

MI Preprint Series

Kyushu University
The Global COE Program
Math-for-Industry Education & Research Hub

Motion and Bäcklund transformations of discrete plane curves

Jun-ichi Inoguchi Kenji
Kajiwara, Nozomu Matsuura and
Yasuhiro Ohta

MI 2010-28

(Received August 30, 2010)

Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

MOTION AND BÄCKLUND TRANSFORMATIONS OF DISCRETE PLANE CURVES

Jun-ichi Inoguchi¹, Kenji Kajiwara², Nozomu Matsuura³ and Yasuhiro Ohta⁴

¹: Department of Mathematical Sciences, Yamagata University,
1-4-12 Kojirakawa-machi, Yamagata 990-8560, Japan.
inoguchi@sci.kj.yamagata-u.ac.jp

²: Faculty of Mathematics, Kyushu University, 744 Motoooka, Fukuoka 819-8581, Japan.
kaji@math.kyushu-u.ac.jp

³: Department of Applied Mathematics, Fukuoka University,
Nanakuma, Fukuoka 814-0180, Japan.
nozomu@fukuoka-u.ac.jp

⁴: Department of Mathematics, Kobe University, Rokko, Kobe 657-8501, Japan.
ohta@math.sci.kobe-u.ac.jp

12 August 2010

Abstract

We construct explicit solutions to discrete motion of discrete plane curves which has been introduced by one of the authors recently. Explicit formulas in terms the τ function are presented. Transformation theory of motions of both smooth and discrete curves is developed simultaneously.

2010 Mathematics Subject Classification: 53A04, 37K25, 37K10, 35Q53.

Keywords and Phrases:

discrete curves; discrete motion; discrete potential mKdV equation; discrete integrable systems; τ function; Bäcklund transformation.

1 Introduction

Differential geometry has a close relationship with the theory of integrable systems. In fact, many integrable differential or difference equations arise as compatibility conditions of some geometric objects. For instance, it is well-known that the compatibility condition of pseudospherical surfaces gives rise to the sine-Gordon equation under the Chebyshev net parametrization. For more information on such connections we refer to a monograph [30] by Rogers and Schief.

The above connection between the differential geometry of surfaces and the integrable systems has been already known in the nineteenth century (although the theory of integrable systems has not yet established). However, it is curious that the link between differential geometry of curves and the integrable systems has been noticed rather recently. Actually G. Lamb [22] and Goldstein–Petrich [11] discovered an interesting connection between integrable systems and differential geometry of plane curves. Namely, they found that the *modified Korteweg-de Vries equation* (mKdV equation in short) appears as the compatibility condition of a certain motion of plane curves. Here a motion of curves means an isoperimetric time evolution of arc-length parametrized plane curves. More

precisely, the compatibility condition implies that the curvature function of a motion should satisfy the mKdV equation. As a result, the angle function of the motion satisfies the *potential modified Korteweg-de Vries equation* (potential mKdV equation, in short).

On the other hand, in the theory of integrable systems, discretization of integrable differential equations preserving the integrability has been paid much attention, after the pioneering work of Ablowitz–Ladik [1] and Hirota [12–16]. Later, Date, Jimbo and Miwa developed a unified algebraic approach from the view of so-called the KP theory [5–9, 21, 26]. For other approaches to the discrete integrable systems, see, for example, [27, 31]. Thus one can expect the existence of discretized differential geometric objects governed by the discrete integrable systems. This idea has been realized by the works of Bobenko–Pinkall [3] and Doliwa [10] where the discrete analogue of classical surface theory has been proposed, and it is now actively studied under the name of discrete differential geometry [4].

On the contrary, the discrete analogue of curves has not been studied well in contrast to discrete surfaces. For instance, Hisakado et al proposed a discretization of arc-length parametrized plane curve [19]. They obtained from the compatibility condition of the motion of curves a certain semi-discrete equation (discrete space variable and continuous time variable) which may be considered as a semi-discretization of the mKdV equation. Hoffmann and Kutz [20] considered discretization of the curvature function. By using their discrete curvature function and Möbius geometry, they obtained another semi-discretization of the mKdV equation. However in both works, discretization of time variable of curve motions was not established.

Recently one of the authors of the present paper formulated a full discretization of motion of discrete curves [25], where the discrete potential mKdV equation proposed by Hirota [17] is deduced as the compatibility condition. In the smooth curve theory, the potential function coincides with the angle function of a curve, the primitive function of the curvature. However in the discrete case, the potential function and the angle function become different objects. In this framework, the primal geometric object is the potential function rather than curvature (see [25] and Section 2 of the present paper). Natural and systematic construction of the discrete motion of the curves is expected by using the theory of discrete integrable systems.

The purpose of the paper is to construct explicit solutions to discrete motion of discrete curves by using the theory of τ function. This paper is organized as follows. In Section 2, we prepare fundamental ingredients of plane curve geometry and motions (isoperimetric time evolutions) of plane curves described by the potential mKdV equation. Next we give a brief review of the discrete motion of discrete curves [25]. In Section 3, we shall give a construction of motions for both smooth and discrete curves by the theory of τ function. More precisely we introduce a system of bilinear equations of Hirota type which can be obtained by a certain reduction of the discrete two-dimensional Toda lattice hierarchy [21, 32, 33]. We shall give a representation formula for curve motions in terms of τ function.

One of the central topics in classical differential geometry is the transformation theory of curves and surfaces. The best known example might be the Bäcklund transformations of pseudospherical surfaces. The original Bäcklund transformation was defined as a tangential line congruence satisfying *constant distance property* and *constant normal angle property* (see [30]). In plane curve geometry, Bäcklund transformations on arc-length parametrized plane curves can be defined as arc-length preserving transformations satisfying constant distance property. Such transformations can be extended to transformations on smooth curve motions via the transformation of solutions to the potential mKdV equation. Motivated by this fact, we shall introduce Bäcklund transforma-

tions for discrete motion of discrete curves in Section 4. In particular we shall give another type of Bäcklund transformations on motions of both smooth and discrete curves, which is related to the discrete sine-Gordon equation. In Section 5, we shall construct and exhibit some explicit solutions of curve motions, namely, the multi-soliton and multi-breather solutions. We also present some pictures of discrete motions of discrete curves. We finally give some explicit formulas for the Bäcklund transformations of both smooth and discrete curve motions via the τ functions.

2 Motion of plane curves

Let $\gamma(x)$ be an arc-length parametrized curve in Euclidean plane \mathbb{R}^2 . Then the Frenet equation of γ is

$$\gamma'' = \begin{bmatrix} 0 & -\kappa \\ \kappa & 0 \end{bmatrix} \gamma'. \quad (2.1)$$

Here $'$ denotes the differentiation with respect to x , and the function κ is the *curvature* of γ . Let us consider the following motion in time t , *i.e.*, isoperimetric time evolution:

$$\frac{\partial}{\partial t} \gamma' = \begin{bmatrix} 0 & \kappa'' + \frac{\kappa^3}{2} \\ -\kappa'' - \frac{\kappa^3}{2} & 0 \end{bmatrix} \gamma'. \quad (2.2)$$

Then the *potential function* $\theta(x, t)$ defined by $\kappa = \theta'$ satisfies the *potential mKdV equation* [11, 22]:

$$\theta_t + \frac{1}{2}(\theta_x)^3 + \theta_{xxx} = 0. \quad (2.3)$$

The function θ is called the *angle function* of γ in differential geometry. Note that γ' can be expressed as

$$\gamma' = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}. \quad (2.4)$$

For any nonzero constant λ , the following set of equations

$$\frac{\partial}{\partial x} \left(\frac{\tilde{\theta} + \theta}{2} \right) = 2\lambda \sin \frac{\tilde{\theta} - \theta}{2}, \quad (2.5)$$

$$\frac{\partial}{\partial t} \left(\frac{\tilde{\theta} + \theta}{2} \right) = -\lambda \{(\theta_x)^2 + 8\lambda^2\} \sin \frac{\tilde{\theta} - \theta}{2} + 2\lambda \theta_{xx} \cos \frac{\tilde{\theta} - \theta}{2} + 4\lambda^2 \theta_x, \quad (2.6)$$

defines a solution $\tilde{\theta}$ to the potential mKdV equation [34]. The solution $\tilde{\theta}$ is called a *Bäcklund transform* of θ .

Definition 2.1 A map $\gamma : \mathbb{Z} \rightarrow \mathbb{R}^2$; $n \mapsto \gamma_n$ is said to be a *discrete curve* of segment length a_n if

$$\left| \frac{\gamma_{n+1} - \gamma_n}{a_n} \right| = 1. \quad (2.7)$$

We introduce the *angle function* Ψ_n of a discrete curve γ by

$$\frac{\gamma_{n+1} - \gamma_n}{a_n} = \begin{bmatrix} \cos \Psi_n \\ \sin \Psi_n \end{bmatrix}. \quad (2.8)$$

A discrete curve γ satisfies

$$\frac{\gamma_{n+1} - \gamma_n}{a_n} = R(K_n) \frac{\gamma_n - \gamma_{n-1}}{a_{n-1}}, \quad (2.9)$$

for $K_n = \Psi_n - \Psi_{n-1}$, where $R(K_n)$ denotes the rotation matrix given by

$$R(K_n) = \begin{pmatrix} \cos K_n & -\sin K_n \\ \sin K_n & \cos K_n \end{pmatrix}. \quad (2.10)$$

Now let us recall the following discrete motion of discrete curve $\gamma_n^m : \mathbb{Z}^2 \rightarrow \mathbb{R}^2$ introduced by Matsuura [25]:

$$\left| \frac{\gamma_{n+1}^m - \gamma_n^m}{a_n} \right| = 1, \quad (2.11)$$

$$\frac{\gamma_{n+1}^m - \gamma_n^m}{a_n} = R(K_n^m) \frac{\gamma_n^m - \gamma_{n-1}^m}{a_{n-1}}, \quad (2.12)$$

$$\frac{\gamma_n^{m+1} - \gamma_n^m}{b_m} = R(W_n^m) \frac{\gamma_{n+1}^m - \gamma_n^m}{a_n}, \quad (2.13)$$

where a_n and b_m are arbitrary functions in n and m , respectively. Compatibility of the system (2.11)–(2.13) implies the existence of the *potential function* Θ_n^m defined by

$$W_m^n = \frac{\Theta_n^{m+1} - \Theta_{n+1}^m}{2}, \quad K_n^m = \frac{\Theta_{n+1}^m - \Theta_{n-1}^m}{2}, \quad (2.14)$$

and it follows that Θ_n^m satisfies the *discrete potential mKdV equation* [17]:

$$\tan\left(\frac{\Theta_{n+1}^{m+1} - \Theta_n^m}{4}\right) = \frac{b_m + a_n}{b_m - a_n} \tan\left(\frac{\Theta_n^{m+1} - \Theta_{n+1}^m}{4}\right). \quad (2.15)$$

Note that the angle function Ψ_n^m can be expressed as

$$\Psi_n^m = \frac{\Theta_{n+1}^m + \Theta_n^m}{2}. \quad (2.16)$$

Remark 2.2 The potential discrete mKdV equation (2.15) has been also known as the superposition formula for the modified KdV equation (2.3) [34] and the sine-Gordon equation [2, 30].

3 τ function representation of plane curves

In this section, we give a representation formula for curve motions in terms of τ function.

Let $\tau_n^m = \tau_n^m(x, t; y)$ be a complex valued function dependent on two discrete variables m and n , three continuous variables x , t and y , which satisfies the following system of bilinear equations:

$$\frac{1}{2}D_x D_y \tau_n^m \cdot \tau_n^m = -(\tau_n^{*m})^2, \quad (3.1)$$

$$D_x^2 \tau_n^m \cdot \tau_n^{*m} = 0, \quad (3.2)$$

$$(D_x^3 + D_t) \tau_n^m \cdot \tau_n^{*m} = 0, \quad (3.3)$$

$$D_y \tau_{n+1}^m \cdot \tau_n^m = -a_n \tau_{n+1}^{*m} \tau_n^{*m}, \quad (3.4)$$

$$D_y \tau_n^{m+1} \cdot \tau_n^m = -b_m \tau_{n+1}^{*m} \tau_n^{*m}, \quad (3.5)$$

$$b_m \tau_n^{*m+1} \tau_{n+1}^m - a_n \tau_{n+1}^{*m} \tau_n^{m+1} + (a_n - b_m) \tau_{n+1}^{*m+1} \tau_n^m = 0. \quad (3.6)$$

Here, $*$ denotes the complex conjugate, and D_x , D_y , D_t are Hirota's *bilinear differential operators* (D -operators) defined by

$$D_x^i D_y^j D_t^k f \cdot g = (\partial_x - \partial_{x'})^i (\partial_y - \partial_{y'})^j (\partial_t - \partial_{t'})^k f(x, y, t) g(x', y', t') \Big|_{x=x', y=y', t=t'}. \quad (3.7)$$

For the calculus of the D -operators, we refer to [18]. In general, the functions satisfying the bilinear equations of Hirota type are called the τ functions.

Theorem 3.1 Let τ_n^m be a solution to eqs.(3.1)–(3.6). Define a real function $\Theta_n^m(x, t; y)$ and an \mathbb{R}^2 -valued function $\gamma_n^m(x, t; y)$ by

$$\Theta_n^m(x, t; y) := \frac{2}{\sqrt{-1}} \log \frac{\tau_n^m}{\tau_n^{*m}}, \quad (3.8)$$

$$\gamma_n^m(x, t; y) := \begin{bmatrix} -\frac{1}{2} (\log \tau_n^m \tau_n^{*m})_y \\ \frac{1}{2\sqrt{-1}} \left(\log \frac{\tau_n^m}{\tau_n^{*m}} \right)_y \end{bmatrix}. \quad (3.9)$$

- (1) For any $m, n \in \mathbb{Z}$ and $y \in \mathbb{R}$, the functions $\theta(x, t) = \Theta_n^m(x, t; y)$ and $\gamma(x, t) = \gamma_n^m(x, t; y)$ satisfy eqs.(2.1)–(2.3).
- (2) For any $x, t, y \in \mathbb{R}$, the functions $\Theta_n^m = \Theta_n^m(x, t; y)$ and $\gamma_n^m = \gamma_n^m(x, t; y)$ satisfy eqs.(2.11)–(2.15).

Proof. (1) Express $\gamma_n^m = {}^t(X_n^m, Y_n^m)$. Then by using eq.(3.1) together with its complex conjugate, we have

$$\begin{aligned} (X_n^m)' &= -\frac{1}{2} \log(\tau_n^{*m} \tau_n^m)_{xy} = -\frac{1}{2} \left[\frac{\frac{1}{2} D_x D_y \tau_n^{*m} \cdot \tau_n^{*m}}{(\tau_n^{*m})^2} + \frac{\frac{1}{2} D_x D_y \tau_n^m \cdot \tau_n^m}{(\tau_n^m)^2} \right] \\ &= \frac{1}{2} \left[\left(\frac{\tau_n^m}{\tau_n^{*m}} \right)^2 + \left(\frac{\tau_n^{*m}}{\tau_n^m} \right)^2 \right] = \cos \Theta_n^m. \end{aligned}$$

Similarly we obtain $(Y_n^m)' = \sin \Theta_n^m$. Differentiating $(\gamma_n^m)' = {}^t(\cos \Theta_n^m, \sin \Theta_n^m)$ by x and noticing that $\kappa = \Theta'$, we obtain eq.(2.1):

$$(\gamma_n^m)'' = (\Theta_n^m)' \begin{pmatrix} -\sin \Theta_n^m \\ \cos \Theta_n^m \end{pmatrix} = \begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix} (\gamma_n^m)'.$$

On the other hand, differentiating $(\gamma_n^m)'$ by t , we have

$$(\gamma_n^m)'_t = (\Theta_n^m)_t \begin{pmatrix} -\sin \Theta_n^m \\ \cos \Theta_n^m \end{pmatrix} = (\Theta_n^m)_t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\gamma_n^m)'.$$

By using the bilinear equations (3.2) and (3.3), $(\Theta_n^m)_t$ can be rewritten as

$$\begin{aligned} (\Theta_n^m)_t &= \frac{2}{\sqrt{-1}} \frac{D_t \tau_n^m \cdot \tau_n^{*m}}{\tau_n^m \tau_n^{*m}} = -\frac{2}{\sqrt{-1}} \frac{D_x^3 \tau_n^m \cdot \tau_n^{*m}}{\tau_n^m \tau_n^{*m}} \\ &= -\frac{2}{\sqrt{-1}} \left[\left(\log \frac{\tau_n^m}{\tau_n^{*m}} \right)_{xxx} + 3 \left(\log \frac{\tau_n^m}{\tau_n^{*m}} \right)_x (\log \tau_n^m \tau_n^{*m})_{xx} + \left\{ \left(\log \frac{\tau_n^m}{\tau_n^{*m}} \right)_x \right\}^3 \right] \\ &= -\frac{2}{\sqrt{-1}} \left[\left(\log \frac{\tau_n^m}{\tau_n^{*m}} \right)_{xxx} - 2 \left\{ \left(\log \frac{\tau_n^m}{\tau_n^{*m}} \right)_x \right\}^3 \right] = -\kappa_{xx} - \frac{\kappa^3}{2} \end{aligned} \quad (3.10)$$

which yields eq.(2.2). Here we have used the relation

$$0 = \frac{D_x^2 \tau_n^m \cdot \tau_n^{*m}}{\tau_n^m \tau_n^{*m}} = (\log \tau_n^m \tau_n^{*m})_{xx} + \left(\log \frac{\tau_n^m}{\tau_n^{*m}} \right)_x^2$$

which is a consequence of eq.(3.2). The potential mKdV equation (2.3) follows immediately from eq.(3.10) by noticing that $\kappa = \Theta'$.

(2) From eq. (3.4) and its complex conjugate we have

$$\left(\log \frac{\tau_{n+1}^m}{\tau_n^m} \right)_y = -a_n \frac{\tau_{n+1}^{*m} \tau_n^{*m}}{\tau_{n+1}^m \tau_n^m}, \quad \left(\log \frac{\tau_{n+1}^{*m}}{\tau_n^{*m}} \right)_y = -a_n \frac{\tau_{n+1}^m \tau_n^m}{\tau_{n+1}^{*m} \tau_n^{*m}}. \quad (3.11)$$

Adding these two equations we obtain

$$(\log \tau_{n+1}^m \tau_{n+1}^{*m})_y - (\log \tau_n^m \tau_n^{*m})_y = -a_n \left(\frac{\tau_{n+1}^{*m} \tau_n^{*m}}{\tau_{n+1}^m \tau_n^m} + \frac{\tau_{n+1}^m \tau_n^m}{\tau_{n+1}^{*m} \tau_n^{*m}} \right), \quad (3.12)$$

which yields

$$\frac{X_{n+1}^m - X_n^m}{a_n} = \cos \Psi_n^m, \quad \Psi_n^m = \frac{1}{\sqrt{-1}} \log \left(\frac{\tau_{n+1}^m \tau_n^m}{\tau_{n+1}^{*m} \tau_n^{*m}} \right) = \frac{\Theta_{n+1}^m + \Theta_n^m}{2}. \quad (3.13)$$

Subtracting the second equation from the first equation in eq.(3.11) we have

$$\frac{Y_{n+1}^m - Y_n^m}{a_n} = \sin \Psi_n^m.$$

Therefore we obtain

$$\frac{\gamma_{n+1}^m - \gamma_n^m}{a_n} = \begin{pmatrix} \cos \Psi_n^m \\ \sin \Psi_n^m \end{pmatrix}. \quad (3.14)$$

which gives eq.(2.11). Next, from eq.(3.14) we see that

$$\frac{\gamma_{n+1}^m - \gamma_n^m}{a_n} = R(\Psi_n^m - \Psi_{n-1}^m) \frac{\gamma_n^m - \gamma_{n-1}^m}{a_{n-1}}, \quad \Psi_n^m - \Psi_{n-1}^m = \frac{\Theta_{n+1}^m - \Theta_{n-1}^m}{2} = K_n^m, \quad (3.15)$$

which is nothing but eq.(2.12). Similarly, starting from eq.(3.5) and its complex conjugate we obtain

$$\frac{\gamma_n^{m+1} - \gamma_n^m}{b_m} = \begin{pmatrix} \cos \Phi_n^m \\ \sin \Phi_n^m \end{pmatrix}, \quad \Phi_n^m = \frac{1}{\sqrt{-1}} \log \left(\frac{\tau_n^{m+1} \tau_n^m}{\tau_n^{*m+1} \tau_n^{*m}} \right) = \frac{\Theta_n^{m+1} + \Theta_n^m}{2}, \quad (3.16)$$

which yields

$$\frac{\gamma_n^{m+1} - \gamma_n^m}{b_m} = R(\Phi_n^m - \Psi_n^m) \frac{\gamma_{n+1}^m - \gamma_n^m}{a_n}, \quad \Phi_n^m - \Psi_n^m = \frac{\Theta_n^{m+1} - \Theta_{n+1}^m}{2} = W_n^m. \quad (3.17)$$

This is equivalent to eq.(2.13).

Finally let us derive the discrete potential mKdV equation (2.15). Dividing eq.(3.6) and its complex conjugate by $\tau_{n+1}^{*m} \tau_n^{*m+1}$ we have

$$\begin{aligned} b_m \exp \left(\frac{\sqrt{-1} \Theta_{n+1}^m}{2} \right) - a_n \exp \left(\frac{\sqrt{-1} \Theta_n^{m+1}}{2} \right) &= -(a_n - b_m) \frac{\tau_{n+1}^{*m+1} \tau_n^m}{\tau_n^{*m+1} \tau_{n+1}^{*m}}, \\ b_m \exp \left(\frac{\sqrt{-1} \Theta_n^{m+1}}{2} \right) - a_n \exp \left(\frac{\sqrt{-1} \Theta_{n+1}^m}{2} \right) &= -(a_n - b_m) \frac{\tau_{n+1}^{m+1} \tau_n^{*m}}{\tau_n^{*m+1} \tau_{n+1}^{*m}}, \end{aligned} \quad (3.18)$$

respectively. Dividing these two equations we obtain

$$\frac{b_m \exp \left(\frac{\sqrt{-1} \Theta_{n+1}^m}{2} \right) - a_n \exp \left(\frac{\sqrt{-1} \Theta_n^{m+1}}{2} \right)}{b_m \exp \left(\frac{\sqrt{-1} \Theta_n^{m+1}}{2} \right) - a_n \exp \left(\frac{\sqrt{-1} \Theta_{n+1}^m}{2} \right)} = \exp \left[-\frac{\sqrt{-1} (\Theta_{n+1}^{m+1} - \Theta_n^m)}{2} \right], \quad (3.19)$$

which is easily verified to be equivalent to eq.(2.15). Thus we have completed the proof of Theorem 3.1. \square

Corollary 3.2 (Representation Formula)

γ_n^m can be expressed in terms of the potential function Θ_n^m as follows:

$$\gamma_n^m(x, t; y) = \begin{bmatrix} \int^x \cos \Theta_n^m(x', t; y) dx' \\ \int^x \sin \Theta_n^m(x', t; y) dx' \end{bmatrix} = \begin{bmatrix} \sum_{n'}^{n-1} a_{n'} \cos \left(\frac{\Theta_{n'}^m(x, t; y) + \Theta_{n'+1}^m(x, t; y)}{2} \right) \\ \sum_{n'}^{n-1} a_{n'} \sin \left(\frac{\Theta_{n'}^m(x, t; y) + \Theta_{n'+1}^m(x, t; y)}{2} \right) \end{bmatrix}. \quad (3.20)$$

Proof. The first equation is a consequence of

$$\frac{\partial}{\partial x} \gamma_n^m(x, t; y) = \begin{bmatrix} \cos \Theta_n^m(x, t; y) \\ \sin \Theta_n^m(x, t; y) \end{bmatrix}, \quad (3.21)$$

and the second equation follows from eq.(3.14). \square

It should be noted here that the bilinear equations (3.1)–(3.6) are derived from the reduction of the following equations,

$$\frac{1}{2}D_x D_y \tau_n^m(s) \cdot \tau_n^m(s) = -\tau_n^m(s+1)\tau_n^m(s-1), \quad (3.22)$$

$$(D_x^2 - D_z) \tau_n^m(s+1) \cdot \tau_n^m(s) = 0, \quad (3.23)$$

$$(D_x^3 + D_t + 3D_x D_z) \tau_n^m(s+1) \cdot \tau_n^m(s) = 0, \quad (3.24)$$

$$D_y \tau_{n+1}^m(s) \cdot \tau_n^m(s) = -a_n \tau_{n+1}^m(s+1)\tau_n^m(s-1), \quad (3.25)$$

$$D_y \tau_n^{m+1}(s) \cdot \tau_n^m(s) = -b_m \tau_{n+1}^m(s+1)\tau_n^m(s-1), \quad (3.26)$$

$$b_m \tau_n^{m+1}(s+1)\tau_{n+1}^m(s) - a_n \tau_{n+1}^m(s+1)\tau_n^{m+1}(s) + (a_n - b_m)\tau_{n+1}^{m+1}(s+1)\tau_n^m(s) = 0, \quad (3.27)$$

for $\tau_n^m(s) = \tau_n^m(x, z, t; y; s)$, which are included in the discrete two-dimensional Toda lattice hierarchy [21, 32, 33]. In fact, imposing the condition

$$\frac{\partial}{\partial z} \tau_n^m(s) = B \tau_n^m(s), \quad \tau_n^m(s+1) = C \tau_n^{*m}(s), \quad B, C \in \mathbb{R}, \quad (3.28)$$

and denoting $\tau_n^m = \tau_n^m(0)$, then eqs.(3.22)–(3.27) yield eqs.(3.1)–(3.6), respectively.

4 Bäcklund transformations

We start with the following fundamental fact on plane curves.

Proposition 4.1 *Let $\gamma(x)$ be an arc-length parametrized curve with angle function $\theta(x)$. Take a nonzero constant λ and a solution $\tilde{\theta}(x)$ to*

$$\left(\frac{\tilde{\theta} + \theta}{2} \right)' = 2\lambda \sin \frac{\tilde{\theta} - \theta}{2}. \quad (4.1)$$

Then

$$\tilde{\gamma}(x) = \gamma(x) + \frac{1}{\lambda} R \left(\frac{\tilde{\theta}(x) - \theta(x)}{2} \right) \gamma'(x) \quad (4.2)$$

is an arc-length parametrized curve with angle function $\tilde{\theta}(x)$. In other words, if $\gamma(x)$ is a solution to eq.(2.1), then $\tilde{\gamma}(x)$ is another solution to eq.(2.1) with $\tilde{\kappa}(x) = \tilde{\theta}'(x)$. The curve $\tilde{\gamma}$ is called a Bäcklund transform of γ .

Proposition 4.1 can be verified easily by direct computation. We next extend the Bäcklund transformation to those of motion of curve.

Proposition 4.2 *Let $\gamma(x, t)$ be a motion of arc-length parametrized curve determined by eqs.(2.2) and (2.3). Take a Bäcklund transform $\tilde{\theta}(x, t)$ defined by eqs.(2.5) and (2.6) of $\theta(x, t)$. Then*

$$\tilde{\gamma}(x, t) = \gamma(x, t) + \frac{1}{\lambda} R \left(\frac{\tilde{\theta}(x, t) - \theta(x, t)}{2} \right) \gamma'(x, t) \quad (4.3)$$

is a motion of arc-length parametrized curve with the angle function $\tilde{\theta}(x, t)$.

Proof. By the preceding Proposition, $\tilde{\gamma}$ satisfies the isoperimetric condition $|\tilde{\gamma}'| \equiv 1$. Computing the t -derivative of $\tilde{\gamma}$ by using (2.6), we can show that $\tilde{\gamma}$ satisfies eq.(2.2) with $\tilde{\kappa} = \tilde{\theta}'$ \square

Now we introduce a Bäcklund transformation of discrete curve.

Proposition 4.3 *Let γ_n be a discrete curve of segment length a_n . Let Θ_n be the potential function defined by*

$$\frac{\gamma_{n+1} - \gamma_n}{a_n} = \begin{bmatrix} \cos \Psi_n \\ \sin \Psi_n \end{bmatrix}, \quad \Psi_n = \frac{\Theta_{n+1} + \Theta_n}{2}. \quad (4.4)$$

For a nonzero constant λ , take a solution $\tilde{\Theta}_n$ to the following equation

$$\tan\left(\frac{\tilde{\Theta}_{n+1} - \Theta_n}{4}\right) = \frac{\frac{1}{\lambda} + a_n}{\frac{1}{\lambda} - a_n} \tan\left(\frac{\tilde{\Theta}_n - \Theta_{n+1}}{4}\right), \quad (4.5)$$

then

$$\tilde{\gamma}_n = \gamma_n + \frac{1}{\lambda} R\left(\frac{\tilde{\Theta}_n - \tilde{\Theta}_{n+1}}{2}\right) \frac{\gamma_{n+1} - \gamma_n}{a_n} \quad (4.6)$$

is a discrete curve with the potential function $\tilde{\Theta}_n$.

Proof. It suffices to show that

$$\frac{\tilde{\gamma}_{n+1} - \tilde{\gamma}_n}{a_n} = \begin{bmatrix} \cos \tilde{\Psi}_n \\ \sin \tilde{\Psi}_n \end{bmatrix}, \quad \tilde{\Psi}_n = \frac{\tilde{\Theta}_{n+1} + \tilde{\Theta}_n}{2} \quad (4.7)$$

for $\tilde{\gamma}_n$ defined by eq.(4.6). This follows from eqs.(4.4) and (4.5). \square

We next extend the Bäcklund transformation to those of motion of discrete curve. In order to do so, we first present the Bäcklund transformation to the discrete potential mKdV equation:

Lemma 4.4 *Let Θ_n^m be a solution to the discrete potential mKdV equation (2.15). A function $\tilde{\Theta}_n^m$ satisfying the following system of equations*

$$\tan\left(\frac{\tilde{\Theta}_{n+1}^m - \Theta_n^m}{4}\right) = \frac{\frac{1}{\lambda} + a_n}{\frac{1}{\lambda} - a_n} \tan\left(\frac{\tilde{\Theta}_n^m - \Theta_{n+1}^m}{4}\right), \quad (4.8)$$

$$\tan\left(\frac{\tilde{\Theta}_n^{m+1} - \Theta_n^m}{4}\right) = \frac{\frac{1}{\lambda} + b_m}{\frac{1}{\lambda} - b_m} \tan\left(\frac{\tilde{\Theta}_n^m - \Theta_n^{m+1}}{4}\right), \quad (4.9)$$

gives another solution to eq.(2.15). We call $\tilde{\Theta}_n^m$ a Bäcklund transform of Θ_n^m .

Proof. First note that eq.(2.15) is equivalent to

$$e^{U_n^{m+1} + U_n^m} - e^{U_{n+1}^{m+1} + U_{n+1}^m} = \frac{a_n}{b_m} \left(e^{U_{n+1}^m + U_n^m} - e^{U_{n+1}^{m+1} + U_n^{m+1}} \right), \quad (4.10)$$

where we put $\frac{\sqrt{-1}\Theta_n^m}{2} = U_n^m$ for notational simplicity. Similarly, eqs.(4.8) and (4.9) are rewritten as

$$e^{\tilde{U}_n^m + U_n^m} - e^{\tilde{U}_{n+1}^m + U_{n+1}^m} = \lambda a_n \left(e^{U_{n+1}^m + U_n^m} - e^{\tilde{U}_{n+1}^m + \tilde{U}_n^m} \right), \quad (4.11)$$

$$e^{\tilde{U}_n^m + U_n^m} - e^{\tilde{U}_n^{m+1} + U_n^{m+1}} = \lambda b_m \left(e^{U_n^{m+1} + U_n^m} - e^{\tilde{U}_n^{m+1} + \tilde{U}_n^m} \right), \quad (4.12)$$

respectively, where $\frac{\sqrt{-1}\tilde{\Theta}_n^m}{2} = \tilde{U}_n^m$. Subtracting eq.(4.12) from eq.(4.11), we have

$$e^{\tilde{U}_n^{m+1}+U_n^{m+1}} - e^{\tilde{U}_{n+1}^m+U_{n+1}^m} = \lambda \left(a_n e^{U_{n+1}^m+U_n^m} - b_m e^{U_n^{m+1}+U_n^m} \right) - \lambda \left(a_n e^{\tilde{U}_{n+1}^m+\tilde{U}_n^m} - b_m e^{\tilde{U}_n^{m+1}+\tilde{U}_n^m} \right). \quad (4.13)$$

Similarly, subtracting eq.(4.12) _{$n \rightarrow n+1$} from eq.(4.11) _{$m \rightarrow m+1$} , we get

$$e^{\tilde{U}_n^{m+1}+U_n^{m+1}} - e^{\tilde{U}_{n+1}^m+U_{n+1}^m} = \lambda \left(a_n e^{U_{n+1}^{m+1}+U_n^{m+1}} - b_m e^{U_{n+1}^m+U_n^m} \right) - \lambda \left(a_n e^{\tilde{U}_{n+1}^{m+1}+\tilde{U}_n^{m+1}} - b_m e^{\tilde{U}_{n+1}^m+\tilde{U}_n^m} \right). \quad (4.14)$$

Subtracting eq. (4.14) from eq.(4.13) yields

$$\begin{aligned} & a_n \left(e^{\tilde{U}_{n+1}^m+\tilde{U}_n^m} - e^{\tilde{U}_{n+1}^{m+1}+\tilde{U}_n^{m+1}} \right) - b_m \left(e^{\tilde{U}_n^{m+1}+\tilde{U}_n^m} - e^{\tilde{U}_{n+1}^{m+1}+\tilde{U}_{n+1}^m} \right) \\ & = a_n \left(e^{U_{n+1}^m+U_n^m} - e^{U_{n+1}^{m+1}+U_n^{m+1}} \right) - b_m \left(e^{U_n^{m+1}+U_n^m} - e^{U_{n+1}^{m+1}+U_{n+1}^m} \right). \end{aligned} \quad (4.15)$$

Now we see that the right hand side of eq.(4.15) vanishes since it is equivalent to eq.(4.10). Then the left hand side gives eq.(2.15) for $\tilde{\Theta}_n^m$. \square

Proposition 4.5 *Let γ_n^m be a discrete motion of discrete curve. Take a Bäcklund transform $\tilde{\Theta}_n^m$ of Θ_n^m defined in Lemma 4.4. Then*

$$\tilde{\gamma}_n^m = \gamma_n^m + \frac{1}{\lambda} R \left(\frac{\tilde{\Theta}_n^m - \Theta_{n+1}^m}{2} \right) \frac{\gamma_{n+1}^m - \gamma_n^m}{a_n} \quad (4.16)$$

is a discrete motion of discrete curve with potential function $\tilde{\Theta}_n^m$. We call $\tilde{\gamma}_n^m$ a Bäcklund transform of γ_n^m .

Proof. It suffices to show that $\tilde{\gamma}_n^m$ satisfies eqs.(2.11)–(2.13) with potential function $\tilde{\Theta}_n^m$, but eqs.(2.11) and (2.12) follow from Proposition 4.3 immediately. Noticing the symmetry in n and m , similar calculations to those in Proposition 4.3 yield

$$\frac{\tilde{\gamma}_n^{m+1} - \tilde{\gamma}_n^m}{b_m} = \begin{bmatrix} \cos \left(\frac{\tilde{\Theta}_n^{m+1} + \tilde{\Theta}_n^m}{2} \right) \\ \sin \left(\frac{\tilde{\Theta}_n^{m+1} + \tilde{\Theta}_n^m}{2} \right) \end{bmatrix} \quad (4.17)$$

by using eq.(4.9). Comparing eqs.(4.7) and (4.17) we obtain

$$\frac{\tilde{\gamma}_n^{m+1} - \tilde{\gamma}_n^m}{b_m} = R \left(\frac{\tilde{\Theta}_n^{m+1} - \tilde{\Theta}_{n+1}^m}{2} \right) \frac{\tilde{\gamma}_{n+1}^m - \tilde{\gamma}_n^m}{a_n}, \quad (4.18)$$

which implies eq.(2.13). \square

It is possible to construct another type of Bäcklund transformations for motions of both smooth and discrete curves by using the symmetry of the potential modified KdV equation (2.3) and discrete potential modified KdV equation (2.15). In fact, if $\theta(x, t)$ is a solution to eq.(2.3), then $-\theta(x, t)$ satisfies the same equation. Combining this symmetry and the Bäcklund transformation defined by eqs.(2.5) and (2.6), we have the following Bäcklund transformation:

Lemma 4.6 Let $\theta(x, t)$ be a solution to the potential modified KdV equation (2.3). For any nonzero constant λ , a function $\bar{\theta}(x, t)$ satisfying the following set of equations

$$\frac{\partial}{\partial x} \left(\frac{\bar{\theta} - \theta}{2} \right) = 2\lambda \sin \frac{\bar{\theta} + \theta}{2}, \quad (4.19)$$

$$\frac{\partial}{\partial t} \left(\frac{\bar{\theta} - \theta}{2} \right) = -\lambda \{(\theta_x)^2 + 8\lambda^2\} \sin \frac{\bar{\theta} + \theta}{2} - 2\lambda \theta_{xx} \cos \frac{\bar{\theta} + \theta}{2} - 4\lambda^2 \theta_x, \quad (4.20)$$

gives another solution to eq.(2.3).

Lemma 4.6 immediately yields the following Bäcklund transformation for $\gamma(x, t)$:

Proposition 4.7 Let $\gamma(x, t)$ be a motion of arc-length parametrized curve determined by eqs.(2.2) and (2.3). Take a Bäcklund transform $\bar{\theta}(x, t)$ of $\theta(x, t)$ defined in Lemma 4.6. Then

$$\bar{\gamma}(x, t) = S \left[\gamma(x, t) + \frac{1}{\lambda} R \left(-\frac{\bar{\theta}(x, t) + \theta(x, t)}{2} \right) \gamma'(x, t) \right], \quad S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (4.21)$$

is a motion of arc-length parametrized curve with angle function $\bar{\theta}(x, t)$.

Note that eqs.(4.19) and (4.20) can be derived from eqs.(2.5) and (2.6) simply by putting $\bar{\theta}(x, t) = -\theta(x, t)$. Moreover, noticing eq.(2.4) and Proposition 4.2, we have

$$\gamma(x, t) + \frac{1}{\lambda} R \left(-\frac{\bar{\theta}(x, t) + \theta(x, t)}{2} \right) \gamma'(x, t) = \begin{bmatrix} \cos(-\bar{\theta}(x, t)) \\ \sin(-\bar{\theta}(x, t)) \end{bmatrix} = \begin{bmatrix} \cos \bar{\theta}(x, t) \\ -\sin \bar{\theta}(x, t) \end{bmatrix}, \quad (4.22)$$

which implies Proposition 4.7.

Similarly, if Θ_n^m is a solution to eq.(2.15), then $-\Theta_n^m$ satisfies the same equation. Therefore Lemma 4.4 and Proposition 4.5 lead to the following Bäcklund transformations:

Lemma 4.8 Let Θ_n^m be a solution to the discrete potential mKdV equation (2.15). A function $\bar{\Theta}_n^m$ satisfying the following system of equations

$$\tan \left(\frac{\bar{\Theta}_{n+1}^m + \Theta_n^m}{4} \right) = \frac{\frac{1}{\lambda} + a_n}{\frac{1}{\lambda} - a_n} \tan \left(\frac{\bar{\Theta}_n^m + \Theta_{n+1}^m}{4} \right), \quad (4.23)$$

$$\tan \left(\frac{\bar{\Theta}_n^{m+1} + \Theta_n^m}{4} \right) = \frac{\frac{1}{\lambda} + b_m}{\frac{1}{\lambda} - b_m} \tan \left(\frac{\bar{\Theta}_n^m + \Theta_n^{m+1}}{4} \right), \quad (4.24)$$

gives another solution to eq.(2.15).

Proposition 4.9 Let γ_n^m be a discrete motion of discrete curve. Take a Bäcklund transform $\bar{\Theta}_n^m$ of Θ_n^m defined in Lemma 4.8. Then

$$\bar{\gamma}_n^m = S \left[\gamma_n^m + \frac{1}{\lambda} R \left(-\frac{\bar{\Theta}_n^m + \Theta_{n+1}^m}{2} \right) \frac{\gamma_{n+1}^m - \gamma_n^m}{a_n} \right], \quad S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (4.25)$$

is a discrete motion of discrete curve with potential function $\bar{\Theta}_n^m$.

Remark 4.10

(1) It may be interesting to point out that eq.(4.23) and eq.(4.24) are rewritten as

$$\sin\left(\frac{\bar{\Theta}_{n+1}^m - \Theta_{n+1}^m - \bar{\Theta}_n^m + \Theta_n^m}{4}\right) = \lambda a_n \sin\left(\frac{\bar{\Theta}_{n+1}^m + \Theta_{n+1}^m + \bar{\Theta}_n^m + \Theta_n^m}{4}\right), \quad (4.26)$$

$$\sin\left(\frac{\bar{\Theta}_n^{m+1} - \Theta_n^{m+1} - \bar{\Theta}_n^m + \Theta_n^m}{4}\right) = \lambda b_m \sin\left(\frac{\bar{\Theta}_n^{m+1} + \Theta_n^{m+1} + \bar{\Theta}_n^m + \Theta_n^m}{4}\right), \quad (4.27)$$

respectively, which are essentially equivalent to the discrete sine-Gordon equation [14].

(2) The Bäcklund transformations described in Propositions 4.2 and 4.5 satisfy “constant distance property”, *i.e.*, $|\bar{\gamma} - \gamma| \equiv 1/\lambda$ or $|\bar{\gamma}_n^m - \gamma_n^m| \equiv 1/\lambda$. These transformations may be regarded as one-dimensional analogue of the original Bäcklund transformations of the pseudospherical surface [30]. On the other hand, the Bäcklund transformations proposed in Propositions 4.7 and 4.9 are characterized by the property $|\bar{\gamma} - S\gamma| = 1/\lambda$.

5 Explicit Solutions

5.1 Solitons and Breathers

For $N \in \mathbb{Z}_{\geq 0}$ we define a function $\tau_n^m(s) = \tau_n^m(x, t; y, z; s)$ by

$$\tau_n^m(s) = \exp\left[-\left(x + \sum_{n'}^{n-1} a_{n'} + \sum_{m'}^{m-1} b_{m'}\right)y\right] \det(f_{s+j-1}^{(i)})_{i,j=1,\dots,N}, \quad (5.1)$$

for $(x, t; y, z) \in \mathbb{R}^4$ and $(m, n, s) \in \mathbb{Z}^3$. Here $f_s^{(i)} = f_s^{(i)}(x, t; y, z; m, n)$ ($i = 1, \dots, N$) satisfies the following linear equations

$$\frac{\partial f_s^{(i)}}{\partial x} = f_{s+1}^{(i)}, \quad \frac{\partial f_s^{(i)}}{\partial z} = f_{s+2}^{(i)}, \quad \frac{\partial f_s^{(i)}}{\partial t} = -4f_{s+3}^{(i)}, \quad \frac{\partial f_s^{(i)}}{\partial y} = f_{s-1}^{(i)}, \quad (5.2)$$

$$\frac{f_s^{(i)}(m, n) - f_s(m, n-1)}{a_n} = f_{s+1}^{(i)}(m, n), \quad \frac{f_s^{(i)}(m, n) - f_s(m-1, n)}{b_m} = f_{s+1}^{(i)}(m, n). \quad (5.3)$$

For $N = 0$, we set $\det(f_{s+j-1}^{(i)})_{i,j=1,\dots,N} = 1$. A typical example for $f_s^{(i)}$ is given by

$$f_s^{(i)} = e^{\eta_i} + e^{\mu_i}, \quad (5.4)$$

$$\begin{cases} e^{\eta_i} = \alpha_i p_i^s \prod_{n'}^{n-1} (1 - a_{n'} p_i)^{-1} \prod_{m'}^{m-1} (1 - b_{m'} p_i)^{-1} e^{p_i x + p_i^2 z - 4p_i^3 t + \frac{1}{p_i} y}, \\ e^{\mu_i} = \beta_i q_i^s \prod_{n'}^{n-1} (1 - a_{n'} q_i)^{-1} \prod_{m'}^{m-1} (1 - b_{m'} q_i)^{-1} e^{q_i x + q_i^2 z - 4q_i^3 t + \frac{1}{q_i} y}, \end{cases} \quad (5.5)$$

where p_i, q_i, α_i and β_i are arbitrary complex constants.

We note that τ_n^m and $f_s^{(i)}$ are functions of continuous variables x, y, z, t and discrete variables m, n, s , but we will indicate only the relevant variables according to the context, for notational simplicity. Then it is well-known that $\tau_n^m(s)$ satisfies the bilinear equations(3.22)–(3.27) [18, 21, 23, 24, 28, 29, 33]. Actually by using the linear relations (5.2) and (5.3), eqs.(3.22)–(3.27) are reduced to the Plücker relations which are quadratic identities of determinants.

It is possible to construct the solutions to the bilinear equations (3.1)–(3.6) by imposing the reduction condition (3.28) on $\tau_n^m(s)$ in eq.(5.1). Those conditions are realized by putting restrictions on parameters of solutions. As an example, we present the multi-soliton and multi-breather solutions:

Proposition 5.1 *Consider the τ function*

$$\tau_n^m = \exp \left[- \left(x + \sum_{n'}^{n-1} a_{n'} + \sum_{m'}^{m-1} b_{m'} \right) y \right] \det \left(f_{j-1}^{(i)} \right)_{i,j=1,\dots,N}, \quad (5.6)$$

$$f_s^{(i)} = e^{\eta_i} + e^{\mu_i}, \quad (5.7)$$

$$\begin{cases} e^{\eta_i} = \alpha_i p_i^s \prod_{n'}^{n-1} (1 - a_{n'} p_i)^{-1} \prod_{m'}^{m-1} (1 - b_{m'} p_i)^{-1} e^{p_i x - 4 p_i^3 t + \frac{1}{p_i} y}, \\ e^{\mu_j} = \beta_j (-p_j)^s \prod_{n'}^{n-1} (1 + a_{n'} p_j)^{-1} \prod_{m'}^{m-1} (1 + b_{m'} p_j)^{-1} e^{-p_j x + 4 p_j^3 t - \frac{1}{p_j} y}. \end{cases} \quad (5.8)$$

(1) *Choosing the parameters as*

$$p_i, \alpha_i \in \mathbb{R}, \quad \beta_i \in \sqrt{-1}\mathbb{R} \quad (i = 1, \dots, N), \quad (5.9)$$

then τ_n^m satisfies the bilinear equations (3.1)–(3.6). This gives the N -soliton solution to eqs. (2.3) and (2.15).

(2) *Taking $N = 2M$, and choosing the parameters as*

$$\begin{aligned} p_i, \alpha_i, \beta_i \in \mathbb{C} \quad (i = 1, \dots, 2M), \quad p_{2k} = p_{2k-1}^* \quad (k = 1, \dots, M), \\ \alpha_{2k} = \alpha_{2k-1}^*, \quad \beta_{2k} = -\beta_{2k-1}^* \quad (k = 1, \dots, M), \end{aligned} \quad (5.10)$$

then τ_n^m satisfies the bilinear equations (3.1)–(3.6). This gives the M -breather solution to eqs. (2.3) and (2.15).

Proof. It is sufficient to show that the conditions in eq.(3.28) are satisfied. We first impose the two-periodicity in s , *i.e.*, $\tau_n^m(s+2) = \text{const.} \times \tau_n^m(s)$. For $\tau_n^m(s)$ in eq.(5.1) with entries given by eqs.(5.4) and (5.5), putting

$$q_i = -p_i, \quad (5.11)$$

we have

$$f_{s+2}^{(i)} = p_i^2 f_s^{(i)}, \quad (5.12)$$

which implies

$$\tau_n^m(s+2) = A_N \tau_n^m(s), \quad A_N = \prod_{i=1}^N p_i^2. \quad (5.13)$$

Note that the condition

$$\frac{\partial \tau_n^m(s)}{\partial z} = B_N \tau_n^m(s), \quad B_N = \sum_{i=1}^N p_i^2, \quad (5.14)$$

is also satisfied simultaneously. Now we consider case (1) and (2) separately:

Case (1). We see from eqs.(5.7) and (5.8) together with eq.(5.9) that

$$f_1^{(i)} = p_i f_0^{(i)*} \quad (5.15)$$

and so

$$\tau_n^m(1) = C_N \tau_n^{*m}(0), \quad C_N = \prod_{i=1}^N p_i \in \mathbb{R}. \quad (5.16)$$

Case (2). We see from eqs.(5.7) and (5.8) together with eq.(5.10) that

$$f_1^{(2k)} = p_{2k-1}^* f_0^{(2k-1)*}, \quad f_1^{(2k-1)} = p_{2k}^* f_0^{(2k)*}, \quad (5.17)$$

and so

$$\tau_n^m(1) = C_N \tau_n^{*m}(0), \quad C_N = (-1)^M \prod_{i=1}^M |p_i|^2 \in \mathbb{R}. \quad (5.18)$$

Therefore we have verified that the conditions in eq.(3.28) are satisfied for both cases. Then putting $\tau_n^m = \tau_n^m(0)$, we obtain the desired result. \square

We present some pictures of motions of the discrete curves. Figure 1 shows the simplest example of curve which corresponds to 1-soliton solution (loop soliton). The next example illustrated

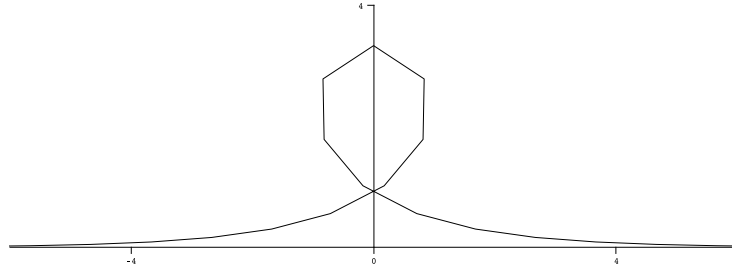


Figure 1. Parameters in eqs.(5.6), (5.7) and (5.8): $N = 1$, $x = 0$, $y = 0$, $\alpha_1 = -1$, $\beta_1 = \sqrt{-1}$, $p_1 = 0.3$, $a_n = 1$, $b_m = 0.5$.

in Figure 2 describes the interaction of two loops which corresponds to the 2-soliton solution. Figures 3 and 4 show the motions which correspond to the 1-breather and 2-breather solutions, respectively.

5.2 Solutions via Bäcklund transformations

In the theory of integrable systems, the Bäcklund transformations are obtained from the shift of a certain discrete independent variable, which also applies to our geometric transformations. We

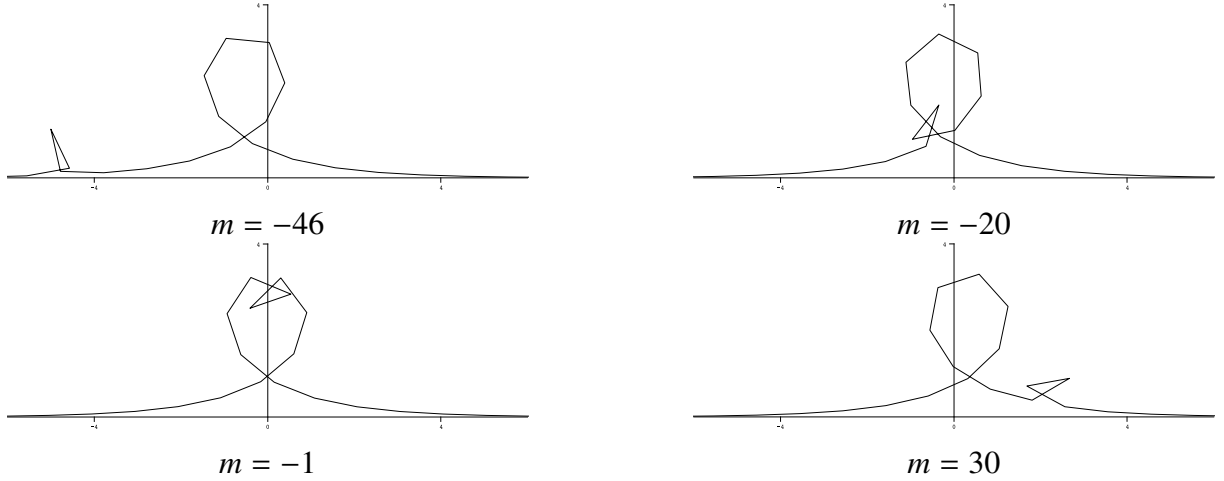


Figure 2. Parameters in eqs.(5.6), (5.7) and (5.8): $N = 2$, $x = 0$, $y = 0$, $\alpha_1 = -1$, $\alpha_2 = 1$, $\beta_1 = -\beta_2 = \sqrt{-1}$, $p_1 = 0.3$, $p_2 = 0.9$, $a_n = 1$, $b_m = 0.5$.

first introduce discrete variables k, l , and regard the determinant size N as an additional discrete variable. We then extend the τ function as $\tau_n^m(k, l, N) = \tau_n^m(x, t; y; k, l, N)$ in the following way:

$$\tau_n^m(k, l, N) = \exp \left[- \left(x + \sum_{n'}^{n-1} a_{n'} + \sum_{m'}^{m-1} b_{m'} + \sum_{k'}^{k-1} c_{k'} + \sum_{l'}^{l-1} \frac{1}{d_{l'}} \right) y \right] \det \left(f_{j-1}^{(i)} \right)_{i,j=1,\dots,N}, \quad (5.19)$$

$$f_s^{(i)} = e^{\eta_i} + e^{\mu_i}, \quad (5.20)$$

$$\left\{ \begin{array}{l} e^{\eta_i} = \alpha_i p_i^s \prod_{n'}^{n-1} (1 - a_{n'} p_i)^{-1} \prod_{m'}^{m-1} (1 - b_{m'} p_i)^{-1} \prod_{k'}^{k-1} (1 - c_{k'} p_i)^{-1} \prod_{l'}^{l-1} \left(1 - \frac{d_{l'}}{p_i} \right)^{-1} \\ \quad \times e^{p_i x - 4 p_i^3 t + \frac{1}{p_i} y}, \\ e^{\mu_j} = \beta_j (-p_j)^s \prod_{n'}^{n-1} (1 + a_{n'} p_j)^{-1} \prod_{m'}^{m-1} (1 + b_{m'} p_j)^{-1} \prod_{k'}^{k-1} (1 + c_{k'} p_j)^{-1} \prod_{l'}^{l-1} \left(1 + \frac{d_{l'}}{p_j} \right)^{-1} \\ \quad \times e^{-p_j x + 4 p_j^3 t - \frac{1}{p_j} y}. \end{array} \right. \quad (5.21)$$

Accordingly, we extend the relevant dependent variables such as Θ and γ in the same way.

Proposition 5.2

- (1) For any $k \in \mathbb{Z}$, $\tilde{\gamma}(x, t) = \gamma(x, t; k+1)$ is a Bäcklund transform of $\gamma(x, t) = \gamma(x, t; k)$ related by eq.(4.3) with $\lambda = \frac{1}{c_k}$.
- (2) For any $k \in \mathbb{Z}$, $\tilde{\gamma}_n^m = \gamma_n^m(k+1)$ is a Bäcklund transform of $\gamma_n^m = \gamma_n^m(k)$ related by eq.(4.16) with $\lambda = \frac{1}{c_k}$.
- (3) For any $N \in \mathbb{Z}_{\geq 0}$, $\tilde{\gamma}(x, t) = \gamma(x, t; N+1)$ is a Bäcklund transform of $\gamma(x, t) = \gamma(x, t; N)$ related by eq.(4.3) with $\lambda = -p_{N+1}$.
- (4) For any $N \in \mathbb{Z}_{\geq 0}$, $\tilde{\gamma}_n^m = \gamma_n^m(N+1)$ is a Bäcklund transform of $\gamma_n^m = \gamma_n^m(N)$ related by eq.(4.16) with $\lambda = -p_{N+1}$.

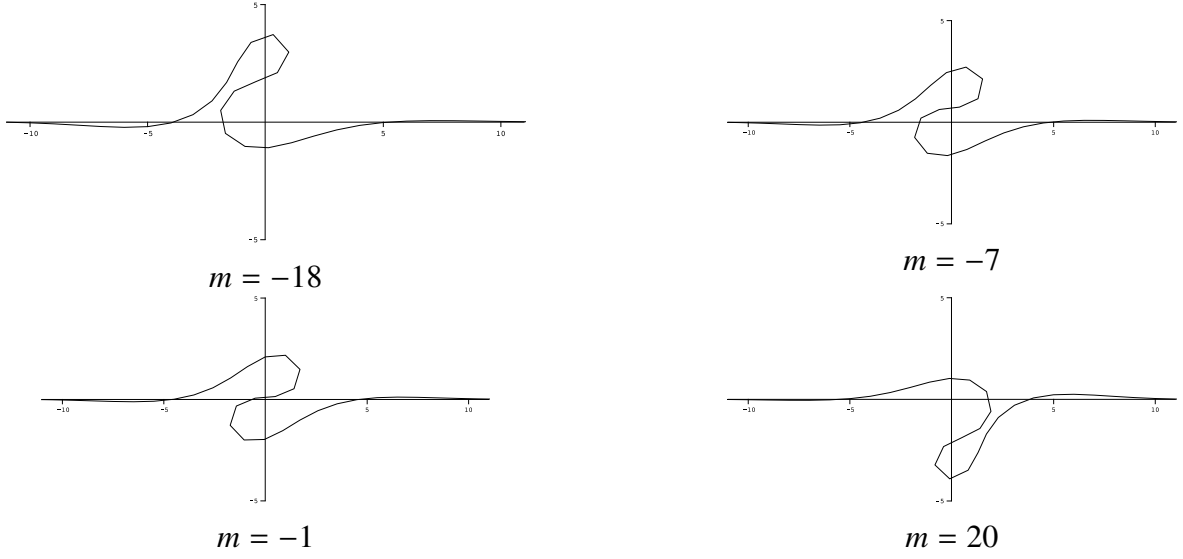


Figure 3. Parameters in eqs.(5.6), (5.7) and (5.8): $N = 2$, $x = 0$, $y = 0$, $\alpha_1 = \alpha_2^* = 1$, $\beta_1 = -\beta_2^* = 1$, $p_1 = p_2^* = 0.2 - 0.2\sqrt{-1}$, $a_n = 1$, $b_m = 1.5$.

- (5) For any $l \in \mathbb{Z}$, $\bar{\gamma}(x, t) = \gamma(x, t; l+1)$ is a Bäcklund transform of $\gamma(x, t) = \gamma(x, t; l)$ related by eq.(4.21) with $\lambda = d_l$.
- (6) For any $l \in \mathbb{Z}$, $\bar{\gamma}_n^m = \gamma_n^m(l+1)$ is a Bäcklund transform of $\gamma_n^m = \gamma_n^m(l)$ related by eq.(4.25) with $\lambda = d_l$.

Proof. We first prove (1) and (2). It follows from eq.(3.4) that the τ function satisfies the bilinear equation

$$D_y \tau_n^m(k+1) \cdot \tau_n^m(k) = -c_k \tau_n^{*m}(k+1) \tau_n^{*m}(k), \quad (5.22)$$

because of the symmetry with respect to the discrete variables m , n and k in eqs.(5.19)–(5.21). Then by the argument similar to that in proof of Theorem 3.1, we see that

$$\frac{\gamma(k+1) - \gamma(k)}{c_k} = \begin{pmatrix} \cos\left(\frac{\theta(k+1) + \theta(k)}{2}\right) \\ \sin\left(\frac{\theta(k+1) + \theta(k)}{2}\right) \end{pmatrix}. \quad (5.23)$$

From eq.(2.4), we have eq.(4.3) with $\tilde{\gamma} = \gamma(k+1)$ and $\tilde{\theta} = \theta(k+1)$

$$\frac{\gamma(k+1) - \gamma(k)}{c_k} = R\left(\frac{\theta(k+1) - \theta(k)}{2}\right) \gamma'(k). \quad (5.24)$$

Similarly from eq.(3.14), we obtain eq.(4.16) with $\tilde{\gamma}_n^m = \gamma_n^m(k+1)$ and $\tilde{\Theta}_n^m = \Theta_n^m(k+1)$

$$\frac{\gamma_n^m(k+1) - \gamma_n^m(k)}{c_k} = R\left(\frac{\Theta_n^m(k+1) - \Theta_{n+1}^m(k)}{2}\right) \frac{\gamma_{n+1}^m(k) - \gamma_n^m(k)}{a_n}, \quad (5.25)$$

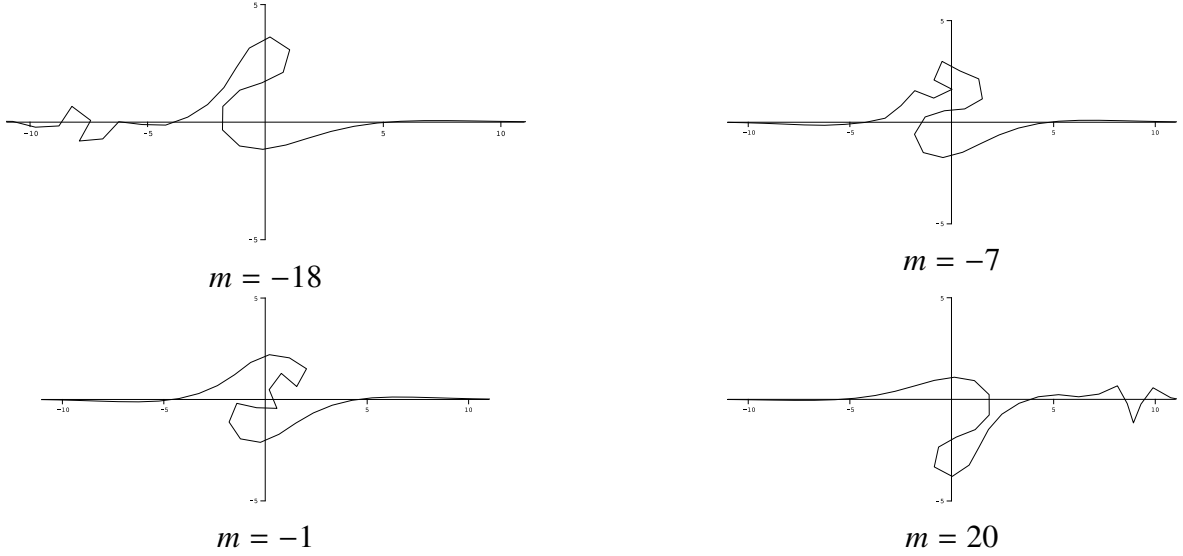


Figure 4. Parameters in eqs.(5.6), (5.7) and (5.8): $N = 4$, $x = 0$, $y = 0$, $\alpha_1 = \alpha_2^* = 1$, $\alpha_3 = \alpha_4^* = \sqrt{-1}$, $\beta_1 = -\beta_2^* = 1$, $\beta_3 = -\beta_4^* = \sqrt{-1}$, $p_1 = p_2^* = 0.2 - 0.2\sqrt{-1}$, $p_3 = p_4^* = 0.8 + 0.8\sqrt{-1}$, $a_n = 1$, $b_m = 1.5$.

which proves (1) and (2). The statements (3)–(4) and (5)–(6) can be proved in much the same way as (1)–(2), by using the bilinear equations

$$D_y \tau_n^m(N+1) \cdot \tau_n^m(N) = \frac{1}{p_{N+1}} \tau_n^{*m}(N) \tau_n^{*m}(N+1), \quad (5.26)$$

$$D_y \tau_n^m(l+1) \cdot \tau_n^{*m}(l) = -\frac{1}{d_l} \tau_n^{*m}(l+1) \tau_n^m(l), \quad (5.27)$$

respectively. These bilinear equations will be proved in Appendix. \square

Remark 5.3 Here we give a physical interpretations of Bäcklund transformations described above. The Bäcklund transforms in (1)–(2) and (5)–(6) correspond to changing the phase of solitons (loops), in other words, positions of solitons. On the other hand, the Bäcklund transforms in (3)–(4) correspond to increasing the number of solitons (loops).

Computing the potential functions of the Bäcklund transforms of the curves, one can verify the following result:

Corollary 5.4

- (1) For any $k \in \mathbb{Z}$, $\tilde{\theta}(x, t) = \theta(x, t; k+1)$ is a Bäcklund transform of $\theta(x, t) = \theta(x, t; k)$ related by eqs.(2.5) and (2.6) with $\lambda = \frac{1}{c_k}$.
- (2) For any $k \in \mathbb{Z}$, $\tilde{\Theta}_n^m = \Theta_n^m(k+1)$ is a Bäcklund transform of $\Theta_n^m = \Theta_n^m(k)$ related by eqs.(4.8) and (4.9) with $\lambda = \frac{1}{c_k}$.
- (3) For any $N \in \mathbb{Z}_{\geq 0}$, $\tilde{\theta}(x, t) = \theta(x, t; N+1)$ is a Bäcklund transform of $\theta(x, t) = \theta(x, t; N)$ related by eqs.(2.5) and (2.6) with $\lambda = -p_{N+1}$.

- (4) For any $N \in \mathbb{Z}_{\geq 0}$, $\widetilde{\Theta}_n^m = \Theta_n^m(N+1)$ is a Bäcklund transform of $\Theta_n^m = \Theta_n^m(N)$ related by eqs.(4.8) and (4.9) with $\lambda = -p_{N+1}$.
- (5) For any $l \in \mathbb{Z}$, $\bar{\theta}(x, t) = \theta(x, t; l+1)$ is a Bäcklund transform of $\theta(x, t) = \theta(x, t; l)$ related by eqs.(4.19) and (4.20) with $\lambda = d_l$.
- (6) For any $l \in \mathbb{Z}$, $\bar{\Theta}_n^m = \Theta_n^m(l+1)$ is a Bäcklund transform of $\Theta_n^m = \Theta_n^m(l)$ related by eqs.(4.23) and (4.24) with $\lambda = d_l$.

A Derivation of bilinear equations (5.26) and (5.27)

In this appendix, we show that the τ function given in eqs.(5.19)–(5.21) actually satisfies the bilinear equations (5.26) and (5.27). For this purpose, we first introduce the generic τ function $\tau_n^m(k, l, N; s) = \tau_n^m(x, t; y, z; k, l, N; s)$ by

$$\tau_n^m(k, l, N; s) = \exp \left[- \left(x + \sum_{n'}^{n-1} a_{n'} + \sum_{m'}^{m-1} b_{m'} + \sum_{k'}^{k-1} c_{k'} + \sum_{l'}^{l-1} \frac{1}{d_{l'}} \right) y \right] \det \left(f_{s+j-1}^{(i)} \right)_{i,j=1,\dots,N}, \quad (\text{A.1})$$

for $(x, t; y, z) \in \mathbb{R}^4$, $(m, n, k, l, s) \in \mathbb{Z}^5$ and $N \in \mathbb{Z}_{\geq 0}$. We require $f_s^{(i)} = f_s^{(i)}(x, t; y, z; m, n; k, l, N)$ ($i = 1, \dots, N$) to satisfy the linear equations (5.2), (5.3) and

$$\frac{f_s^{(i)}(k, l) - f_s(k-1, l)}{c_k} = f_{s+1}^{(i)}(k, l), \quad \frac{f_s^{(i)}(k, l) - f_s(k, l-1)}{d_l} = f_{s-1}^{(i)}(k, l). \quad (\text{A.2})$$

A typical example for $f_s^{(i)}$ is given by

$$f_s^{(i)} = e^{\eta_i} + e^{\mu_i}, \quad (\text{A.3})$$

$$\left\{ \begin{array}{l} e^{\eta_i} = \alpha_i p_i^s \prod_{n'}^{n-1} (1 - a_{n'} p_i)^{-1} \prod_{m'}^{m-1} (1 - b_{m'} p_i)^{-1} \prod_{k'}^{k-1} (1 - c_{k'} p_i)^{-1} \prod_{l'}^{l-1} \left(1 - \frac{d_{l'}}{p_i} \right)^{-1} e^{p_i x - 4p_i^3 t + \frac{1}{p_i} y}, \\ e^{\mu_j} = \beta_j (-p_j)^s \prod_{n'}^{n-1} (1 - a_{n'} q_j)^{-1} \prod_{m'}^{m-1} (1 - b_{m'} q_j)^{-1} \prod_{k'}^{k-1} (1 - c_{k'} q_j)^{-1} \prod_{l'}^{l-1} \left(1 - \frac{d_{l'}}{q_j} \right)^{-1} \\ \quad \times e^{q_j x - 4q_j^3 t + \frac{1}{q_j} y}, \end{array} \right. \quad (\text{A.4})$$

where p_i, q_i, α_i and β_i are arbitrary complex constants. We put

$$\sigma_n^m(y; k, l, N; s) = \det \left(f_{s+j-1}^{(i)} \right)_{i,j=1,\dots,N}. \quad (\text{A.5})$$

Proposition A.1 σ satisfies the following bilinear equations:

$$D_y \sigma_n^m(N+1; s) \cdot \sigma_n^m(N; s) = \sigma_n^m(N; s+1) \sigma_n^m(N+1; s-1), \quad (\text{A.6})$$

$$\left(D_y - \frac{1}{d_l} \right) \sigma_n^m(l+1; s) \cdot \sigma_n^m(l; s+1) = -\frac{1}{d_l} \sigma_n^m(l+1; s+1) \sigma_n^m(k; s). \quad (\text{A.7})$$

We apply the determinantal technique in order to prove Proposition A.1. The bilinear equations are reduced to the Plücker relations which are quadratic identities of determinants whose columns are appropriately shifted. To this end, we construct such formulas that express the determinants in the Plücker relations in terms of derivative or shift of discrete variable of $\sigma_n^m(k, l, N; s)$ by using the linear relations of the entries. For the details of the technique, we refer to [18, 23, 24, 28, 29].

We introduce a notation

$$\sigma_n^m(l, N; s) = | 0_l, 1_l, \dots, N-2_l, N-1_l |, \quad (\text{A.8})$$

where “ j_l ” denotes the column vector

$$j_l = \begin{bmatrix} f_{s+j}^{(1)}(l) \\ \vdots \\ f_{s+j}^{(N)}(l) \end{bmatrix}. \quad (\text{A.9})$$

Lemma A.2 *The following formulas hold:*

$$\partial_y \sigma_n^m(l, N; s) = | -1, 1, \dots, N-2, N-1 |, \quad (\text{A.10})$$

$$\sigma_n^m(l+1, N; s) = | 0_{l+1}, 1, \dots, N-2, N-1 |, \quad (\text{A.11})$$

$$d_l \sigma_n^m(l+1, N; s) = | 1_{l+1}, 1, \dots, N-2, N-1 |, \quad (\text{A.12})$$

$$-(d_l \partial_y - 1) \sigma_n^m(l+1, N; s) = | 0, 1_{l+1}, 2, \dots, N-2, N-1 |. \quad (\text{A.13})$$

Note that the subscript of column vectors are shown only when l is shifted for notational simplicity.

Proof. Eq. (A.10) can be verified by direct calculation by using the fourth equation in eq.(5.2). We have

$$\sigma_n^m(l+1, N; s) = | 0_{l+1}, 1_{l+1}, \dots, N-2_{l+1}, N-1_{l+1} |. \quad (\text{A.14})$$

Adding the $(N-1)$ -th column multiplied by d_{l+1} to the N -th column and using eq.(A.2), we have

$$\sigma_n^m(l+1, N; s) = | 0_{l+1}, 1_{l+1}, \dots, N-2_{l+1}, N-1_l |. \quad (\text{A.15})$$

Similarly, adding the $(i-1)$ -th column multiplied by d_{l+1} to the i -th column and using eq.(A.2) for $i = N-1, \dots, 2$, we obtain

$$\sigma_n^m(l+1, N; s) = | 0_{l+1}, 1, \dots, N-2, N-1 |, \quad (\text{A.16})$$

which is eq. (A.11). Multiplying d_{l+1} to the first column of eq. (A.11) and using eq.(A.2), we obtain eq. (A.12). Finally, differentiating eq.(A.12) with respect to y yields

$$\begin{aligned} d_l \partial_y \sigma_n^m(l+1, N; s) &= | 0_{l+1}, 1, 2, \dots, N-2, N-1 | + | 1_{l+1}, 0, 2, \dots, N-2, N-1 | \\ &= \sigma_n^m(l+1, N; s) - | 0, 1_{l+1}, 2, \dots, N-2, N-1 |, \end{aligned} \quad (\text{A.17})$$

which is equivalent to eq.(A.13). This completes the proof. \square

Proof of Proposition A.1 Consider the Plücker relation (see, for example, [29]),

$$\begin{aligned} 0 = & | -1, 0, 1, \dots, N-2 | \times | 1, \dots, N-2, N-1, \phi | \\ & + | 0, 1, \dots, N-2, N-1 | \times | -1, 1, \dots, N-2, \phi | \\ & - | 0, 1, \dots, N-2, \phi | \times | -1, 1, \dots, N-2, N-1 |, \end{aligned} \quad (\text{A.18})$$

where ϕ is a column vector given by

$$\phi = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \quad (\text{A.19})$$

By using eqs.(A.8) and (A.10), expanding the determinant with respect to the column ϕ , eq.(A.18) is rewritten as

$$0 = \sigma_n^m(N; s-1) \sigma_n^m(N-1; s+1) + \sigma_n^m(N; s) \partial_y \sigma_n^m(N-1; s) - \sigma_n^m(N-1; s) \partial_y \sigma_n^m(N; s), \quad (\text{A.20})$$

which implies eq.(A.6). Similarly, applying Lemma A.2 on the Plücker relation

$$\begin{aligned} 0 = & | -1, 0, 1, \dots, N-2 | \times | 0_{l+1}, 1, \dots, N-2, N-1 | \\ & - | 0_{l+1}, 0, 1, \dots, N-2 | \times | -1, 1, \dots, N-2, N-1 | \\ & - | 0, 1, \dots, N-2, N-1 | \times | -1, 0_{l+1}, 1, \dots, N-2 |, \end{aligned} \quad (\text{A.21})$$

we obtain

$$\begin{aligned} 0 = & \sigma_n^m(l; s-1) \times \sigma_n^m(l+1; s) - d_l \sigma_n^m(l+1; s-1) \times \partial_y \sigma_n^m(l; s) \\ & - \sigma_n^m(l; s) \times \left[-(d_l \partial_y - 1) \sigma_n^m(l+1; s-1) \right], \end{aligned} \quad (\text{A.22})$$

which is equivalent to eq.(A.7). This completes the proof. \square

From Proposition A.1 and eq.(A.1), we see that $\tau_n^m(k, l, N; s)$ satisfies

$$D_y \tau_n^m(N+1; s) \cdot \tau_n^m(N; s) = \tau_n^m(N; s+1) \tau_n^m(N+1; s-1), \quad (\text{A.23})$$

$$D_y \tau_n^m(l+1; s) \cdot \tau_n^m(l; s+1) = -\frac{1}{d_l} \tau_n^m(l+1; s+1) \tau_n^m(k; s). \quad (\text{A.24})$$

We finally obtain eqs.(5.26) and (5.27) from eqs.(A.23) and (A.24), respectively, by imposing the reduction condition (3.28).

Acknowledgements

One of the authors (K. K) would like to thank Professor Tim Hoffmann for giving a series of introductory lectures on discrete differential geometry at Kyushu University. This works is partially supported by JSPS Grant-in-Aid for Scientific Research No. 19340039, 21540067, 21656027 and 22656026.

References

- [1] M. J. Ablowitz B. Prinari and A.D. Trubatch, Discrete and continuous nonlinear Schrödinger systems (Cambridge University Press, 2004).
- [2] L. Bianchi, Sulla trasformazione di Bäcklund per le superficie pseudosferiche, *Rend. Lincei* **5**(1892) 3–12.
- [3] A. Bobenko and U. Pinkall, Discrete surface with constant negative Gaussian curvature and the Hirota equation, *J. Differential Geom.* **43**(1996) 527–611.
- [4] A.I. Bobenko and Y.B. Suris, Discrete differential geometry (American Mathematical Society, 2008).
- [5] E. Date, M. Jimbo and T. Miwa, Method for generating discrete soliton equations.I, *J. Phys. Soc. Jpn.* **51**(1982) 4116–4124.
- [6] E. Date, M. Jimbo and T. Miwa, Method for generating discrete solitons equations.II, *J. Phys. Soc. Jpn.* **51**(1982) 4125–4131.
- [7] E. Date, M. Jimbo and T. Miwa, Method for generating discrete soliton equations.III, *J. Phys. Soc. Jpn.* **53**(1983) 388–393.
- [8] E. Date, M. Jimbo and T. Miwa, Method for generating discrete soliton equations.IV, *J. Phys. Soc. Jpn.* **53**(1983) 761–765.
- [9] E. Date, M. Jimbo and T. Miwa, Method for generating discrete soliton equations.V, *J. Phys. Soc. Jpn.* **53**(1983) 766–771.
- [10] A. Doliwa, Geometric discretization of the Toda system, *Phys. Lett.* **A234**(1997)187–192.
- [11] R. E. Goldstein and D. M. Petrich, The Korteweg-de Vries hierarchy as dynamics of closed curves in the plane, *Phys. Rev. Lett.* **67** (1991) 3203–3206.
- [12] R. Hirota, Nonlinear partial difference equations. I. A difference analogue of the Korteweg-de Vries equation, *J. Phys. Soc. Jpn.* **43**(1977) 4116–4124.
- [13] R. Hirota, Nonlinear partial difference equations. II. Discrete-time Toda equation, *J. Phys. Soc. Jpn.* **43**(1977) 2074–2078.
- [14] R. Hirota, Nonlinear partial difference equations. III. Discrete sine-Gordon equation, *J. Phys. Soc. Jpn.* **43**(1977) 2079–2086.
- [15] R. Hirota, Nonlinear partial difference equations. IV. Bäcklund transformation for the discrete-time Toda equation, *J. Phys. Soc. Jpn.* **45**(1978) 321–332.
- [16] R. Hirota, Nonlinear partial difference equations. V. Nonlinear equations reducible to linear equations, *J. Phys. Soc. Jpn.* **46**(1979) 312–319.
- [17] R. Hirota, Discretization of the potential modified KdV equation, *J. Phys. Soc. Jpn.* **67**(1998) 2234–2236.

- [18] R. Hirota, The direct method in soliton theory. Cambridge Tracts in Mathematics **155** (Cambridge University Press, 2004)
- [19] M. Hisakado, K. Nakayama and M. Wadati, Motion of discrete curves in the plane, J. Phys. Soc. Jpn. **64** (1995) 2390–2393.
- [20] T. Hoffmann and N. Kutz, Discrete curves in $\mathbb{C}P^1$ and the Toda lattice, Stud. Appl. Math. **113** (2004) 31–55.
- [21] M. Jimbo and T. Miwa, Solitons and infinite dimensional Lie algebras, Publ. RIMS **19**(1983) 943-1001.
- [22] G. Lamb Jr., Solitons and the motion of helical curves, Phys. Rev. Lett. **37** (1976) 235–237.
- [23] K. Maruno, K. Kajiwara and M. Oikawa, Casorati determinant solution for the discrete-time relativistic Toda lattice equation, Phys. Lett. **A241**(1998) 335–343.
- [24] K. Maruno and Y. Ohta, Casorati determinant form of dark soliton solutions of the discrete nonlinear Schrödinger equation, J. Phys. Soc. Jpn. **75**(2006) 054002.
- [25] N. Matsuura, Discrete KdV and discrete modified KdV equations arising from motions of discrete planar curves, preprint.
- [26] T. Miwa, On Hirota’s difference equations, Proc. Japan Acad. Ser. A Math. Sci. **58**(1982) 9–12.
- [27] F.W. Nijhoff and H. Capel, The discrete KdV equation, Acta Appl. Math. **39**(1995) 133–158.
- [28] Y. Ohta, R. Hirota, S. Tsujimoto and T. Imai, Casorati and discrete Gram type determinant representations of solutions to the discrete KP hierarchy, J. Phys. Soc. Jpn. **62**(1993) 1872–1886.
- [29] Y. Ohta, K. Kajiwara, J. Matsukidaira and J. Satsuma, Casorati determinant solution for the relativistic Toda lattice equation, J. Math. Phys. **34**(1993) 5190–5204.
- [30] C. Rogers and W.K. Schief, Bäcklund and Darboux transformations: geometry and modern applications in soliton theory, Cambridge texts in applied mathematics (Cambridge University Press, 2002).
- [31] Y. B. Suris, The problem of integrable discretization (Birkhäuser, 2003).
- [32] S. Tsujimoto, On a discrete analogue of the two-dimensional Toda lattice hierarchy, Publ. RIMS **38**(2002) 113-133.
- [33] K. Ueno and K. Takasaki, Toda lattice hierarchy, Group representations and systems of differential equations, Adv. Stud. Pure Math. **4** (Kinokuniya, 1982) 1–95.
- [34] M. Wadati, Bäcklund transformation for solutions of the modified Korteweg-de Vries equation, J. Phys. Soc. Jpn. **36**(1974) 1498.

List of MI Preprint Series, Kyushu University

The Global COE Program
Math-for-Industry Education & Research Hub

MI

- MI2008-1 Takahiro ITO, Shuichi INOKUCHI & Yoshihiro MIZOGUCHI
Abstract collision systems simulated by cellular automata
- MI2008-2 Eiji ONODERA
The initial value problem for a third-order dispersive flow into compact almost Hermitian manifolds
- MI2008-3 Hiroaki KIDO
On isosceles sets in the 4-dimensional Euclidean space
- MI2008-4 Hirofumi NOTSU
Numerical computations of cavity flow problems by a pressure stabilized characteristic-curve finite element scheme
- MI2008-5 Yoshiyasu OZEKI
Torsion points of abelian varieties with values in infinite extensions over a p -adic field
- MI2008-6 Yoshiyuki TOMIYAMA
Lifting Galois representations over arbitrary number fields
- MI2008-7 Takehiro HIROTSU & Setsuo TANIGUCHI
The random walk model revisited
- MI2008-8 Silvia GANDY, Masaaki KANNO, Hirokazu ANAI & Kazuhiro YOKOYAMA
Optimizing a particular real root of a polynomial by a special cylindrical algebraic decomposition
- MI2008-9 Kazufumi KIMOTO, Sho MATSUMOTO & Masato WAKAYAMA
Alpha-determinant cyclic modules and Jacobi polynomials

- MI2008-10 Sangyeol LEE & Hiroki MASUDA
Jarque-Bera Normality Test for the Driving Lévy Process of a Discretely Observed Univariate SDE
- MI2008-11 Hiroyuki CHIHARA & Eiji ONODERA
A third order dispersive flow for closed curves into almost Hermitian manifolds
- MI2008-12 Takehiko KINOSHITA, Kouji HASHIMOTO and Mitsuhiro T. NAKAO
On the L^2 a priori error estimates to the finite element solution of elliptic problems with singular adjoint operator
- MI2008-13 Jacques FARAUT and Masato WAKAYAMA
Hermitian symmetric spaces of tube type and multivariate Meixner-Pollaczek polynomials
- MI2008-14 Takashi NAKAMURA
Riemann zeta-values, Euler polynomials and the best constant of Sobolev inequality
- MI2008-15 Takashi NAKAMURA
Some topics related to Hurwitz-Lerch zeta functions
- MI2009-1 Yasuhide FUKUMOTO
Global time evolution of viscous vortex rings
- MI2009-2 Hidetoshi MATSUI & Sadanori KONISHI
Regularized functional regression modeling for functional response and predictors
- MI2009-3 Hidetoshi MATSUI & Sadanori KONISHI
Variable selection for functional regression model via the L_1 regularization
- MI2009-4 Shuichi KAWANO & Sadanori KONISHI
Nonlinear logistic discrimination via regularized Gaussian basis expansions
- MI2009-5 Toshiro HIRANOUCI & Yuichiro TAGUCHI
Flat modules and Groebner bases over truncated discrete valuation rings

- MI2009-6 Kenji KAJIWARA & Yasuhiro OHTA
Bilinearization and Casorati determinant solutions to non-autonomous 1+1 dimensional discrete soliton equations
- MI2009-7 Yoshiyuki KAGEI
Asymptotic behavior of solutions of the compressible Navier-Stokes equation around the plane Couette flow
- MI2009-8 Shohei TATEISHI, Hidetoshi MATSUI & Sadanori KONISHI
Nonlinear regression modeling via the lasso-type regularization
- MI2009-9 Takeshi TAKAISHI & Masato KIMURA
Phase field model for mode III crack growth in two dimensional elasticity
- MI2009-10 Shingo SAITO
Generalisation of Mack's formula for claims reserving with arbitrary exponents for the variance assumption
- MI2009-11 Kenji KAJIWARA, Masanobu KANEKO, Atsushi NOBE & Teruhisa TSUDA
Ultradiscretization of a solvable two-dimensional chaotic map associated with the Hesse cubic curve
- MI2009-12 Tetsu MASUDA
Hypergeometric q -functions of the q -Painlevé system of type $E_8^{(1)}$
- MI2009-13 Hidenao IWANE, Hitoshi YANAMI, Hirokazu ANAI & Kazuhiro YOKOYAMA
A Practical Implementation of a Symbolic-Numeric Cylindrical Algebraic Decomposition for Quantifier Elimination
- MI2009-14 Yasunori MAEKAWA
On Gaussian decay estimates of solutions to some linear elliptic equations and its applications
- MI2009-15 Yuya ISHIHARA & Yoshiyuki KAGEI
Large time behavior of the semigroup on L^p spaces associated with the linearized compressible Navier-Stokes equation in a cylindrical domain

- MI2009-16 Chikashi ARITA, Atsuo KUNIBA, Kazumitsu SAKAI & Tsuyoshi SAWABE
Spectrum in multi-species asymmetric simple exclusion process on a ring
- MI2009-17 Masato WAKAYAMA & Keitaro YAMAMOTO
Non-linear algebraic differential equations satisfied by certain family of elliptic functions
- MI2009-18 Me Me NAING & Yasuhide FUKUMOTO
Local Instability of an Elliptical Flow Subjected to a Coriolis Force
- MI2009-19 Mitsunori KAYANO & Sadanori KONISHI
Sparse functional principal component analysis via regularized basis expansions and its application
- MI2009-20 Shuichi KAWANO & Sadanori KONISHI
Semi-supervised logistic discrimination via regularized Gaussian basis expansions
- MI2009-21 Hiroshi YOSHIDA, Yoshihiro MIWA & Masanobu KANEKO
Elliptic curves and Fibonacci numbers arising from Lindenmayer system with symbolic computations
- MI2009-22 Eiji ONODERA
A remark on the global existence of a third order dispersive flow into locally Hermitian symmetric spaces
- MI2009-23 Stjepan LUGOMER & Yasuhide FUKUMOTO
Generation of ribbons, helicoids and complex scherk surface in laser-matter Interactions
- MI2009-24 Yu KAWAKAMI
Recent progress in value distribution of the hyperbolic Gauss map
- MI2009-25 Takehiko KINOSHITA & Mitsuhiro T. NAKAO
On very accurate enclosure of the optimal constant in the a priori error estimates for H_0^2 -projection

- MI2009-26 Manabu YOSHIDA
Ramification of local fields and Fontaine's property (Pm)
- MI2009-27 Yu KAWAKAMI
Value distribution of the hyperbolic Gauss maps for flat fronts in hyperbolic three-space
- MI2009-28 Masahisa TABATA
Numerical simulation of fluid movement in an hourglass by an energy-stable finite element scheme
- MI2009-29 Yoshiyuki KAGEI & Yasunori MAEKAWA
Asymptotic behaviors of solutions to evolution equations in the presence of translation and scaling invariance
- MI2009-30 Yoshiyuki KAGEI & Yasunori MAEKAWA
On asymptotic behaviors of solutions to parabolic systems modelling chemo-taxis
- MI2009-31 Masato WAKAYAMA & Yoshinori YAMASAKI
Hecke's zeros and higher depth determinants
- MI2009-32 Olivier PIRONNEAU & Masahisa TABATA
Stability and convergence of a Galerkin-characteristics finite element scheme of lumped mass type
- MI2009-33 Chikashi ARITA
Queueing process with excluded-volume effect
- MI2009-34 Kenji KAJIWARA, Nobutaka NAKAZONO & Teruhisa TSUDA
Projective reduction of the discrete Painlevé system of type $(A_2 + A_1)^{(1)}$
- MI2009-35 Yosuke MIZUYAMA, Takamasa SHINDE, Masahisa TABATA & Daisuke TAGAMI
Finite element computation for scattering problems of micro-hologram using DtN map

- MI2009-36 Reiichiro KAWAI & Hiroki MASUDA
Exact simulation of finite variation tempered stable Ornstein-Uhlenbeck processes
- MI2009-37 Hiroki MASUDA
On statistical aspects in calibrating a geometric skewed stable asset price model
- MI2010-1 Hiroki MASUDA
Approximate self-weighted LAD estimation of discretely observed ergodic Ornstein-Uhlenbeck processes
- MI2010-2 Reiichiro KAWAI & Hiroki MASUDA
Infinite variation tempered stable Ornstein-Uhlenbeck processes with discrete observations
- MI2010-3 Kei HIROSE, Shuichi KAWANO, Daisuke MIIKE & Sadanori KONISHI
Hyper-parameter selection in Bayesian structural equation models
- MI2010-4 Nobuyuki IKEDA & Setsuo TANIGUCHI
The Itô-Nisio theorem, quadratic Wiener functionals, and 1-solitons
- MI2010-5 Shohei TATEISHI & Sadanori KONISHI
Nonlinear regression modeling and detecting change point via the relevance vector machine
- MI2010-6 Shuichi KAWANO, Toshihiro MISUMI & Sadanori KONISHI
Semi-supervised logistic discrimination via graph-based regularization
- MI2010-7 Teruhisa TSUDA
UC hierarchy and monodromy preserving deformation
- MI2010-8 Takahiro ITO
Abstract collision systems on groups
- MI2010-9 Hiroshi YOSHIDA, Kinji KIMURA, Naoki YOSHIDA, Junko TANAKA & Yoshihiro MIWA
An algebraic approach to underdetermined experiments

- MI2010-10 Kei HIROSE & Sadanori KONISHI
Variable selection via the grouped weighted lasso for factor analysis models
- MI2010-11 Katsusuke NABESHIMA & Hiroshi YOSHIDA
Derivation of specific conditions with Comprehensive Groebner Systems
- MI2010-12 Yoshiyuki KAGEI, Yu NAGAFUCHI & Takeshi SUDO
Decay estimates on solutions of the linearized compressible Navier-Stokes equation around a Poiseuille type flow
- MI2010-13 Reiichiro KAWAI & Hiroki MASUDA
On simulation of tempered stable random variates
- MI2010-14 Yoshiyasu OZEKI
Non-existence of certain Galois representations with a uniform tame inertia weight
- MI2010-15 Me Me NAING & Yasuhide FUKUMOTO
Local Instability of a Rotating Flow Driven by Precession of Arbitrary Frequency
- MI2010-16 Yu KAWAKAMI & Daisuke NAKAJO
The value distribution of the Gauss map of improper affine spheres
- MI2010-17 Kazunori YASUTAKE
On the classification of rank 2 almost Fano bundles on projective space
- MI2010-18 Toshimitsu TAKAESU
Scaling limits for the system of semi-relativistic particles coupled to a scalar bose field
- MI2010-19 Reiichiro KAWAI & Hiroki MASUDA
Local asymptotic normality for normal inverse Gaussian Lévy processes with high-frequency sampling
- MI2010-20 Yasuhide FUKUMOTO, Makoto HIROTA & Youichi MIE
Lagrangian approach to weakly nonlinear stability of an elliptical flow

- MI2010-21 Hiroki MASUDA
Approximate quadratic estimating function for discretely observed Lévy driven SDEs with application to a noise normality test
- MI2010-22 Toshimitsu TAKAESU
A Generalized Scaling Limit and its Application to the Semi-Relativistic Particles System Coupled to a Bose Field with Removing Ultraviolet Cutoffs
- MI2010-23 Takahiro ITO, Mitsuhiko FUJIO, Shuichi INOKUCHI & Yoshihiro MIZOGUCHI
Composition, union and division of cellular automata on groups
- MI2010-24 Toshimitsu TAKAESU
A Hardy's Uncertainty Principle Lemma in Weak Commutation Relations of Heisenberg-Lie Algebra
- MI2010-25 Toshimitsu TAKAESU
On the Essential Self-Adjointness of Anti-Commutative Operators
- MI2010-26 Reiichiro KAWAI & Hiroki MASUDA
On the local asymptotic behavior of the likelihood function for Meixner Lévy processes under high-frequency sampling
- MI2010-27 Chikashi ARITA & Daichi YANAGISAWA
Exclusive Queueing Process with Discrete Time
- MI2010-28 Jun-ichi INOBUCHI, Kenji KAJIWARA, Nozomu MATSUURA & Yasuhiro OHTA
Motion and Bäcklund transformations of discrete plane curves