma : C \times \mathbb{R}P^1 \to M_b$ be the map given by $\gamma(p,q) = (\gamma_q(p), q)$, where $\gamma_q : C \to \mathbb{R}$ is the orthogonal projection defined before. Then $\mathcal{PV}(C) = (\mathrm{Im}(\gamma), \mathrm{Crit}(\gamma))$. Further, (p,q) is a singular point of γ if and only if q is the tangent direction of C at p.*

Example 1. In Figure 2, we show a panorama view of a curve, taken as the pair of image and critical locus of the projection γ of the twisted tube: (a) shows a curve, (b) shows a slice of the tube at $q = 0$, projection γ and the critical values, (c) shows the same for another slice, and (d) is the panorama view, where both sides should be reversely identified.

A point $p \in C$ is an *inflexion* if the tangent line has 3-point contact with C at p. A point $p \in C$ is a *higher inflexion* if the tangent line has at least 4-point contact with C at p. In the following, we consider C, in addition to the general requirement stated in the introduction, in the set $\mathcal{F}$ of curves defined as below:

- Any curve $C \in \mathcal{F}$ has only finitely many inflexions, and no higher inflexions.
- Any two inflexions of $C \in \mathcal{F}$ do not have the same tangent line.
- The tangent line at an inflexion of $C \in \mathcal{F}$ has no other tangent point with C.
- There is no self-tangent point in C.
- Any straight line is tangent to C at most two points.

It is known that the set of curves satisfying the first assumption is open and dense ([2]). Since such curves can be moved to satisfy the remaining assumptions by slight perturbations, the set $\mathcal{F}$ is also open and dense.

Proposition 2. *For $C \in \mathcal{F}$, the map γ is stable.*

Remark: A stable map (see *e.g.*, [4] for definition) between 2-dimensional manifolds can be characterised by having only cusps and folds as singular points, and by having critical locus, or the image of the singular points, in a certain general position. Any smooth map can be moved to a stable map, by a perturbation. Refer to [4] for details.

Proof. Each singular point of γ is a contact point of C with parallel lines to $\mathrm{Ker}\,\gamma_p$ (Proposition 1). By the first assumption of $\mathcal{F}$, γ at a singular point is written in the local form $(g(t,u),u)$ centred at the singular point $(t,u)=(0,0)$, where $g(t,u)$ is either $t^2\cos u - t\sin u$ or $t^3\cos u - t\sin u$ (refer to Figure 3). In the first case, $g = \frac{\partial g}{\partial t} = 0$ but $\frac{\partial^2 g}{\partial t^2} \neq 0$, which implies that the point is a fold, and in the second case, $g = \frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial t^2} = 0$ but $\frac{\partial^3 g}{\partial t^3} \neq 0$, which implies that the point is a cusp ([4]). The remaining assumptions imply that γ restricted to the set of folds is an immersion which may have normal crossings, and further that a cusp shares its γ-image with none of the other singular points. Then by a singularity theory, one can see that γ is stable (its multi-jet extensions are transverse stable, in detail) from these observations (refer to the remark before). □

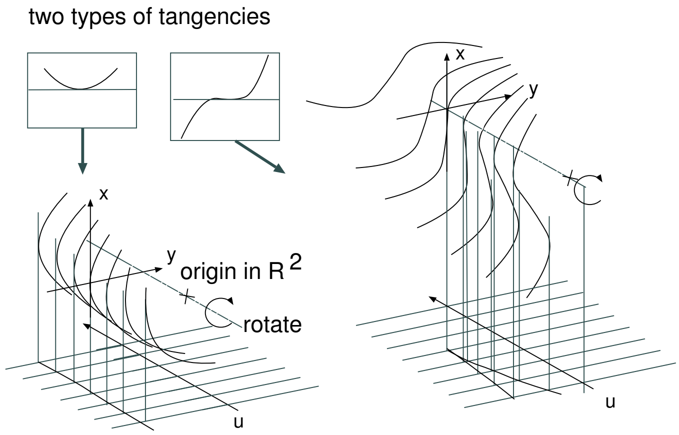

Figure 3: Projections of a twisted tube

The second part of Proposition 1 shows that there is a correspondence

(1) $\mathcal{D}: C \to \mathrm{PV}_0(C) \subset M_b\ ; \quad p \mapsto (d,q) \in M_b,$

by which a point $p \in C$ corresponds to the signed-distance $d = \gamma_q(p)$ from the origin $O \in \mathbb{R}^2$ and the direction q, of the tangent line at p. As mentioned in the above proof of Proposition 2, a basic theory of singularity shows that $\mathrm{PV}_0(C)$ is a smooth 1-dimensional submanifold except at a finite number of points of two kinds; a *normal crossing* defined by $xy=0$ and a *cusp* defined by $x^3 - y^2 = 0$ for some local coordinates x, y of $\mathbb{R}^2$. We will give a fundamental observation on $\mathcal{D}$, of our feature extraction.

A tangent line to $C \in \mathcal{F}$ is said to be *bi-tangent* if it is tangent to C at two different points. Such two points are referred to as bi-tangent points. The bi-tangent lines and the inflexions are basic features on the shape of C, but are generally difficult to find out directly from C without calculation on the defining equation, a parametrisation, *etc*, of the curve. The theorem below shows that one can find them easier, from the panorama view.

Theorem 1. *For a curve $C \in \mathcal{F}$:*

1. *The correspondence $\mathcal{D}$ in (1) is bijective, except exactly at the bi-tangent points.*

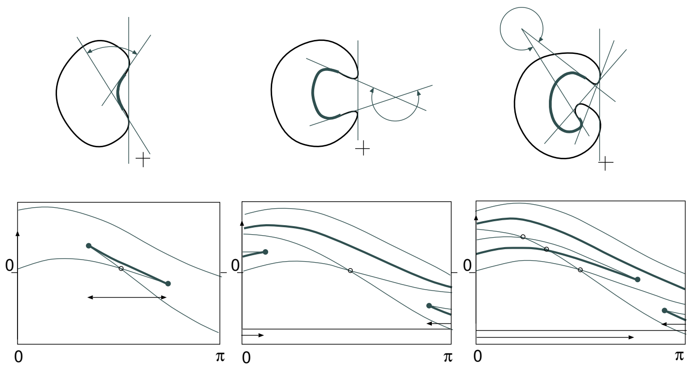

Figure 4: Basic features of curves detected by panorama views

2. *A pair of bi-tangent points $p, p' \in C$ of a bi-tangent line corresponds to a normal crossing of* $\mathrm{PV}_0(C)$.

3. *An inflexion of C corresponds to a cusp of* $\mathrm{PV}_0(C)$.

Remark: The correspondence $\mathcal{D}$ is classically called the projective dual, in more general settings (refer to [1]), and the correspondences in the theorem are stated in [6].

Proof. We show first the surjectivity of $\mathcal{D}$. Since $\mathrm{PV}_0(C)$ is the critical locus of γ by Proposition 1, a point (d,q) is in $\mathrm{PV}_0(C)$ if and only if (p,q) is a singular point of γ for some $p \in C$. The second part of the proposition shows that (d,q) is the correspondent of p by $\mathcal{D}$.

Second, let (p,q) and (p',q) be singular points of γ with the same image; $\gamma(p,q) = \gamma(p',q) = (d,q)$. There are at most two such points p and p' for a point (d,q) in $\mathrm{PV}_0(C)$, by stability of γ. In fact, the correspondence of the singular points of γ to the critical locus is 2-to-1 at each normal crossing, while it is 1-to-1 at the other points, as mentioned. On the other hand, this relation is equivalent to p and p' being the bi-tangent points of the line $\gamma_q^{-1}(d)$. This implies the first assertion. Note that the above argument shows the second assertion also.

It is mentioned in the proof of Proposition 2 that $\gamma(p,q)$ is a cusp if p is an inflexion, while it is a fold otherwise, which implies the third assertion. □

Example 2. Bi-tangent lines and inflexions of curves can be detected from panorama views, as shown in Figure 4. One can also measure the angles indicated in the figures (an angle between tangent lines at inflexions swept by the tangent directions of the bold arc in C between the inflexions), by following the moves of q between two cusps of the panorama views.

A simple closed curve C is *convex* if any chord between distinct two points on C is inside the compact set enclosed by C.

Corollary 1. *A simple closed curve $C \in \mathcal{F}$ is convex if and only if* $\mathrm{PV}_0(C)$ *contains no cusps.*

Remark: This claims that convexity and inflexion-freeness are equivalent for C, by the theorem.

Proof. The convexity of C is equivalent to the map γ in Proposition 1 being at most 2-to-1, since for a chord between two points in C, all the intersections of its extended line with C have the same value $a \in \mathrm{Im}(\gamma)$.

Assume that there is a cusp in $\mathrm{PV}_0(C)$. A local observation of a cusp tells us that the number of inverse images $\gamma^{-1}(a)$ varies as a moves around the cusp, and attains any of $k, k+1, k+2$, for a certain non-negative integer k (refer to Figure 3). Especially, γ is not at most 2-to-1, and hence C is not convex.

Second, assume that there are no cusps in $\mathrm{PV}_0(C)$. Then $\mathrm{PV}_0(C)$ is decomposed into smooth arcs of a finite number, when it is cut at $q = \pi$. Each arc corresponds to the monotonous move of the tangent directions of C from 0 to π, and hence the number of arcs is twice of the tangent winding number of C; the last one is 1. Hence $\mathrm{PV}_0(C)$ has two sub-arcs. Recall that $\gamma_q(C)$ is an interval of non-zero finite length, for any $q \in \mathbb{R}P^1$. It implies that $\mathcal{PV}(C)$ is the pair of closed Möbius strip and its boundary. It is now clear that γ is at most 2-to-1. □

Remark: One can show also that C *in the corollary is convex if and only if* $\mathrm{PV}_0(C)$ *contains no crossings*, by using the materials in the above proof. But it requires another lemma that *if* $\mathrm{PV}_0(C)$ *has a cusp, then it has at least one crossing*, which we omit to prove here.

3. Hough transform applied to a plane curve

In this section, we review the Hough transform in the style of [3], but applied to a plane curve $C \in \mathcal{F}$. We will show its equivalence to the panorama view.

For each point $p = (x, y) \in C \subset \mathbb{R}^2$, draw a sine-curve s_p in a (d, q)-plane $\mathbb{R} \times [0, \pi]$ defined by

$$(2) \qquad d = x \cos q + y \sin q = \sqrt{x^2 + y^2}\, \sin(q - \alpha),$$

for $\sin\alpha = -\frac{x}{\sqrt{x^2+y^2}}$ and $\cos\alpha = \frac{y}{\sqrt{x^2+y^2}}$. By identification $(d, 0) \sim (-d, \pi)$, we regard it drawn on M_b. Denote by $\mathrm{H}(C)$ the union $\cup_p s_p \subset M_b$, which we call the *Hough image* and by $\mathrm{H}_0(C)$ the envelope of the one-parameter family $\{s_p; p \in C\}$, which we call the *Hough locus*. Denote by $\mathcal{H}(C)$ the pair $(\mathrm{H}(C), \mathrm{H}_0(C))$, which we call the *Hough transform*.

In case it is applied to a straight line L in place of C, the result $\mathrm{H}(L)$ has a point P where infinitely many sine-curves passing through. In fact, L can be written in the form of (2), where d and q represent the signed distance from $O \in \mathbb{R}^2$ and the normal direction of L, respectively. This means that the point $P = (d, q)$ in M_b is common for all sine-curves defined for points (x, y)'s on L. It is the principle of the straight line segment detection. Namely, for a collection of points $\mathcal{P}$, or a picture, one is required to find an accumulation point in $\mathcal{H}(\mathcal{P})$ to extract such a segment in $\mathcal{P}$.

Example 3. Hough transform applied to a line and a curve (Figure 5). In each figure, both sides of the box should be reversely identified. The shading is made by overdrawing sine-curves. In this example, the conventional coordinate system in image analysis is chosen.

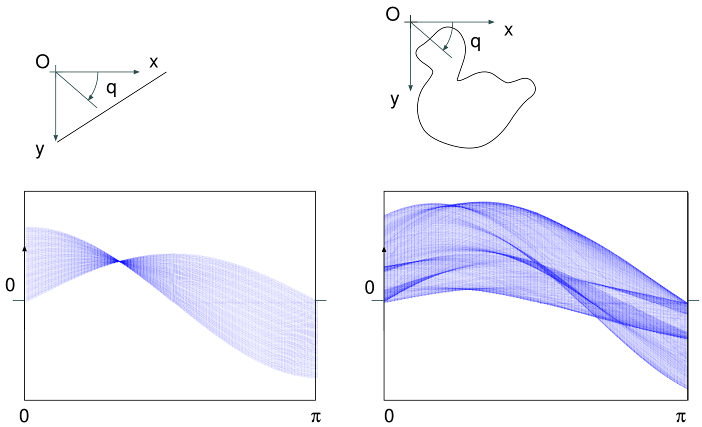

Figure 5: Hough transform of a line and a curve

Lemma 1. *For a simple closed curve $C \in \mathcal{F}$, the two pairs $\mathcal{H}(C)$ and $\mathcal{PV}(C)$ coincide.*

Remark: Precisely speaking, the two pairs $\mathcal{H}(C)$ and $\mathcal{PV}(C)$ have a difference of the $\frac{\pi}{2}$-shift to the q-direction (see the proof below).

Proof. First we see the correspondence between $\mathrm{H}(C)$ and $\mathrm{PV}(C)$. A point $(d, q) \in M_b$ is in $\mathrm{H}(C)$ if and only if the equation (2) holds for a point $p = (x, y) \in C$, which means that p is projected onto the point d on the straight line through $O \in \mathbb{R}^2$ of the direction and orientation given by $(\cos q, \sin q)$, by an orthogonal projection. Namely, $d = \gamma_{q'}(p)$, where $q' = q - \frac{\pi}{2}$ (see Figure 1), and hence $(d, q') \in \mathrm{PV}(C)$.

Second, we see the correspondence between $\mathrm{H}_0(C)$ and $\mathrm{PV}_0(C)$. Assume p to be a local parameter of the curve C, and consider the graph of d in equation (2) regarded as a function of (p, q), where x and y are also regarded as functions of p; the graph is sliced into sine-curves s_p's. Then a point (d, q) is in $\mathrm{H}_0(C)$ if and only if it is a critical value of the projection of the graph to the (d, q)-plane, which is equivalent to the vanishing of $\dfrac{\partial d}{\partial p}$ at (p, q) (refer to Figure 6). On the other hand, $\gamma(p, q') = (d, q')$ and its Jacobian matrix is

$$J_\gamma(p, q') = \begin{pmatrix} \frac{\partial d}{\partial p} & \frac{\partial d}{\partial q'} \\ 0 & 1 \end{pmatrix}_{(p,q')}$$

and hence the above vanishing is equivalent to (p, q') being a singular point of γ. Thus Proposition 1 implies the correspondence between $\mathrm{H}_0(C)$ and $\mathrm{PV}_0(C)$. □

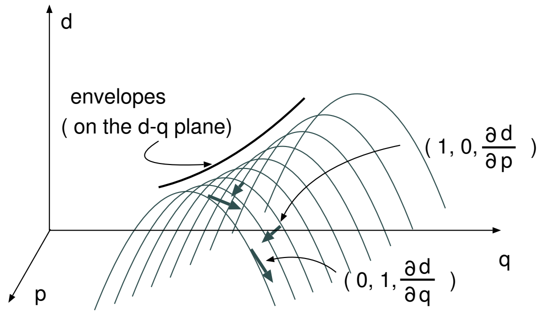

Figure 6: Envelopes as critical locus of a projection

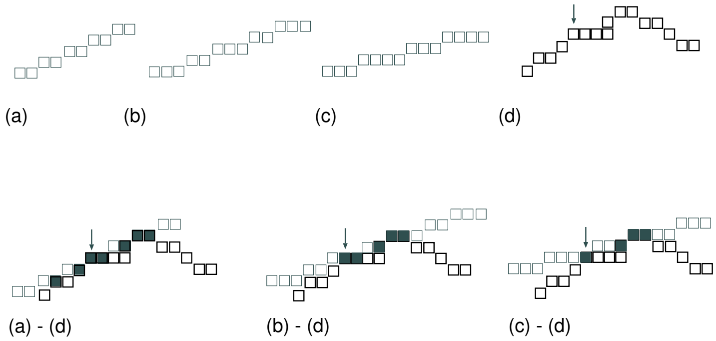

Figure 7: Difficulty of the tangency judgement in the digital world (solid boxes are common for (d) and the line)

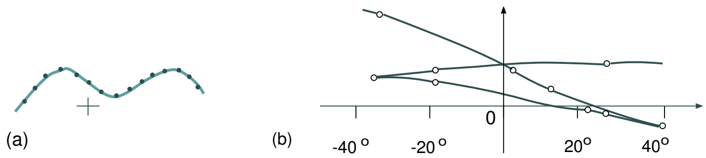

Figure 8: A dot-sequence on a curve (a) and its panorama view (b)

4. Feasibility and robustness of the panorama view method

The projective dual to a plane curve C is the correspondence by which distance to the origin of $\mathbb{R}^2$ and the direction of the tangent line are associated to a point $p \in C$, or exactly the $\mathcal{D}$ in (1). Hough transform, projective dual, and the panorama view applied to a plane curve are essentially the same process, as mentioned. However, as to the feasibility, there is a big difference between the Hough transform and the panorama view: to get the Hough locus $\mathrm{H}_0(C)$ for $C \in \mathcal{F}$, we need to detect envelopes from $\mathrm{H}(C)$, which is by itself an interesting but a difficult problem.

The difference between the projective dual and the panorama view is smaller. In a phrase, projective dual in its definition is driven by points on a curve, while the panorama view is driven by angles. It is just a difference in process, but is disregarded in effect, when applied to a digital curve C as mentioned below.

In taking the projective dual by following the definition faithfully, one is at first required to follow the pixels on C in order. One is then requested to find a tangent line l at each point of C. It is but difficult, because a line of general slope is substituted by a sequence of horizontal edges in the digital world, which makes the tangency judgement at a point in C to be not straightforward. Figure 7 shows an uncertainty of the notion of tangency, where it is not clear which line ((a), (b), or (c)) is tangent to (d) at the marked pixel. Another idea to find l will be to do it asymptotically. Namely, we measure the slope between two points on a curve and examine its trends when the points become nearer. But the second part is again not clear (note that the slope between two adjoining pixels is one of the four values $0, \pm 1$, or ∞). In contrast, the panorama view method by Proposition 1 requires no tracing of C, and we need to judge the tangency only with the horizontal lines (after suitable rotation of C), which can be dealt only with the addresses of the pixels on C.

Because of the merits above, a panorama view is robust for breaks of C at some disappeared points: one can guess $\mathrm{PV}_0(C)$ even by a panorama view of a finite number of points on C, as an extreme case (Figure 8, where (b) is the panorama view of the piecewise linear curve obtained by connecting the dots in order, refer to Example 7 in the next section for definition; the curve in (b) is broken into pieces of curves, each of which is a sub-arc of a sine-curve produced by a dot in (a)). These merits also contribute to save the computation costs, as understood easily: it is an advantage in implementation in embedded systems.

Another remarkable property of the panorama view is its natural correspondence to the critical locus of stable maps (Propositions 1 and 2). The critical locus of a stable map has a simple structure, as it is made of isolated cusps, normal crossings and arcs between them, as mentioned. We have previously made use of this merit in Section 2 (*e.g.*, in the proof of Theorem 1 and of Corollary 1), but one can make further use of it, *e.g.*, a rough understanding of a curve (refer to Example 4 in the next section), and a completion of the critical locus of a panorama view with a missing part.

5. Some applications

In [6], an application of the Hough transform to find the convex hull of C is mentioned. We add here a few examples of applications of the Hough transform by panorama view.

Example 4. Discrimination of curves; The three curves in Figure 9 are all concave and have four inflexions. We can distinguish one from the other by using the panorama views. The first two can be distinguished by the topological types of the critical locus $\mathrm{PV}_0(C)$. The difference of the first and the last one is more subtle; $\mathrm{PV}_0(C)$'s have the same topological type. But they can be distinguished by

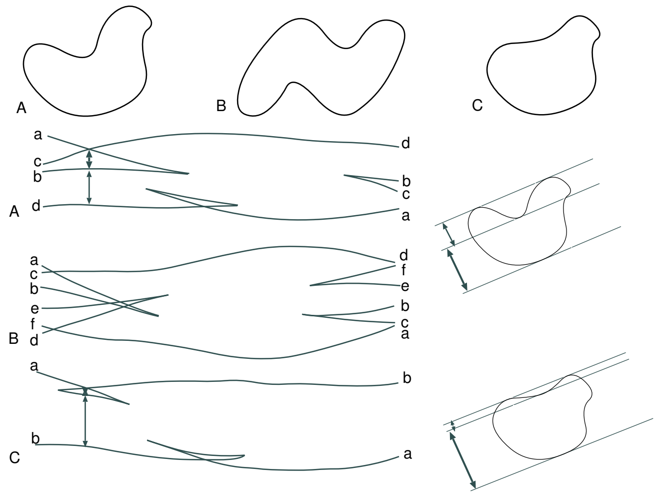

Figure 9: Discrimination of curves by panorama views

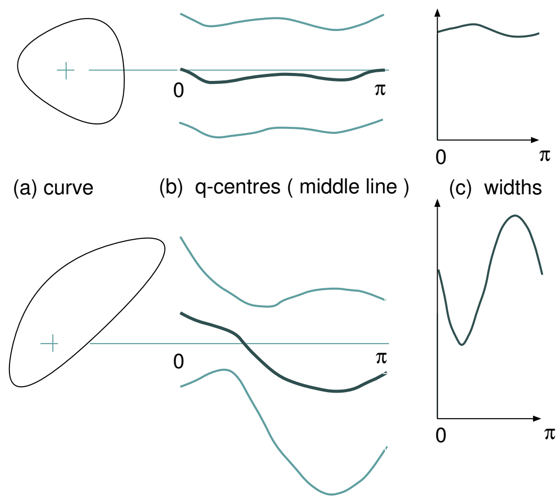

Figure 10: q-centres and widths of convex curves

the topological types of the branched coverings $\mathrm{PV}_0(C) \to \mathbb{R}P^1$, or also by the "relative depth of concavities" (the ratio of two vertical lines marked in the panorama views).

In general, for $C \in \mathcal{F}$, one can read from the above branched covering the switches of the $+$ and $-$ of the direction of the move of the tangent direction as a point p traces C. The order and lengths of the $+$ and $-$-intervals will be also helpful to the discrimination of two curves.

Example 5. Roundness of C; We estimate the roundness of C by two aspects; *stability of q-centres* and *constancy of widths* (Figure 10). Let $v(q)$ be the middle point of $\gamma_q(C)$, which we refer to as the q-centre. Consider the points $(\cos q, \sin q, v(q)), q \in \mathbb{R}P^1$ on a cylinder in $\mathbb{R}^3$, where q is now regarded as an angle in $[0, \pi]$. One can approximately calculate the optimal plane in $\mathbb{R}^3$ nearest to these points and can take the mean distance between the points and the plane, which we call the stability of q-centres. If C is a circle, then q-centres draw a sine-curve in $\mathbb{R} \times [0, \pi]$ (note: they draw a straight line, in case C is centred at the origin of $\mathbb{R}^2$), hence the previous points in $\mathbb{R}^3$ are on a plane, and thus the stability is zero. Constancy of width is the variance of length of $\gamma_q(C)$ when q moves throughout $\mathbb{R}P^1$. In the figure, one can see q-centres and widths, for two curves.

Example 6. Optimal arrangements of C; We can find a "minimum" box that contains C, where what minimum stands for depends on what is demanded. For some purpose, the area minimising box is preferred, and for another, a box that best approximates a square is preferred, *etc.* The edge lengths of a container box are the width of $\gamma_q(C)$ and $\gamma_{q'}(C)$, where $q' = q + \frac{1}{2}\pi$. One can find the optimal q approximately by scanning these lengths on q's. In Figure 11, we show (a) a curve, (b) various containers, (c) the best square-approximating box, and (d) the area minimizing one.

Example 7. Panorama views of polygons; We can take a

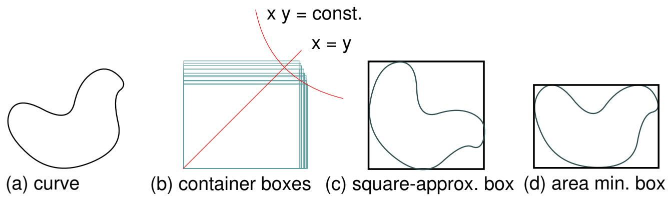

Figure 11: Finding optimum container boxes

panorama view also for a polygon P. A critical point of γ_q restricted to the boundary ∂P stands for the point $p \in \partial P$ that either of $\gamma_q(p) \le \gamma_q(t)$ or $\gamma_q(p) \ge \gamma_q(t)$ holds for any t in a small neighbourhood of p in ∂P. An *inflexion edge* and *bi-tangency at vertices* are defined as in Figure 12, as substitutions of an inflexion and bi-tangent points in the smooth case, respectively. The first one is defined by using the sides where the neighbouring edges lie, and the second by a chord between two vertices that passes through the outer-turn regions at each vertex, where the last region at a vertex is the region swept by an edge when it is turned around the vertex to the position of the adjoining edge so that the outward direction is preserved (refer to the figure). These features can be extracted from the panorama view, by using a topological version of Theorem 1.

We end this section by posing two problems on further applications of the panorama views:

Problems:

1. (piecewise linearisation of C) For a simple closed curve C, find a set of points on C of the smallest number such that the polygon boundary S obtained by connecting these points following the order in C has the branching $\mathrm{PV}_0(S) \to \mathbb{R}P^1$ of the the same topological type as $\mathrm{PV}_0(C) \to \mathbb{R}P^1$.

2. (coding of C) Find a coding of a curve C by combina-

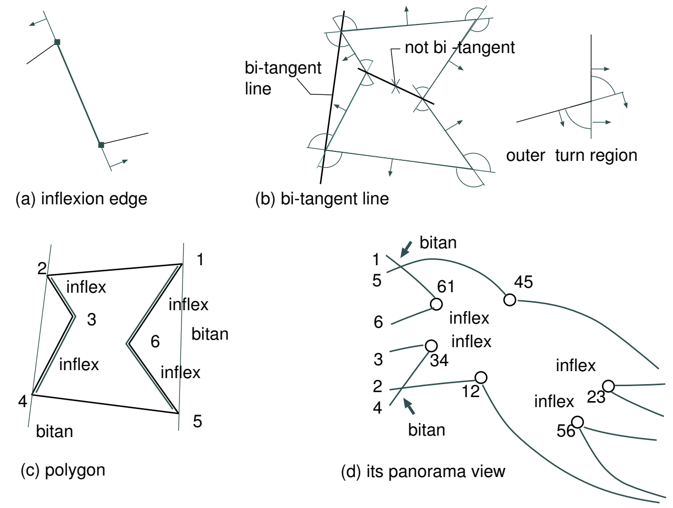

Figure 12: Feature extractions of a polygon by a panorama view

torial descriptions of the branched covering $\mathrm{PV}_0(C) \to \mathbb{R}P^1$.

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