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Reiichiro Kawai
& Hiroki Masuda

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Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

Exact Simulation of Finite Variation Tempered Stable Ornstein-Uhlenbeck Processes

REIICHIRO KAWAI* AND HIROKI MASUDA†

Abstract

We develop an exact yet simple simulation algorithm for a wide class of Ornstein-Uhlenbeck processes with a tempered stable stationary distribution of finite variation. We derive the exact transition probability of tempered stable Ornstein-Uhlenbeck processes between consecutive times due to the homogeneous Markovian autoregressive structure. Random element involved can be divided into independent tempered stable and compound Poisson distributions, each of which can be generated in the exact sense with acceptance-rejection methods, respectively, with stable and gamma proposal distributions. Our algorithm proves useful for the simulations of bilateral tempered stable Ornstein-Uhlenbeck processes and normal tempered stable processes. The effectiveness of the proposed algorithm is discussed relative to the existing approximative method based on infinite series representation of sample paths.

Keywords: acceptance-rejection method, Lévy process, Ornstein-Uhlenbeck processes, subordinator, transition probability, tempered stable process.
2010 Mathematics Subject Classification: 68U20, 62E15, 65C10, 60E07.

1 Introduction

The class of non-Gaussian Ornstein-Uhlenbeck processes has long been of both theoretical and practical interest. From a theoretical point of view, on one hand, this class is closely related to the self-decomposable infinitely divisible distribution. Several interesting properties are known,

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*Email Address: reiichiro.kawai@gmail.com. Postal Address: Department of Mathematics, University of Leicester, Leicester LE1 7RH, UK.
†Email Address: hiroki@math.kyushu-u.ac.jp. Postal Address: Graduate School of Mathematics, Kyushu University, Fukuoka 819-0395, Japan.

such as the explicit relation between the Lévy measures of the stationary distribution and the underlying Lévy process and the representation of entire trajectory using the series representation of underlying Lévy process, to mention just a few. (For details, see Section 17 of Sato [15], Masuda [11] and references therein.) On the other hand, in practice, non-Gaussian Ornstein-Uhlenbeck processes have been used in mathematical physics under the name of exponentially correlated colored noise, and more recently in financial economics and mathematical finance (for example, Barndorff-Nielsen and Shephard [2, 3] and Benth et al. [4]). Due to the growing practical interest, many authors have proposed statistical inference methods for the non-Gaussian Ornstein-Uhlenbeck processes. (See, for example, Brockwell et al. [5] and Jongbloed et al. [9].)

The main purpose of this paper is to develop an exact simulation algorithm for a wide class of Ornstein-Uhlenbeck processes of finite variation with a tempered stable stationary distribution. In particular, the flexibility and mathematical tractability of the tempered stable distribution makes this class more attractive than the other classes of Ornstein-Uhlenbeck processes. It is well known that the exact simulation of its *entire trajectory* over a finite horizon is only possible with the infinite series representation of tempered stable Lévy processes. (See, for example, Barndorff-Nielsen and Shephard [2, 3] and Rosiński [14].) This method often requires extremely expensive computing effort, especially when the convergence of the infinite series is very slow. In addition, the series representation is no longer an exact simulation method as soon as the infinite sum is truncated. The exact simulation algorithm we develop in this paper is designed to simulate arbitrary discrete time skeleton of the trajectory. Thanks to the homogeneous Markovian autoregressive structure of Ornstein-Uhlenbeck processes, its transition probability between consecutive times can be derived in closed form, in a similar manner to Zhang and Zhang [17]. Random element involved can be divided into independent tempered stable and compound Poisson components, each of which can be simulated exactly with acceptance-rejection methods, respectively, with non-tempered stable and gamma proposal distributions. It turns out that our approach is easily applicable to the settings of bilateral tempered stable Ornstein-Uhlenbeck processes of finite variation and of normal tempered stable processes.

The rest of this paper is organized as follows. Section 2 summarizes background material on stable and tempered stable subordinators and on tempered stable Ornstein-Uhlenbeck processes. In Section 3, we derive the exact transition probability and show that random element is divided into independent tempered stable and compound Poisson components. In Section 4, we discuss acceptance-rejection methods for simulation of tempered stable and compound Poisson distributions involved in the exact transition. We illustrate in Section 5 the effectiveness of our exact simulation algorithm relative to the existing approximative method based on infinite series representation. Finally, Section 6 concludes.

2 Preliminaries

Let us begin this preliminary section with the notations which will be used throughout the paper. We denote by $\mathbb{R}$ the one dimensional Euclidean space with the norm $|\cdot|$, $\mathbb{R}_+ := (0,+\infty)$ and $\mathbb{R}_- := (-\infty,0)$. Let $\mathbb{N}$ be the collection of positive integers with $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. We denote by $\stackrel{\mathscr{L}}{=}$ and $\stackrel{\mathscr{L}}{\to}$, respectively, identity and convergence in law. We denote by $\Gamma(a,b)$ the gamma distribution with density $b^a/\Gamma(a)x^{a-1}e^{-bx}$. We fix $(\Omega,\mathscr{F},\mathbb{P})$ as our underlying probability space. We say that the stochastic process $\{Y_t : t\geq 0\}$ in $\mathbb{R}$ is a subordinator (without drift) if it is a non-decreasing Lévy process with characteristic function

$$\mathbb{E}\left[e^{iyY_t}\right] = \exp\left[t\int_{\mathbb{R}_+}\left(e^{iyz}-1\right)\nu(dz)\right], \tag{2.1}$$

where ν is a Lévy measure defined on $\mathbb{R}_+$ satisfying $\int_0^1 z\nu(dz) < +\infty$. Finally, let us note that $\Gamma(-a) < 0$ for $a\in(0,1)$.

2.1 Stable Subordinator

Let $\{L_t^{(\mathrm{s})} : t\geq 0\}$ be a stable subordinator with characteristic function

$$\mathbb{E}\left[e^{iyL_t^{(\mathrm{s})}}\right] = \exp\left[t\int_{\mathbb{R}_+}\left(e^{iyz}-1\right)\frac{a}{z^{\alpha+1}}dz\right] = \exp\left[ta\Gamma(-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)|y|^\alpha\left(1-i\tan\frac{\pi\alpha}{2}\mathrm{sgn}(y)\right)\right], \tag{2.2}$$

with $\alpha\in(0,1)$ and $a>0$. For each $t>0$, the marginal $L_t^{(\mathrm{s})}$ has a stable distribution on $\mathbb{R}_+$ and $\mathbb{E}[(L_t^{(\mathrm{s})})^\theta]$ is finite if $\theta\in(0,\alpha)$, while is infinite if $\theta\geq\alpha$. Throughout this paper, we denote by $S(\alpha,a)$ the distribution of $L_1^{(\mathrm{s})}$ when (2.2) is satisfied. Clearly, it holds that for each $t>0$, $\mathscr{L}(L_t^{(\mathrm{s})}) = S(\alpha,ta)$. The distribution $S(\alpha,a)$ can be simulated in the exact sense through the well known representation, due to Chambers et al. [6],

$$S(\alpha,a)\stackrel{\mathscr{L}}{=}\left(\frac{a\Gamma(1-\alpha)}{\alpha\cos(V)}\right)^{\frac{1}{\alpha}}\sin(\alpha(V+\pi/2))\left(\frac{\cos\left(V-\alpha\left(V+\pi/2\right)\right)}{E}\right)^{\frac{1-\alpha}{\alpha}}, \tag{2.3}$$

where V is a uniform random variable on $(-\pi/2,\pi/2)$ and E is a standard exponential random variable independent of V. The distribution $S(\alpha,a)$ has a C^∞-density on $\mathbb{R}+$ given in form of convergent series

$$f_{S(\alpha,a)}(x) := \frac{1}{\pi(-a\Gamma(-\alpha))^{1/\alpha}}\sum_{k=1}^{+\infty}(-1)^{k-1}\sin(k\pi\alpha)\frac{\Gamma(k\alpha+1)}{k!}\left(\frac{x}{(-a\Gamma(-\alpha))^{1/\alpha}}\right)^{-k\alpha-1}. \tag{2.4}$$

See Zolotarev [18] for more details on the stable distribution.

2.2 Tempered Stable Subordinator

Consider the *exponentially tempered* stable Lévy density

$$\nu(z) = a\frac{e^{-bz}}{z^{\alpha+1}}, \quad z \in \mathbb{R}_+, \tag{2.5}$$

where $a > 0$, $b > 0$ and $\alpha \in (0,1)$. The associated subordinator $\{L_t^{(\mathrm{ts})} : t \geq 0\}$ (without drift) is often called the *tempered stable subordinator*, with characteristic function

$$\mathbb{E}\left[e^{iyL_t^{(\mathrm{ts})}}\right] = \exp\left[t\int_{\mathbb{R}_+}\left(e^{iyz}-1\right)\nu(z)dz\right] = \exp\left[ta\Gamma(-\alpha)\left((b-iy)^{\alpha}-b^{\alpha}\right)\right]. \tag{2.6}$$

Throughout this paper, we denote by $TS(\alpha,a,b)$ the distribution of $L_1^{(\mathrm{ts})}$ defined on $\mathbb{R}_+$, which we call tempered stable distribution, when (2.6) is satisfied. Clearly, it holds that for each $t > 0$, $\mathscr{L}(L_t^{(\mathrm{ts})}) = TS(\alpha,ta,b)$. The tempered stable distribution has a C^{∞}-density on $\mathbb{R}_+$ as well, with a simple yet very insightful relation to the density of a stable distribution

$$f_{TS(\alpha,a,b)}(x) := e^{-bx-a\Gamma(-\alpha)b^{\alpha}} f_{S(\alpha,a)}(x). \tag{2.7}$$

This property serves as a key building block later. The class of tempered stable distributions is first proposed by Tweedie [16]. Barndorff-Nielsen and Shephard [3] studies the tempered stable subordinator and the so-called normal tempered stable distributions, that is a normal variance-mean mixture of the positive tempered stable distribution, with a view towards financial economics. Several featuring properties of tempered stable processes are revealed by Rosiński [14], such as a stable-like behavior over short intervals, the absolute continuity with respect to its short-range limiting stable subordinator (Proposition 4.1), an aggregational Gaussianity and an infinite series representation in closed form

$$\left\{L_t^{(\mathrm{ts})} : t \in [0,T]\right\} \stackrel{\mathscr{L}}{=} \left\{\sum_{k=1}^{+\infty}\left[\left(\frac{\alpha\Gamma_k}{aT}\right)^{-1/\alpha} \wedge \frac{V_k U_k^{1/\alpha}}{b}\right]\mathbb{1}(T_k \in [0,t]) : t \in [0,T]\right\}, \tag{2.8}$$

which was first introduced by Rosiński in the discussion part of Barndorff-Nielsen and Shephard [2]. Here, $\{\Gamma_k\}_{k\in\mathbb{N}}$ is a sequence of standard Poisson arrivals, $\{T_k\}_{k\in\mathbb{N}}$ is a sequence of iid uniform random variables on $[0,T]$, $\{V_k\}_{k\in\mathbb{N}}$ is a sequence of iid standard exponential random variables and $\{U_k\}_{k\in\mathbb{N}}$ is a sequence of iid uniform random variables on $[0,1]$. All those random sequences are mutually independent. Note that the kernel of series representation is not unique. In fact, there are a different series representation derived through the rejection method of Rosiński [13] and yet another representation numerically obtained in Imai and Kawai [8].

2.3 Ornstein-Uhlenbeck Processes with Tempered Stable Stationary Distribution

Consider the stochastic process $\{Y_t : t \geq 0\}$ defined in form of stochastic differential equation

$$dY_t = -\lambda Y_t dt + dZ_{\lambda t}, \tag{2.9}$$

where $\lambda > 0$ and $\{Z_t : t \geq 0\}$ is a subordinator, or in canonical form

$$Y_t = e^{-\lambda t} Y_0 + e^{-\lambda t} \int_0^{\lambda t} e^s dZ_s. \tag{2.10}$$

The process of this type is called a Lévy-driven Ornstein-Uhlenbeck process and is used, for example, to model the squared volatility in a stochastic volatility model of Barndorff-Nielsen and Shephard [2].

The Lévy density (2.5) forms a self-decomposable Lévy measure. By the arguments in Section 17 of Sato [15], there exists an Ornstein-Uhlenbeck process $\{Y_t : t \geq 0\}$ whose marginal has the infinitely divisible distribution with the tempered stable Lévy density (2.5), if the initial state Y_0 is chosen to have the same distribution to the stationary infinitely divisible distribution. In particular, the Ornstein-Uhlenbeck process with inverse Gaussian stationary marginal ($\alpha = 1/2$) is often abbreviated to IG-OU and is applied in Benth [4] to stochastic volatility modeling of [2] for volatility and variance swap valuations.

Let $w(z)$ be the Lévy density of the marginal Z_1 and let $u(z)$ be the Lévy density of the stationary marginal Y_1. If $u(z)$ is differentiable, then the Lévy densities $w(z)$ and $u(z)$ are related as

$$w(z) = -u(z) - z\frac{\partial}{\partial z}u(z) = a\left(\frac{\alpha}{z} + b\right)\frac{e^{-bz}}{z^\alpha}. \tag{2.11}$$

This implies that the underlying subordinator $\{Z_t : t \geq 0\}$ is the superposition of a tempered stable subordinator and a compound Poisson process. With the infinite series representation (2.8) of tempered stable subordinators, we can equate in law as

$$\begin{aligned}\{Y_t : t \in [0,T]\} \stackrel{\mathscr{L}}{=} \Bigg\{ e^{-\lambda t} Y_0 + \sum_{k=1}^{+\infty} e^{\lambda(T_k - t)} \left[\left(\frac{\Gamma_k}{aT}\right)^{-1/\alpha} \wedge \frac{V_k U_k^{1/\alpha}}{b}\right] \mathbb{1}\left(T_k \in [0,t]\right) \\ + \sum_{k=1}^{+\infty} e^{\widetilde{\Gamma}_k - \lambda t} G_k \mathbb{1}\left(\widetilde{\Gamma}_k \in [0, \lambda t]\right) : t \in [0,T] \Bigg\},\end{aligned} \tag{2.12}$$

where $\{\widetilde{\Gamma}_k\}_{k\in\mathbb{N}}$ is a sequence of Poisson arrivals with intensity $a\Gamma(1-\alpha)b^\alpha$, independent of $\{\Gamma_k\}_{k\in\mathbb{N}}$, and $\{G_k\}_{k\in\mathbb{N}}$ is a sequence of iid random variables with gamma distribution $\Gamma(1-\alpha, b)$.

In fact, we can readily extend to the bilateral finite variation setting by superpositioning two independent subordinators in the opposite directions, by setting $Z_t := Z_t^+ - Z_t^-$ in the definition (2.9) or (2.10), where $\{Z_t^\pm : t \geq 0\}$ are independent subordinators with suitable laws. This setting will be considered in Corollary 3.2.

3 Exact Transition Probability

The main purpose of this paper is to develop an algorithm for the exact simulation of arbitrary discrete time skeleton

$$Y_0, Y_\Delta, Y_{2\Delta}, \cdots,$$

of a tempered stable Ornstein-Uhlenbeck process (2.9), with a positive time increment Δ. (In principle, time increments do not need to be equidistant and can be set different positive values for different steps.) To this end, we first derive the exact transition probability of the random sequence $\{Y_{k\Delta}\}_{k\in\mathbb{N}_0}$.

Theorem 3.1. *For each $n \in \mathbb{N}_0$, it holds that given $Y_{n\Delta}$,*

$$Y_{(n+1)\Delta} \stackrel{\mathscr{L}}{=} e^{-\lambda\Delta} Y_{n\Delta} + \eta_0 + \sum_{k=1}^{N_\Delta} \eta_k,$$

where N_Δ and $\eta_0, \eta_1, \ldots$ are independent random variables specified as follows.

- $\eta_0 \sim TS(\alpha, a(1-e^{-\alpha\lambda\Delta}), b)$.
- *N_Δ denotes the Poisson random variable with intensity $-a(1-e^{-\alpha\lambda\Delta})\Gamma(-\alpha)b^\alpha$.*
- *$\{\eta_k\}_{k\in\mathbb{N}}$ is a sequence of iid random variables with common probability density*

$$\nu_\Delta(x) := \frac{1}{(1-e^{\alpha\lambda\Delta})\Gamma(-\alpha)b^\alpha} x^{-1-\alpha} \left(e^{-bx} - e^{-be^{\lambda\Delta}x}\right), \quad x \in \mathbb{R}_+. \tag{3.1}$$

Proof. Due to the homogeneous Markovian autoregressive structure of (2.10), it holds that for each $n \in \mathbb{N}_0$,

$$\begin{aligned} Y_{(n+1)\Delta} &= e^{-\lambda\Delta} Y_{n\Delta} + \int_{n\Delta}^{(n+1)\Delta} e^{-\lambda((n+1)\Delta - s)} dZ_{\lambda s} \\ &=: e^{-\lambda\Delta} Y_{n\Delta} + \varepsilon_{\Delta,n+1} \\ &\stackrel{\mathscr{L}}{=} e^{-\lambda\Delta} Y_{n\Delta} + \int_0^{\lambda\Delta} e^{-s} dZ_s, \end{aligned}$$

where the identity in law holds by the independence and stationarity of increments of the underlying subordinator $\{Z_t : t \geq 0\}$. This implies that $\{\varepsilon_{\Delta,k}\}_{k\in\mathbb{N}}$ is simply a sequence of iid random variables with common distribution $F_\Delta := \mathscr{L}(\int_0^{\lambda\Delta} e^{-s}dZ_s)$. It thus suffices to consider the conditional distribution $\mathscr{L}(Y_\Delta|Y_0)$ of the first increment. Note that by definition, this distribution is infinitely divisible.

Let $w(z)$ be the Lévy density of Z_1 given by (2.11). By the Lévy-integral transform of the characteristic function, we get

$$\begin{aligned}\ln\mathbb{E}\left[e^{iy\varepsilon_{\Delta,1}}\right] &= \int_0^{\lambda\Delta} \ln\mathbb{E}\left[e^{iye^{-s}Z_1}\right] ds \\ &= \int_{\mathbb{R}_+} \left(e^{iyz}-1\right)\left(\int_0^{\lambda\Delta} e^s w\left(e^s z\right) ds\right) dz \\ &=: \int_{\mathbb{R}_+} \left(e^{iyz}-1\right) w_\Delta(z) dz.\end{aligned}$$

Note that $w_\Delta(z)$ indicates the Lévy density of the distribution F_Δ. Observe that for each $z \in \mathbb{R}_+$,

$$\begin{aligned}w_\Delta(z) &= az^{-1-\alpha}\int_0^{\lambda\Delta} (\alpha + be^s z)\, e^{-\alpha s} e^{-be^s z} ds \\ &= az^{-1-\alpha}\left(e^{-bz} - e^{-\alpha\lambda\Delta} e^{-be^{\lambda\Delta}z}\right) \\ &= a\left(1-e^{-\alpha\lambda\Delta}\right) z^{-1-\alpha} e^{-bz} + ae^{-\alpha\lambda\Delta} z^{-1-\alpha}\left(e^{-bz} - e^{-be^{\lambda\Delta}z}\right) \\ &=: w_{\Delta,1}(z) + w_{\Delta,2}(z),\end{aligned}$$

where the second equality holds by $(\partial/\partial s)(-e^{-\alpha s}e^{-be^s z}) = (\alpha + be^s z)e^{-\alpha s}e^{-be^s z}$. Clearly, the function $w_{\Delta,1}(z)$ is the Lévy density of $TS(\alpha, a(1-e^{-\alpha\lambda\Delta}), b)$. Moreover, since

$$\int_{\mathbb{R}_+} w_{\Delta,2}(z) dz = a\left(e^{-\alpha\lambda\Delta} - 1\right)\Gamma(-\alpha) b^\alpha < +\infty,$$

the function $w_{\Delta,2}(z)$ acts as the Lévy density of the compound Poisson component. This completes the proof. □

Let us below describe a direct extension to the bilateral finite variation setting. We omit the proof to avoid overloading the paper with lengthy details of routine nature.

Corollary 3.2. *For each $n \in \mathbb{N}_0$, it holds that given $Y_{n\Delta}$,*

$$Y_{(n+1)\Delta} \stackrel{\mathscr{L}}{=} e^{-\lambda\Delta} Y_{n\Delta} + \eta_0^+ + \sum_{k=1}^{N_\Delta^+} \eta_k^+ + \eta_0^- + \sum_{k=1}^{N_\Delta^-} \eta_k^-,$$

where N_Δ^+, N_Δ^-, η_0^+, η_1^+, $\ldots$, η_0^-, η_1^-, $\ldots$ are mutually independent random variables specified as follows

- $\eta_0^+ \sim TS(\alpha_+, a_+(1-e^{-\alpha_+\lambda\Delta}), b_+)$ *and* $\eta_0^- \sim TS(\alpha_-, a_-(1-e^{-\alpha_-\lambda\Delta}), b_-)$.
- N_Δ^+ *and* N_Δ^- *are Poisson random variables with intensities* $a_+(e^{-\alpha_+\lambda\Delta}-1)\Gamma(-\alpha_+)b_+^{\alpha_+}$ *and* $a_-(e^{-\alpha_-\lambda\Delta}-1)\Gamma(-\alpha_-)b_-^{\alpha_-}$*, respectively.*
- $\{\eta_k^+\}_{k\in\mathbb{N}}$ *and* $\{\eta_k^-\}_{k\in\mathbb{N}}$ *are sequences of iid random variables with common probability densities*

$$v_\Delta^+(x) := \frac{1}{(1-e^{\alpha_+\lambda\Delta})\Gamma(-\alpha_+)b_+^{\alpha_+}} x^{-1-\alpha_+}\left(e^{-b_+x}-e^{-b_+e^{\lambda\Delta}x}\right), \quad x\in\mathbb{R}_+,$$
$$v_\Delta^-(x) := \frac{1}{(1-e^{\alpha_-\lambda\Delta})\Gamma(-\alpha_-)b_-^{\alpha_-}} |x|^{-1-\alpha_-}\left(e^{-b_-|x|}-e^{-b_-e^{\lambda\Delta}|x|}\right), \quad x\in\mathbb{R}_-,$$

respectively.

4 Exact Simulation Using Acceptance-Rejection Methods

Due to the exact transitions of Theorem 3.1 and Corollary 3.2, the exact simulation of random elements involved enables one to simulate exactly the discrete time skeleton $\{Y_{k\Delta}\}_{k\in\mathbb{N}}$ in a recursive manner. The random elements to be generated are the tempered stable random variable η_0 and the random variables N_Δ and $\{\eta_k\}_{k\in\mathbb{N}}$ in the compound Poisson component.

Let us begin with the exact simulation of $\eta_0 \sim TS(\alpha, a(1-e^{-\alpha\lambda\Delta}), b)$ of Theorem 3.1. An efficient exact simulation method for the case $\alpha = 0.5$, that is the inverse Gaussian, is well known due to Michael et al. [12]. For the general case of $\alpha\in(0,1)$, the best route would be the acceptance-rejection method based on the representation (2.3) of the stable distribution and the ratio of the two densities; for each $x\in\mathbb{R}_+$,

$$\frac{f_{TS(\alpha,a,b)}(x)}{f_{S(\alpha,a)}(x)} = e^{-bx-a\Gamma(-\alpha)b^\alpha} \le e^{-a\Gamma(-\alpha)b^\alpha}, \tag{4.1}$$

where the density functions $f_{S(\alpha,a)}(x)$ and $f_{TS(\alpha,a,b)}(x)$ are given respectively by (2.4) and (2.7). The acceptance-rejection method for the generation of the random variable η_0 is then as simple as

Algorithm 1;
Step 1. Generate U as uniform $(0,1)$ and V as $S(\alpha, a(1-e^{-\alpha\lambda\Delta}))$ through (2.3).
Step 2. If $U \le e^{-bV}$, let $\eta_0 \leftarrow V$. Otherwise, return to Step 1.

Clearly, this algorithm works more efficiently when the acceptance rate $e^{a(1-e^{-\alpha\lambda\Delta})\Gamma(-\alpha)b^\alpha}$ at Step 2 is closer to 1. This happens when $b\downarrow 0$ and/or $\Delta\downarrow 0$. The case $b\downarrow 0$ is obvious since then the tempered stable distribution approaches to its stable proposal distribution. In practice, we only have control on the time interval Δ. To account for the case $\Delta\downarrow 0$, we employ the short-range behavior of tempered stable subordinators, which is rigorously proved first by Rosiński [14].

Proposition 4.1. *Let* $\{L_t^{(\mathrm{s})} : t \geq 0\}$ *and* $\{L_t^{(\mathrm{ts})} : t \geq 0\}$ *be Lévy processes respectively with* $S(\alpha, a)$ *and* $TS(\alpha, a, b)$*. It holds that as* $h \downarrow 0$*,* $h > 0$*,*

$$\left\{h^{-1/\alpha} L_{ht}^{(\mathrm{ts})} : t \geq 0\right\} \to \left\{L_t^{(\mathrm{s})} : t \geq 0\right\},$$

where the convergence of random processes holds in the weak sense in the space $\mathbb{D}([0, +\infty); \mathbb{R}_+)$ *of càdlàg functions equipped with the Skorohod topology.*

This convergence result implies that a tempered stable marginal over a very short time is very close to a stable distribution. The acceptance rate is then close to 1 as well. To illustrate this phenomenon, we compare in Table 1 percentiles of a tempered stable marginal and its stable proposal marginal at time $t = 0.1$, 0.01 and 0.001. The percentiles are estimated by Monte Carlo methods based on 3000000 iid replications. The acceptance rates of acceptance-rejection algorithm are respectively 0.7192, 0.9676 and 0.9967. Clearly, the tempered stable distribution tends to the stable proposal distribution as t is smaller. It is worth pointing out an obvious merit of Algorithm 1 in the implementation of the Euler-Maruyama scheme for more general stochastic differential equations driven by a tempered stable subordinator, in which the time increment Δ is often desired to be taken arbitrarily small. (See Baeumer and Meerschaert [1] and Devroye [7] for more details.)

	20%	40%	60%	80%	90%	95%	97%	98%	99%
$L_1^{(\mathrm{s})}$	5.25	6.77	9.29	16.43	31.51	65.17	115.65	186.04	425.94
$h_1^{-1/\alpha} L_{h_1}^{(\mathrm{ts})}$	5.25	6.76	9.27	16.31	30.92	62.44	107.76	167.31	350.53
$h_2^{-1/\alpha} L_{h_2}^{(\mathrm{ts})}$	5.22	6.68	9.05	15.36	27.07	48.73	74.51	103.28	172.17
$h_3^{-1/\alpha} L_{h_3}^{(\mathrm{ts})}$	5.00	6.19	7.89	11.47	16.33	22.76	28.52	33.73	44.02

Table 1: Percentile comparison of scaled marginals $h^{-1/\alpha} L_h^{(\mathrm{ts})}$ of $TS(0.8, 1.0, 0.5)$ and the 0.8-stable proposal distribution $L_1^{(\mathrm{s})}$ for different scales $(h_1, h_2, h_3) = $ (1e-3, 1e-2, 1e-1).

Remark 4.2. Let X_S and X_{TS} be random variables respectively with distributions $S(\alpha, a)$ under the probability measure $\mathbb{Q}$ and $TS(\alpha, a, b)$ under $\mathbb{P}$. It is a straightforward application of Theorem 33.3 of Sato [15] to evaluate an expected value related to tempered stable random variables by the density transform

$$\mathbb{E}_{\mathbb{P}}\left[\Phi(X_{TS})\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_{\mathscr{G}} \Phi(X_S)\right],$$

with $\Phi : \mathbb{R}_+ \to \mathbb{R}$ such that $\mathbb{E}_{\mathbb{P}}[|\Phi(X_{TS})|] < +\infty$. Here, the Radon-Nykodym derivative is given in closed form $(d\mathbb{P}/d\mathbb{Q})|_{\mathscr{G}} = e^{-bX_S}/\mathbb{E}_{\mathbb{Q}}[e^{-bX_S}]$, $\mathbb{Q}$-*a.s.*, where $\mathscr{G}$ is the minimal σ-field generated by the random variable X_S. (This density transform formulation is found useful in the computation of Greeks under an asset price model driven by tempered stable processes. See Kawai and Takeuchi [10] for details.) This method does not require the acceptance-rejection method. In fact, the tempered stable random variable X_{TS} is not even generated in this formulation. However, this is only valid for the evaluation of expectations, and the estimator variance $\mathrm{Var}_{\mathbb{Q}}((d\mathbb{P}/d\mathbb{Q})|_{\mathscr{G}}\Phi(X_S))$ is typically greater than the original one $\mathrm{Var}_{\mathbb{P}}(\Phi(X_{TS}))$, provided that both variances are well defined, due to $\mathbb{E}_{\mathbb{Q}}[e^{-bX_{TS}}] \ll \mathbb{E}_{\mathbb{P}}[e^{-bX_S}]$. Those observations discourage the use of this approach in the Monte Carlo framework. $\square$

We next consider the generation of the random variables N_Δ and $\{\eta_k\}_{k\in\mathbb{N}}$ in the compound Poisson component. We do not consider N_Δ since Poisson random number generator is available in most mathematical tools. Recall that $\{\eta_k\}_{k\in\mathbb{N}}$ has a common probability density $v_\Delta(x)$ given by (3.1). In a similar manner to Lemma 1 of [17], observe that

$$v_\Delta(x) \le \alpha \frac{e^{\lambda\Delta}-1}{e^{\alpha\lambda\Delta}-1}\left(\frac{b^{1-\alpha}}{\Gamma(1-\alpha)}x^{(1-\alpha)-1}e^{-bx}\right) =: C(\Delta)\, g_1(x), \quad x\in\mathbb{R}_+,$$

where $C(\Delta) := \alpha(e^{\lambda\Delta}-1)/(e^{\alpha\lambda\Delta}-1) \ge 1$ and $g_1(x)$ is the density of the gamma distribution $\Gamma(1-\alpha, b)$. Also, it holds that

$$\frac{v_\Delta(x)}{C(\Delta)g_1(x)} = \frac{1-e^{b(1-e^{\lambda\Delta})x}}{bx} =: g_2(x), \quad x\in\mathbb{R}_+.$$

This suggests the following acceptance-rejection algorithm for the simulation of the random variable η_1.

Algorithm 2;
Step 1: Generate U as uniform $(0,1)$ and V as $\Gamma(1-\alpha, b)$.
Step 2: If $U \le g_2(V)$, let $\eta_1 \leftarrow V$. Otherwise, return to Step 1.

The acceptance rate here is $1/C(\Delta)$, which approaches to 1 almost linearly when Δ is close to the origin.

Let us remark here that with this exact simulation method for the tempered stable distribution, we can readily derive an exact simulation algorithm for the multivariate, possibly skewed, normal tempered stable distribution. To the best of our knowledge, the infinite series representation has also been the only simulation tool for the normal tempered stable distribution and process.

Next, we consider the simulation of random elements appearing in Corollary 3.2. In principle, the recipe is almost the same, that is, Algorithm 1 is applicable to η_0^+ and η_0^-, while Algorithm 2 to $\{\eta_k^+\}_{k\in\mathbb{N}}$ and $\{\eta_k^-\}_{k\in\mathbb{N}}$. When $\alpha_+ = \alpha_- =: \alpha$, however, we can go a little further. The proposal distribution for the sum $\eta_0^+ + \eta_0^-$ is then a stable distribution and thus can be simulated as a single stable random variable. Write $\eta := \eta_0^+ + \eta_0^-$, where $\eta_0^\pm \sim TS(\alpha, \widetilde{a}_\pm, b_\pm)$, $\widetilde{a}_\pm := a_\pm(1 - e^{-\alpha\lambda\Delta})$, and moreover $c := e^{-\Gamma(-\alpha)(\widetilde{a}_+ b_+^\alpha + \widetilde{a}_- b_-^\alpha)}$. The density f_η of the random variable η is written as convolution

$$\begin{aligned}
f_\eta(x) &= \int_{\mathbb{R}} f_{TS(\alpha,\widetilde{a}_+,b_+)}(x-y) f_{TS(\alpha,\widetilde{a}_-,b_-)}(-y)dy \\
&= c\int_{-\infty}^{x\wedge 0} e^{-b_+(x-y)} e^{b_- y} f_{S(\alpha,\widetilde{a}_+)}(x-y) f_{S(\alpha,\widetilde{a}_-)}(-y)dy \\
&\le c e^{-b_+ x} e^{(b_+ + b_-)(x\wedge 0)} \int_{-\infty}^{x\wedge 0} f_{S(\alpha,\widetilde{a}_+)}(x-y) f_{S(\alpha,\widetilde{a}_-)}(-y)dy \\
&= c\left(e^{-b_+ x}\mathbb{1}(x>0) + e^{-b_-|x|}\mathbb{1}(x<0)\right)\int_{\mathbb{R}} f_{S(\alpha,\widetilde{a}_+)}(x-y) f_{S(\alpha,\widetilde{a}_-)}(-y)dy \\
&=: c\, g_3(x) \int_{\mathbb{R}} f_{S(\alpha,\widetilde{a}_+)}(x-y) f_{S(\alpha,\widetilde{a}_-)}(-y)dy,
\end{aligned}$$

where the last integral is the density of the (bilateral) stable distribution with no drift, $S(\alpha, \widetilde{a}_+, \widetilde{a}_-)$ say, with Lévy density

$$\frac{\widetilde{a}_+}{z^{1+\alpha}}\mathbb{1}(z>0) + \frac{\widetilde{a}_-}{|z|^{1+\alpha}}\mathbb{1}(z<0).$$

This stable proposal distribution can also be simulated in a similar manner to (2.3). (See Chambers et al. [6] for details.) We can then apply the following acceptance-rejection algorithm, similar to Algorithm 1, to generate η with the bilateral stable proposal distribution $S(\alpha, \widetilde{a}_+, \widetilde{a}_-)$.

Algorithm 3;
Step 1. Generate U as uniform $(0,1)$ and V as $S(\alpha, \widetilde{a}_+, \widetilde{a}_-)$.
Step 2. If $U \le g_3(V)$, let $\eta \leftarrow V$. Otherwise, return to Step 1.

Note that the acceptance rate at Step 2 is $c^{-1} = e^{\Gamma(-\alpha)(1-e^{-\alpha\lambda\Delta})(a_+ b_+^\alpha + a_- b_-^\alpha)}$, which increases to 1 as Δ tends to zero. An important remark here is that the generation of η by Algorithm 3 may not always outperform that of η_+ and η_- through the implementation of Algorithm 1 twice. To describe this, fix $\alpha_\pm = \alpha \in (0,1)$ and $a_\pm = b_\pm = 1$. Also, let $N_1(\lambda\Delta)$ and $N_2(\lambda\Delta)$ be the expected loop numbers required, respectively, for the generation of η_+ and η_- by Algorithm 1 twice and for the generation of η by Algorithm 3 once, where $N_1(s) := 2e^{-(1-e^{-\alpha s})\Gamma(-\alpha)}$ and $N_2(s) := e^{-2(1-e^{-\alpha s})\Gamma(-\alpha)}$. Then, observe that

$$N_1(s) - N_2(s)\begin{cases} \ge 0, & \text{if } s \in \left[0, -\frac{1}{\alpha}\ln\left(1 + \frac{\ln 2}{\Gamma(-\alpha)}\right)\right], \\ < 0, & \text{otherwise.}\end{cases}$$

Hence, Algorithm 3 is of practical use, rather than the implementation of Algorithm 1 twice, only if $\Delta \le -(\lambda\alpha)^{-1}\ln(1 + \ln 2/\Gamma(-\alpha))$. Note that the boundary is strictly decreasing in α and tends to zero as $\alpha \uparrow 1$.

All Algorithm 1, 2 and 3 approach to the perfect acceptance method as $\Delta \downarrow 0$. This indicates that the simulation of each increment, that is $Y_{(k+1)\Delta}$ given $Y_{k\Delta}$ say, can be made as efficient as possible by setting Δ arbitrarily small. However, in simulating an entire sample path of $\{Y_t : t \in [0,T]\}$, a finer division of the interval $[0,T]$ with smaller Δ increases the required computing effort in a proportional manner.

5 Numerical Illustration

We provide in Figure 1 typical sample paths of tempered stable Ornstein-Uhlenbeck processes based on the exact transitions given in Theorem 3.1 and the acceptance-rejection methods described in Algorithm 1 and 2. The model parameters are set $\lambda = 0.5$, $a = b = 1$ and $\alpha = 0.4, 0.6$ and 0.8. For simplicity, we set the initial state $Y_0 = a\Gamma(1-\alpha)b^{\alpha-1}$, that is the mean of the stationary distribution $TS(\alpha,a,b)$. Sample paths are simulated over the time interval either $[0,100]$ or $[0,200]$, where time increments are kept $\Delta = 0.1$ in common. This means that 1000 and 2000 recursive increments are needed, respectively, for intervals $[0,100]$ and $[0,200]$. It is often preferable to take Δ small and T large in the context of asymptotic statistics for discretely observed Ornstein-Uhlenbeck processes. (See, for example, [5] and [9].)

The computing times required for the implementation of 2000 recursive increments by R software are 0.20, 0.25 and 0.34 seconds, respectively, for $\alpha = 0.4, 0.6$ and 0.8. (Computing times can be reduced further by using a low-level language such as C, rather than high-level ones such as R and MATLAB.) In principle, this difference in computing time comes from acceptance rates in Algorithm 1 and 2. In our parameter setting, the acceptance rates in Algorithm 1 (one sample of η_0) are 0.929, 0.896 and 0.800, respectively, while in Algorithm 2 (one sample of η_1), the acceptance rates are, respectively, 0.985, 0.990 and 0.995. Clearly, the acceptance rate of η_0 virtually dominates that of η_1, due to very small means 0.074, 0.109 and 0.225 of the Poisson random variable N_Δ. It is worth mentioning that the sample paths generation tends to work more efficiently when λ, Δ, a and b are chosen smaller.

Finally, let us comment in brief on the existing simulation method based on the infinite series representation (2.12) for comparison. The simulation use of infinite series representations entails a truncation of the infinite summation. We have observed through numerical experiments that under the same parameter setting as above, approximately 4000 summands are needed to obtain sensible sample paths over time interval [0, 200]. (Our observation here is based upon Monte Carlo estimation of the mean $\mathbb{E}[Y_{200}] \approx Y_0$ and the variance $\mathrm{Var}(Y_{200}) \approx a\Gamma(2-\alpha)b^{\alpha-2}$.) Although results are different for different parameter settings and for different criteria and although generalizing solely based on numerical experiments is somewhat risky, it seems fair to claim that our exact simulation algorithm outperforms the approximative method based on series representation, considering many kinds of random sequences to be generated and all the other operations such as taking minimum, sorting the series by $\{T_k\}_{k\in\mathbb{N}}$, counting arrivals $\{\widetilde{\Gamma}_k\}_{k\in\mathbb{N}}$ and monitoring at discrete times in the representation (2.12). In particular, some of those operations may require a tremendous amount of computing time in high-level languages based on matrix operations.

6 Concluding Remarks

In this paper, we have developed an exact yet simple simulation algorithm for a wide class of Ornstein-Uhlenbeck processes of finite variation with a tempered stable stationary distribution based on the exact transition probability between consecutive observations. We have adopted acceptance-rejection methods to simulate tempered stable and compound Poisson distributions, respectively, with stable and gamma proposal distribution. Our algorithm proves applicable to the simulations of bilateral tempered stable Ornstein-Uhlenbeck processes and normal tempered stable processes. The Euler-Maruyama scheme for general stochastic differential equations driven by a tempered stable subordinator is also within our scope. We have also illustrated that our exact simulation algorithm works more efficiently relative to the existing approximative simulation method based on infinite series representation of sample paths.

As future research, it would be interesting to extend to the infinite variation setting. In this case, no practical exact simulation method, such as an acceptance-rejection algorithm, of the tempered stable distribution of infinite variation is known. Moreover, it would still be worthwhile to study infinite series representation of non-Gaussian Ornstein-Uhlenbeck processes, such as (2.8), from a numerical point of view. These topics will be investigated in subsequent papers.

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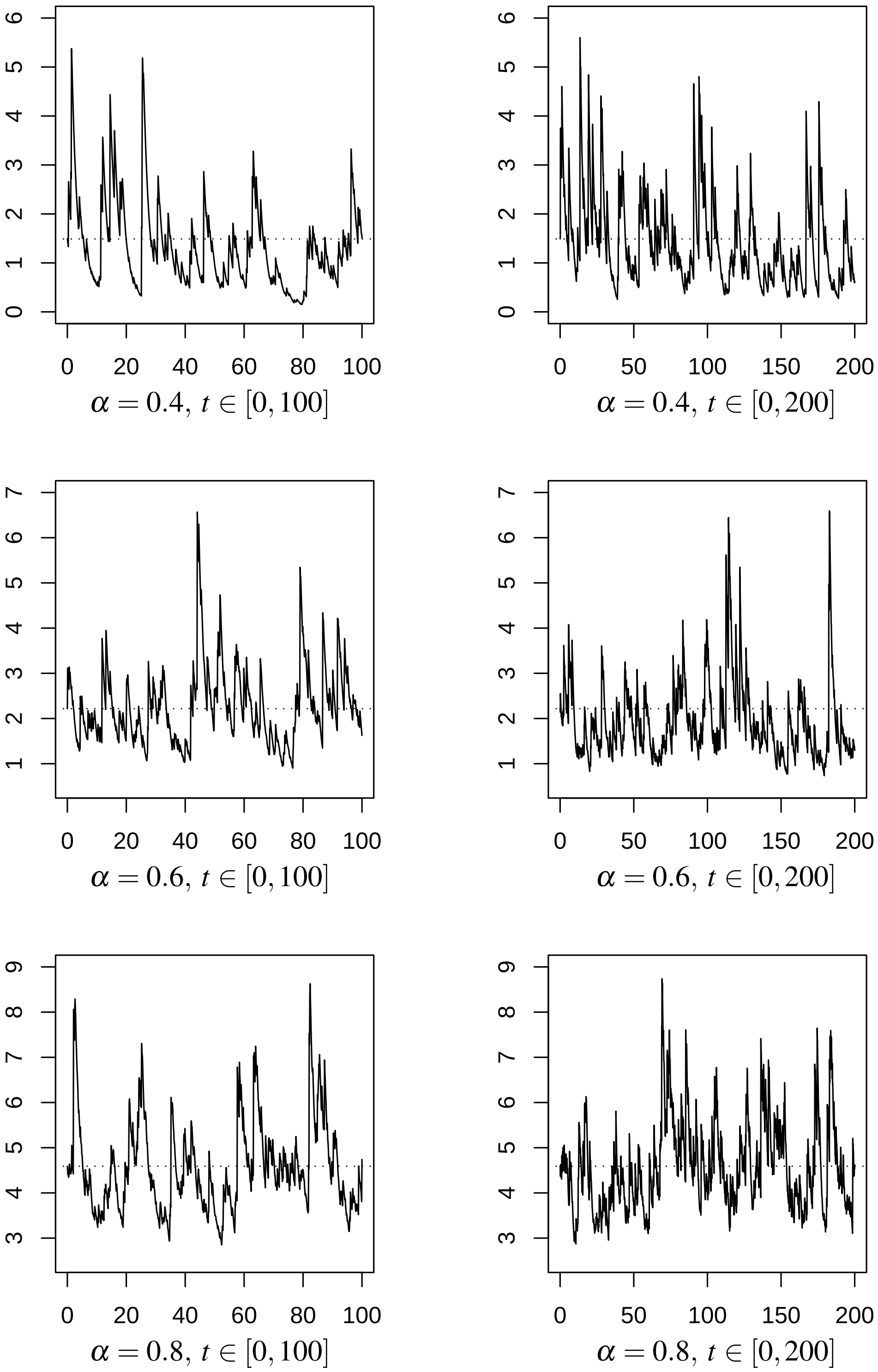

Figure 1: Typical sample paths of tempered stable Ornstein-Uhlenbeck processes through exact simulation algorithm. The model parameters are set $Y_0 = a\Gamma(1-\alpha)b^{\alpha-1}$, $a = b = 1$ and $\lambda = 0.5$. The horizontal dashed lines indicate the initial state Y_0.

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