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Composition, Union and Division of Cellular Automata on Groups

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We introduce the notion of 'Composition', 'Union' and 'Division' of cellular automata on groups. A kind of notions of compositions was investigated by Sato (1994) and Manzini (1998) for linear cellular automata, we extend the notion to general cellular automata on groups and investigated their properties. We observe the all unions and compositions generated by one-dimensional 2-neighborhood cellular automata over $\mathbf{Z}_2$ including non-linear cellular automata. Next we prove that the composition is right-distributive over union, but is not left-distributive. Finally, we conclude by showing reformulation of our definition of cellular automata on group which admit more than three states. We also show our formulation contains the representation using formal power series for linear cellular automata in Manzini (1998).

Keywords: Cellular automata, Groups, Models of computation, Automata

1 Introduction

The study of cellular automata was initiated by von Neumann (1983) and have been developed by many researchers as a good computational model for physical systems simulation. Recently cellular automata have been investigated in various fields including computer science, biology, physics, since they provide simple and powerful models for parallel computation and natural phenomena.

In this paper, we investigate cellular automata on groups as a formal model of computation. To compose simple cellular automata into a complex cellular automaton, we introduce the notion of 'Composition' of cellular automata on groups. The notion of automata on groups was first treated as a special case for automata on graphs (Caley graphs) which represent groups in Róka (1994); Rémila (1998). Watanabe and Noguchi (1982) investigated the decomposition of finite automata from the view point of spatial networks using groups. Pries et al. (1986) investigated cellular automata as a tool for implementing hardware algorithms in VLSI. They considered configurations decided by a cellular automaton as a group and divided configurations into simple configurations using group properties. Sato (1994) introduced group structured linear cellular automata and the star operation of local transition rules. The star operation is a kind of composition of cellular automata but the definition of it is different from ours. Manzini (1998) also investigated the linear cellular automata using the formal power series and their product to find inverse local transition functions. The product of formal power series are equal to our composition of cellular automata for linear cases. An abstract collision system in Ito et al. (2008) is considered as an extension of a cellular automaton, the notion of 'composition' for an abstract collision system on G-sets is investigated in Ito (2010).

This paper follow on from Fujio (2008). He introduced the composition of cellular automata on groups in order to reduce a complex behaved dynamics into simpler ones. We introduce a formal definition of cellular automata on group over $\mathbf{Z}_2$. In our framework, operations on cellular automata 'Union', 'Division' and 'Composition' are introduced. Unions of all 2-neighborhood cellular automata are investigated. Compositions of all 2-neighborhood cellular automata are also investigated and determined the subset of 3-neighborhood cellular automata which are generated by composing two 2-neighborhood cellular automata. Next we prove that the composition is right-distributive over union, but is not left-distributive. Finally, we conclude by showing reformulation of our definition of cellular automata on group which admit more than three states. We also show our formulation contains the representation using formal power series for linear cellular automata in Manzini (1998).

2 Cellular Automata on Groups

Definition 1 *Let G be a group. A cellular automaton on C is a triple $C = (G, V, V')$ of a group G, subsets $V \subset G$ and $V' \subset 2^V$. For V', we define functions $l_{V'} : 2^V \to \{\phi, \{e\}\}$ by*

$$l_{V'}(X) = \begin{cases} \phi & (X \notin V') \\ \{e\} & (X \in V'), \end{cases}$$

and $F_C : 2^G \to 2^G$ by $F_C(\mathbf{c}) = \bigcup_{g \in G} g l_{V'}(g^{-1}\mathbf{c} \cap V)$. We call the map l_V a local transition function and F_C a global transition function.

Proposition 2 *Let $C_1 = (G, V_1, V_1')$ and $C_2 = (G, V_2, V_2')$ be cellular automata. If*

$$e \in F_{C_1}(\mathbf{c}) \quad \Leftrightarrow \quad e \in F_{C_2}(\mathbf{c}) \ \ \textit{(for any } \mathbf{c} \in 2^G\textit{)}$$

then $F_{C_1} = F_{C_2}$

Proof. Since $F_{C_1}(\mathbf{c}) = \{g \in G \,|\, l_{V_1'}(g^{-1}\mathbf{c} \cap V_1) = \{e\}\} = \{g \in G \,|\, g^{-1}\mathbf{c} \cap V_1 \in V_1'\}$, we have $g \in F_{C_1}(\mathbf{c}) \Leftrightarrow g^{-1}\mathbf{c} \cap V_1 \in V_1' \Leftrightarrow e \in F_{C_1}(g^{-1}\mathbf{c}) \Leftrightarrow e \in F_{C_2}(g^{-1}\mathbf{c}) \Leftrightarrow g^{-1}\mathbf{c} \cap V_2 \in V_2' \Leftrightarrow g \in F_{C_2}(\mathbf{c})$.
□

In the followings, we consider the set of all integers $\mathbf{Z}$ as an additive group $\mathbf{Z} = (\mathbf{Z}, +, 0)$. So usual one dimensional cellular automata with 2-states are represented as cellular automata on the group $\mathbf{Z}$. We define 2-neighborhood and 3-neighborhood 2-states cellular automata in the next definition.

Definition 3 *For $k \geq 1$ and $n \in \{0, 1, \cdots, 2^{2^k} - 1\}$, we define cellular automata $CA(k, n)$ on $\mathbf{Z}$ by $CA(k, n) = (\mathbf{Z}, V, V_n')$ where $V = \{0, 1, \cdots, k-1\}$, and V_n' is the subset of 2^V which satisfies $n = \sum_{X \in V_n'} 2^{\sum_{i \in X} 2^i}$.*

We note $CA(1, 0) = (\mathbf{Z}, \{0\}, \phi)$ and $CA(1, 1) = (\mathbf{Z}, \{0\}, \{0\})$.

Example 4 *Since* $6 = 2+2^2 = 2^{2^0}+2^{2^1}$*, we have* $CA(2,6) = (\mathbf{Z}, \{0,1\}, \{\{0\},\{1\}\}\})$*. Since* $90 = 2+2^3+2^4+2^6 = 2^{2^0}+2^{2^0+2^1}+2^{2^2}+2^{2^1+2^2}$*, we have* $CA(3,90) = (\mathbf{Z}, \{0,1,2\}, \{\{0\},\{2\},\{0,1\},\{1,2\}\})$*. The elements* X *in* V'_n *represents the state of neighborhood which induce the next states '1'. For a rule number 90, we have the following table:*

Neighborhood	*111*	*110*	*101*	*100*	*011*	*010*	*001*	*000*
$X \in V'_n$	$\{0,1,2\}$	$\{1,2\}$	$\{0,2\}$	$\{2\}$	$\{0,1\}$	$\{1\}$	$\{0\}$	ϕ
$l'_V(X)$	ϕ	$\{e\}$	ϕ	$\{e\}$	$\{e\}$	ϕ	$\{e\}$	ϕ

The configuration $\mathbf{c} \subset \mathbf{Z}$ represents places where the state is 1. Since $n \in F_C(\mathbf{c}) \Leftrightarrow l_{V'}(n^{-1}\mathbf{c} \cap V) = \{e\} \Leftrightarrow n^{-1}\mathbf{c} \cap V \in V' \Leftrightarrow \mathbf{c} \cap nV \in nV'$, the next state at n is 1 if $\mathbf{c} \cap nV \in nV'$. For 3-neighborhood case we are choosing $V = \{0,1,2\}$, the left-hand side of the state is changing. It seems to be better that we choose $V = \{-1,0,1\}$ but it is not convenient for even-neighborhood case. Our numbered 3-neighborhood cellular automata $CA(3,n)$ is a shifted version of usual numbered elementary cellular automata. Later, we define a cellular automaton SHIFT which represent a shift operation and a operator 'composition' ($\diamond$) of two cellular automata. After that the usual numbered elementary cellular automata is represented as SHIFT$\diamond CA(3,n)$.

Example 5 *SHIFT* $= (\mathbf{Z}, \{-1,0\}, \{\{-1\},\{-1,0\}\})$ *is a cellular automata on group* $\mathbf{Z}$*.*

$\mathbf{Z}^2$ is also considered as a group, so it is easy to represent a multi-dimensional cellular automata such as The Game of Life (Berlekamp et al. (1982)) as a cellular automata on a group.

Example 6 *LIFE* $= (\mathbf{Z}^2, V_{LIFE}, V'_{LIFE})$ *is a cellular automata on group* $\mathbf{Z}^2$*, where*

$$\begin{aligned} V_{LIFE} &= \{\begin{pmatrix}-1\\-1\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}+1\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}+1\\0\end{pmatrix}, \begin{pmatrix}-1\\+1\end{pmatrix}, \begin{pmatrix}0\\+1\end{pmatrix}, \begin{pmatrix}+1\\+1\end{pmatrix}\}, \text{ and} \\ V'_{LIFE} &= \{v \in 2^V \mid (\#v = 3) \vee (\#v = 4 \wedge \begin{pmatrix}0\\0\end{pmatrix} \in v)\}. \end{aligned}$$

We note that $\#v$ *is the number of elements in a set* v*.*

One dimensional cellular automaton on $\mathbf{Z}$ is embedded into the two dimensional cellular automaton on $\mathbf{Z}^2$. We define two natural embeddings EX and EY in the following.

Definition 7 *For a cellular automata* $C = (G, V, V')$*, we define a cellular automata* $EX(C)$ *on* $\mathbf{Z}^2$ *by* $EX(C) = (\mathbf{Z}^2, V_{EX(C)}, V'_{EX(CA)})$ *where*

$$\begin{aligned} V_{EX(C)} &= \{\begin{pmatrix}x\\0\end{pmatrix} \mid x \in V\}, \text{ and} \\ V'_{EX(C)} &= \{\{\begin{pmatrix}x\\0\end{pmatrix} \mid x \in X\} \mid X \in V'\}. \end{aligned}$$

We also define a cellular automata $EY(C)$ *on* $\mathbf{Z}^2$ *by* $EY(C) = (\mathbf{Z}^2, V_{EY(C)}), V'_{EY(C)})$ *where*

$$\begin{aligned} V_{EY(C)} &= \{\begin{pmatrix}0\\x\end{pmatrix} \mid x \in V\}, \text{ and} \\ V'_{EY(C)} &= \{\{\begin{pmatrix}0\\x\end{pmatrix} \mid x \in X\} \mid X \in V'\}. \end{aligned}$$

$n\backslash m$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15
2	2	3	2	3	6	7	6	7	10	11	10	11	14	15	14	15
3	3	3	3	3	7	7	7	7	11	11	11	11	15	15	15	15
4	4	5	6	7	4	5	6	7	12	13	14	15	12	13	14	15
5	5	5	7	7	5	5	7	7	13	13	15	15	13	13	15	15
6	6	7	6	7	6	7	6	7	14	15	14	15	14	15	14	15
7	7	7	7	7	7	7	7	7	15	15	15	15	15	15	15	15
8	8	9	10	11	12	13	14	15	8	9	10	11	12	13	14	15
9	9	9	11	11	13	13	15	15	9	9	11	11	13	13	15	15
10	10	11	10	11	14	15	14	15	10	11	10	11	14	15	14	15
11	11	11	11	11	15	15	15	15	11	11	11	11	15	15	15	15
12	12	13	14	15	12	13	14	15	12	13	14	15	12	13	14	15
13	13	13	15	15	13	13	15	15	13	13	15	15	13	13	15	15
14	14	15	14	15	14	15	14	15	14	15	14	15	14	15	14	15
15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15

Fig. 1: Table of unions : $CA(2,n) \cup CA(2,m)$

Definition 8 *Let* $1 \le k < k'$, $0 \le x \le k' - k$ *and* $CA(k,n) = (\mathbf{Z}, V, V')$. $CA(k,n)^{k'}_x$ *is defined by* $CA(k,n)^{k'}_x = (\mathbf{Z}, \{0,1,\cdots,k'-1\}, V'')$ *where*

$$\begin{aligned} V'' &= \{s_1 \cup v \cup s_2 \mid s_1 \in S_1, s_2 \in S_2, v \in V'\}, \\ S_1 &= \begin{cases} \{\phi\} & (x = 0) \\ 2^{\{0,\cdots,x-1\}} & (x > 0) \end{cases}, \\ S_2 &= 2^{\{x+1,\cdots,k\}} \end{aligned}$$

We note that $F_{CA(k,n)^{k'}_0} = F_{CA(k,n)}$ and $F_{CA(k,n)^{k'}_1} = SHIFT \diamond F_{CA(k,n)}$.

Definition 9 (Union) *Let* $C_1 = (G, V_1, V_1')$ *and* $C_2 = (G, V_2, V_2')$ *be cellular automata on G. The union* $C_1 \cup C_2$ *of* C_1 *and* C_2 *is defined by* $C_1 \cup C_2 = (G, V_1 \cup V_2, V_1' \cup V_2')$.

Definition 10 (Division) *Let* $C = (G, V, V')$ *be a cellular automaton on G. If there exist* $C_1 = (G, V_1, V_1')$ *and* $C_2 = (G, V_2, V_2')$ *be cellular automata on G such that* $V = V_1 \cup V_2$ *and* $V' = V_1' \cup V_2'$, *then we call* C_1 *and* C_2 *are division of* C *and* C *is dividable by* C_1 *and* C_2.

Example 11 *The class of all 2-neighborhood cellular automata* $\{CA(2,n) \mid n = 0,..,15\}$ *is generated by* $\{CA(2,0)$, $CA(2,1)$, $CA(2,2)$, $CA(2,4)$, $CA(2,8)\}$ *using 'union' operations. For example,* $CA(2,13)$ *is dividable by* $CA(2,1)$, $CA(2,4)$, *and* $CA(2,8)$. *Fig 1 is the table of unions for* $CA(2,n)$ *(*$n = 0,..,15$*).*

Definition 12 (Composition) *Let* $C_1 = (G, V_1, V_1')$ *and* $C_2 = (G, V_2, V_2')$ *be cellular automata on G. The composition* $C_1 \diamond C_2$ *of* C_1 *and* C_2 *is defined by* $C_1 \diamond C_2 = (G, V_1 \cdot V_1, V_1' \diamond V_2')$ *where*

$$\begin{aligned} V_1 \cdot V_2 &= \{v_1 v_2 \in G \mid v_1 \in V_1, v_2 \in V_2\} \text{ and} \\ V_1' \diamond V_2' &= \{X \in 2^{V_1 \cdot V_2} \mid \{v \in V_1 \mid v^{-1}X \cap V_2 \in V_2'\} \in V_1'\}. \end{aligned}$$

Example 13 *Let* $V = \{0,1\}$ *and* $V' = \{\{0\},\{1\}\})$. *We have* $V \cdot V = \{0,1,2\}$. *Since* $1^{-1}\{0,1\} \cap \{0,1\}$ $\{0-1, 1-1\} \cap \{0,1\} = \{0\}$ *and* $0^{-1}\{1,2\} \cap \{0,1\} = \{1-0, 2-0\} \cap \{0,1\} = \{1\}$, *we have* $V' \diamond V' = \{\{0\},\{2\},\{0,1\},\{1,2\}\}$. *So we have* $CA(2,6) \diamond CA(2,6) = CA(3,90)$.

$n\backslash m$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	255	236	209	192	139	136	129	128	55	36	17	0	3	0	1	0
2	0	16	34	48	68	68	66	64	8	24	34	48	12	12	2	0
3	255	252	243	240	207	204	195	192	63	60	51	48	15	12	3	0
4	0	2	12	12	48	34	24	8	64	66	68	68	48	34	16	0
5	255	238	221	204	187	170	153	136	119	102	85	68	51	34	17	0
6	0	18	46	60	116	102	90	72	72	90	102	116	60	46	18	0
7	255	254	255	252	255	238	219	200	127	126	119	116	63	46	19	0
8	0	1	0	3	0	17	36	55	128	129	136	139	192	209	236	255
9	255	237	209	195	139	153	165	183	183	165	153	139	195	209	237	255
10	0	17	34	51	68	85	102	119	136	153	170	187	204	221	238	255
11	255	253	243	243	207	221	231	247	191	189	187	187	207	221	239	255
12	0	3	12	15	48	51	60	63	192	195	204	207	240	243	252	255
13	255	239	221	207	187	187	189	191	247	231	221	207	243	243	253	255
14	0	19	46	63	116	119	126	127	200	219	238	255	252	255	254	255
15	255	255	255	255	255	255	255	255	255	255	255	255	255	255	255	255

Fig. 2: Table of compositions : $CA(2,n)\diamond CA(2,m)$

Example 14 *The rule numbers of the 3-neighborhood cellular automata generated by composing 2-neighborhood cellular automata is* $\{0, 1, 2, 3, 8, 12, 15, 16, 17, 18, 19, 24, 34, 36, 46, 48, 51, 55, 60, 63, 64, 66, 68, 72, 85, 90, 102, 116, 119, 126, 127, 128, 129, 136, 139, 153, 165, 170, 183, 187, 189, 191, 192, 195, 200, 204, 207, 209, 219, 221, 231, 236, 237, 238, 239, 240, 243, 247, 252, 253, 254, 255\}$. *There are 62 kinds of 3-neighborhood cellular automata. Fig 2 is the table of compositions for* $CA(2,n)$ *(*$n = 0, .., 15$*).*

Lemma 15 *Let* $C = (G, V, V')$ *be a cellular automaton and* $V_0 \subset G$*. For any* $\mathbf{c} \in 2^G$*,*

$$F_C(\mathbf{c}) \cap V_0 = F_C(\mathbf{c} \cap (V_0 \cdot V)) \cap V_0$$

Proof. We have $F_C(\mathbf{c}) \cap V_0 = \{v_0 \in V_0 \,|\, v_0^{-1}\mathbf{c} \cap V \in V'\} = \{v_0 \in V_0 \,|\, \mathbf{c} \cap v_0 V \in v_0 V'\} = \{v_0 \in V_0 \,|\, (\mathbf{c} \cap V_0 \cdot V) \cap v_0 V \in v_0 V'\} = F_C(\mathbf{c} \cap (V_0 \cdot V)) \cap V_0$. □

The composition of cellular automata corresponds to find a cellular automaton which global transition function is the composition of global transition functions of original cellular automata.

Theorem 16 (Fujio (2008))

$$F_{C_1} \circ F_{C_2} = F_{C_1 \diamond C_2}$$

Proof. Since $F_{C_2}(\mathbf{c}) \cap V_1 = \{v \in V_1 \,|\, v^{-1}\mathbf{c} \cap V_2 \in V_2'\}$, we have

$$\begin{aligned}
e \in F_{C_1}(F_{C_2}(\mathbf{c})) \quad &\Leftrightarrow \quad F_{C_2}(\mathbf{c}) \cap V_1 \in V_1' \\
&\Leftrightarrow \quad F_{C_2}(\mathbf{c} \cap V_1 \cdot V_2) \cap V_1 \in V_1' \text{ (by Proposition. 2)} \\
&\Leftrightarrow \quad \{v_1 \in V_1 \,|\, v^{-1}(\mathbf{c} \cap V_1 \cdot V_2) \cap V_2 \in V_2'\} \in V_1' \\
&\Leftrightarrow \quad \mathbf{c} \cap V_1 \cdot V_2 \in V_1' \diamond V_2' \\
&\Leftrightarrow \quad e \in F_{C_1 \diamond C_2}(\mathbf{c})
\end{aligned}$$

□

Theorem 17 *Let* $C_1 = (G, V, V_1')$*,* $C_2 = (G, V, V_2')$ *and* $C_3 = (G, V_3, V_3')$ *be cellular automata on a group G. Then,*

$$(C_1 \cup C_2) \diamond C_3 = (C_1 \diamond C_3) \cup (C_2 \diamond C_3)$$

Proof. First, we note

$$\begin{aligned}(C_1 \cup C_2)\diamond C_3 &= (G, V\cdot V_3, (V_1' \cup V_2')\diamond V_3')\text{ , and}\\ (C_1\diamond C_3) \cup (C_2\diamond C_3) &= (G, V\cdot V_3, (V_2'\diamond V_1') \cup (V_3'\diamond V_1')).\end{aligned}$$

Next, we have

$$\begin{aligned}(V_1' \cup V_2')\diamond V_3' &= \{X \in 2^{V\cdot V_3} \mid \{v \in V \mid v^{-1}X \cap V_3 \in V_3'\} \in V_1' \cup V_2'\}\\ &= \{X \in 2^{V\cdot V_3} \mid \{v \in V \mid v^{-1}X \cap V_3 \in V_3'\} \in V_1'\}\\ &\quad \cup\{X \in 2^{V\cdot V_3} \mid \{v \in V \mid v^{-1}X \cap V_3 \in V_3'\} \in V_2'\}\\ &= (V_1'\diamond V_3') \cup (V_2'\diamond V_3')\end{aligned}$$

□

We note that $C_1\diamond(C_2 \cup C_3) = (C_1\diamond C_2) \cup (C_1\diamond C_3)$ does not always holds for cellular automata C_1, C_2 and C_3. For example $CA(2,6)\diamond(CA(2,2) \cup CA(2,4)) = CA(2,6)\diamond CA(2,6) = CA(3,90)$, and $(CA(2,6) \cup CA(2,2))\diamond(CA(2,6) \cup CA(2,4)) = CA(3,46)\diamond CA(3,116)) = CA(3,126)$.

Proposition 18 *Let $CA(1,n)^{k_1}_x$, $CA(k_2,n_2)$ and $CA(k_2,n_3)$ be cellular automata on $\mathbf{Z}$, where $0 \le x < k_1$, and $n = 0,1$. Then,*

$$CA(1,n)^{k_1}_x\diamond(CA(k_2,n_2) \cup CA(k_2,n_3)) = (CA(1,n)^{k_1}_x\diamond CA(k_2,n_2)) \cup (CA(1,n)^{k_1}_x\diamond CA(k_2,n_3)).$$

Proof. Let $V_1 = \{0,\cdots,k_1-1\}$, $V_1' = \{X \in 2^V \mid x \in X\})$, $\bar{V_1'} = \{X \in 2^V \mid x \notin X\})$, $CA(k_2,n_2) = (\mathbf{Z}, V_2, V_2')$, and $CA(k_2,n_3) = (\mathbf{Z}, V_2, V_3')$. First, we note

$$\begin{aligned}CA(1,0)^{k_1}_x &= (\mathbf{Z}, V_1, \bar{V_1'}),\\ CA(1,1)^{k_1}_x &= (\mathbf{Z}, V_1, V_1'),\\ CA(1,0)^{k_1}_x\diamond(CA(k_2,n_2) \cup CA(k_2,n_3)) &= (\mathbf{Z}, V_1\cdot V_2, V_1'\diamond(V_2' \cup V_3')),\text{ and}\\ (CA(1,n)^{k_1}_x\diamond CA(k_2,n_2)) \cup (CA(1,n)^{k_1}_x\diamond CA(k_2,n_3)) &= (\mathbf{Z}, V_1\cdot V_2, (V_1'\diamond V_2') \cup (V_1'\diamond V_3')).\end{aligned}$$

Since

$$\begin{aligned}V_1'\diamond(V_2' \cup V_3') &= \{X \in 2^{V_1\cdot V_2} \mid \{v \in V \mid v^{-1}X \cap V_2 \in (V_2' \cup V_3')\} \in V_1'\}\\ &= \{X \in 2^{V_1\cdot V_2} \mid x^{-1}X \cap V_2 \in (V_2' \cup V_3')\},\text{ and}\\ (V_1'\diamond V_2') \cup (V_1'\diamond V_3') &= \{X \in 2^{V_1\cdot V_2} \mid \{v \in V \mid v^{-1}X \cap V_2 \in V_2'\} \in V_1'\}\\ &\quad \cup\{X \in 2^{V_1\cdot V_2} \mid \{v \in V \mid v^{-1}X \cap V_2 \in V_3'\} \in V_1'\}\\ &= \{X \in 2^{V_1\cdot V_2} \mid x^{-1}X \cap V_2 \in V_2'\}\\ &\quad \cup\{X \in 2^{V_1\cdot V_2} \mid x^{-1}X \cap V_2 \in V_3'\},\end{aligned}$$

we have $V_1'\diamond(V_2' \cup V_3') = (V_1'\diamond V_2') \cup (V_1'\diamond V_3')$, and

$$CA(1,1)^{k_1}_x\diamond(CA(k_2,n_2) \cup CA(k_2,n_3)) = (CA(1,1)^{k_1}_x\diamond CA(k_2,n_2)) \cup (CA(1,1)^{k_1}_x\diamond CA(k_2,n_3)).$$

Similarly, we can prove

$$CA(1,0)_x^{k_1}\diamond(CA(k_2,n_2)\cup CA(k_2,n_3)) = (CA(1,0)_x^{k_1}\diamond CA(k_2,n_2))\cup(CA(1,0)_x^{k_1}\diamond CA(k_2,n_3)).$$

□

Example 19 *We note* $CA(3,3) = (\mathbf{Z},\ \{0,1,2\},\ \{\phi,\ \{0\}\})$ *and* $CA(3,102) = (\mathbf{Z},\ \{0,1,2\},\ \{\{0\},\ \{1\},\ \{0,2\},\ \{1,2\}\})$. *The composition* $CA(3,3)\diamond CA(3,102) = (\{0,1,2,3,4\},\ \{\{1\},\{0,1\},\{1,4\},\{0,1,4\},\ \{3\},\{0,3\},\{3,4\},\{0,3,4\}\})$. *Since* $\{\{1\},\{0,1\},\{1,4\},\{0,1,4\},\ \{3\},\{0,3\},\{3,4\},\{0,3,4\}\}) = \bigcup\{\{\{0\}\cup s,\ s\cup\{4\},\ \{0\}\cup s\cup\{4\}\}\ |s\in\{\{1\},\{3\}\}\}$, *we have* $CA(3,3)\diamond CA(3,102) = CA(3,18)_1^5$. *(cf. Fig 3, Fig 4, Fig 5)*

Fig. 3: An example of configurations of $CA(3,3)$

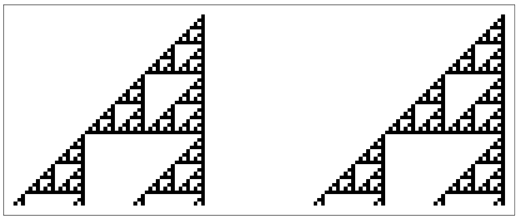

Fig. 4: An example of configurations of $CA(3,102)$

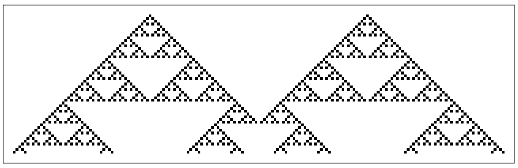

Fig. 5: An example of configurations of $CA(3,18) = CA(3,3)\diamond CA(3,102)$

Example 20 *A* 2*-neighborhood cellular automaton is considered as* 3*-neighborhood cellular automaton and* 3*-neighborhood cellular automaton is also considered as* 5*-neighborhood cellular automaton. The followings is an observation of the embeddings and compositions.*

$$\begin{aligned}
CA(2,1) &= (\mathbf{Z},\{0,1\},\{\phi\})\\
CA(2,1)_0^3 &= (\mathbf{Z},\{0,1,2\},\{\phi,\{2\}\})\\
&= CA(3,17)\\
CA(2,1)\diamond CA(2,1) &= (\mathbf{Z},\{0,1,2\},\{\{0,1\},\{0,2\},\{1,2\},\{1\}\})\\
&= CA(3,236)\\
CA(3,17)\diamond CA(3,17) &= (\mathbf{Z},\{0,1,2,3,4,5\},V')\\
&= CA(5,3974950124) = CA(3,236)_0^5\\
V' &= \bigcup\{\{s,s\cup\{3\},s\cup\{4\},s\cup\{3,4\}\}|s\in CA(3,236)\}
\end{aligned}$$

3 Generalization

A subset V of G is considered as a characteristic function $V : G \to 2$ where $2 = \{0, 1\}$. That is V is a function which values are

$$V(g) = \begin{cases} 0 & (g \notin V) \\ 1 & (g \in V). \end{cases}$$

Sometimes V is represented as an injection $i_V : V \to G$ where $i_V(g) = g$.

Extending our 2-states cellular automata on groups to many-states cellular automata on groups, we replace the set $2 = \{0, 1\}$ to a finite set S.

Definition 21 *Let G be a group, S a finite set. A generalized cellular automaton on G is a four-tuple $C = (G, S, i_V, V')$ of the group G, an injection $i_v : V \to G$, and a function $V' : S^V \to S$ where S^V is the set of all functions from V to S. A configuration $\mathbf{c} : G \to S$ is a function. The global transition function $F_C : S^G \to S^G$ is defined by $F_C(\mathbf{c})(g) = V'(\mathbf{c} \circ g \circ i_V)$.*

Proposition 22 *Let G be a group, and $S = 2 = \{0, 1\}$. The global function $F_C : 2^G \to 2^G$ is the same as defined in Definition 1. That is $F_C(\mathbf{c}) = \{g \in G \,|\, V'(\mathbf{c} \circ g \circ i_V) = 1\} = \bigcup_{g \in G} g \cdot l_{V'}(g^{-1} \cdot \mathbf{c} \cap V)$.*

Proof. For $g \in G$, we have

$$\begin{aligned}
g \in \bigcup_{g \in G} g \cdot l_{V'}(g^{-1} \cdot \mathbf{c} \cap V) \quad &\Leftrightarrow \quad l_{V'}(g^{-1} \cdot \mathbf{c} \cap V) = \{e\} \\
&\Leftrightarrow \quad g^{-1} \cdot \mathbf{c} \cap V \in V' \\
&\Leftrightarrow \quad g^{-1}\{x \,|\, \mathbf{c}(x) = 1\} \cap V \in V' \\
&\Leftrightarrow \quad \{g^{-1}x \,|\, \mathbf{c}(x) = 1\} \cap V \in V' \\
&\Leftrightarrow \quad \{v \,|\, \mathbf{c}(gv) = 1\} \cap V \in V' \text{ (cf. } (x = gv)) \\
&\Leftrightarrow \quad \{v \,|\, \mathbf{c}(gv) = 1, v \in V\} \in V' \\
&\Leftrightarrow \quad \{v \,|\, \mathbf{c} \circ g \circ i_V(v) = 1\} \in V' \\
&\Leftrightarrow \quad V'(\mathbf{c} \circ g \circ i_V) = 1 \\
&\Leftrightarrow \quad g \in F_C(\mathbf{c}).
\end{aligned}$$

□

Example 23 *Let $G = \mathbf{Z}$, $S = \mathbf{Z}_m$, and $V = \{-r, -r+1, \cdots, 0, \cdots, +r\}$. For a polynomial $f(X) = \sum_{i=-r}^{+r} a_i X^i$, ($a_i \in \mathbf{Z}_m$), we define the function $V'_{f(X)} : \mathbf{Z}_m^V \to \mathbf{Z}_m$ by $V'(x_{-r}, x_{-r+1}, \cdots, x_0, \cdots, x_{+r})$ $= \sum_{i=-r}^{+r} a_{-i} x_i$, ($(x_{-r}, x_{-r+1}, \cdots, x_0, \cdots, x_{+r}) \in \mathbf{Z_m}^V$). A configuration $\mathbf{c} \in \mathbf{Z}_m^{\mathbf{Z}}$ is represented as a formal power series $\sum c_i X^i$ where $c_i = \mathbf{c}(i)$ (cf. Sato (1994); Manzini (1998)). Since $\mathbf{c} \circ j \circ i_V(i) =$*

$\mathbf{c}(j+i) = c_{j+i}$, *and* $\mathbf{c} \circ j \circ i_V = (c_{j-r}, c_{j-r+1}, \cdots, c_j, \cdots, c_{j+r})$, *we have*

$$\begin{aligned}
(\sum \mathbf{c}(i)X^i)f(X) &= (\sum c_i X^i)f(X) \\
&= (\sum c_i X^i)(\sum_{i'=-r}^{+r} a_{i'} X^{i'}) \\
&= (\sum c_i X^i)(\sum_{i'=-r}^{+r} a_{-i'} X^{-i'}) \\
&= \sum(\sum_{i'=-r}^{+r} c_i a_{-i'} X^{i-i'}) \\
&= \sum((\sum_{i'=-r}^{+r} a_{-i'} c_{j+i'})X^j) \text{ (cf. } j = i - i'\text{)} \\
&= \sum(V'(c_{j-r}, c_{j-r+1}, \cdots, c_j, \cdots, c_{j+r})X^j) \\
&= \sum(V'(\mathbf{c} \circ j \circ i_V)X^j). \\
&= \sum(F_C(\mathbf{c})(j)X^j).
\end{aligned}$$

The transition of the cellular automaton $C = (\mathbf{Z}, \mathbf{Z}_m, i_V, V'_{f(X)})$ *is corresponding to the product of polynomials (the formal power series).*

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