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Abstract

It is proved that the ramification filtration of the absolute Galois group of a complete discrete valuation field with perfect residue field is characterized in terms of Fontaine's property (P_m) .

1 Introduction

Let K be a complete discrete valuation field with perfect residue field k of characteristic $p > 0$, \mathcal{O}_K its valuation ring, v_K its valuation normalized by $v_K(K^\times) = \mathbb{Z}$, K^{alg} a fixed algebraic closure of K and \bar{K} the separable closure of K in K^{alg} . In this paper, we construct a certain decreasing filtration of the absolute Galois group $G_K := \text{Gal}(\bar{K}/K)$ to measure the ramification of extensions of K . If L is a finite separable extension of K , we denote by \mathcal{O}_L the integral closure of \mathcal{O}_K in L . For an algebraic extension E of K and a real number m , we put $\mathfrak{a}_{E/K}^m := \{x \in \mathcal{O}_E \mid v_K(x) \geq m\}$, which is an ideal of \mathcal{O}_E . For a finite separable extension L/K and a real number m , we consider the following property studied in [Fo]:

(P_m) For any algebraic extension E/K , if there exists an \mathcal{O}_K -algebra homomorphism $\mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K}^m$, then there exists a K -embedding $L \hookrightarrow E$.

For a finite separable extension L of K , we put

$$m_{L/K} := \inf\{m \in \mathbb{R} \mid (P_m) \text{ is true for } \tilde{L}/K\},$$

where \tilde{L} is the Galois closure of L over K . The property (P_m) is stable under composition of extensions of K . Hence we can define two filtrations of G_K as follows: For a real number m , we denote by $\bar{K}_{< m}$ (resp. $\bar{K}_{\leq m}$) the composite field of all finite Galois extensions L of K such that $m_{L/K} < m$ (resp. $m_{L/K} \leq m$). We define two closed normal subgroups $G_K^{\geq m}$ and $G_K^{> m}$ of G_K by

$$G_K^{\geq m} := \text{Gal}(\bar{K}/\bar{K}_{< m}), \quad G_K^{> m} := \text{Gal}(\bar{K}/\bar{K}_{\leq m}).$$

These filtrations satisfies $\bigcap_m G_K^{\geq m} = 1$, $G_K^{\geq 0} = G_K$. Moreover, $G_K^{\geq 1}$ is the inertia subgroup of G_K and $G_K^{> 1}$ is the wild inertia subgroup of G_K (cf. Thm. 2.6, Rem. 2.7).

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On the other hand, we denote by $G_K^{(m)}$ the m -th upper numbering ramification group in the sense of [Fo]. Namely, we put $G_K^{(m)} := G_K^{m-1}$, where the latter is the upper numbering ramification group defined in [Se]. In addition, we put $G_K^{(m+)} := \bigcup_{m' > m} G_K^{(m')}$. These define two decreasing filtrations of G_K and they are well-known in the classical ramification theory.

We denote by $\bar{K}_{(m)}$ (resp. $\bar{K}_{(m+)}$) the fixed field of \bar{K} by $G_K^{(m)}$ (resp. $G_K^{(m+)}$). Our main result in this paper is:

Theorem 1.1. *For a real number m , we have $\bar{K}_{< m} = \bar{K}_{(m)}$ and $\bar{K}_{\leq m} = \bar{K}_{(m+)}$, so that*

$$G_K^{\geq m} = G_K^{(m)}, \quad G_K^{> m} = G_K^{(m+)}.$$

We prove this theorem by showing the equality $m_{L/K} = u_{L/K}$ for a finite Galois extension L of K , where $u_{L/K}$ is the greatest upper ramification break of L/K in the sense of [Fo].

The property (P_m) is useful for obtaining ramification bounds for certain Galois representations ([CL], [Fo], [Ha1]). Indeed, Fontaine proved the following: in the case where the characteristic of K is 0, for an integer $n \geq 1$, if we denote by \mathcal{G} a finite flat group scheme over \mathcal{O}_K killed by p^n , then the ramification of $\mathcal{G}(\bar{K})$ is bounded by m if $m > e(n + \frac{1}{p-1})$, where e is the absolute ramification index of K ([Fo], Thm. A). This is extended to the imperfect residue field case by Hattori ([Ha2], Thm. 7). Our equality $m_{L/K} = u_{L/K}$ was used in [Ha1], Proposition 5.6, to improve the ramification bound for semi-stable torsion representation.

As an interesting and analogous result, Colmez endowed Fontaine's p -adic period ring B_{dR} with a filtration by means of "valuation of convergence", so that $B_{\text{dR}}^{(m)} \cap \bar{K} = \bar{K}_{(m+)}$ for a non-negative real number m ([Co], Thm. 0.1).

In Section 2, we study some properties of (P_m) and the number $m_{L/K}$. By using these results, we define our filtration of G_K and deduce its properties (i)–(iv) above. In Section 3, after recalling the classical ramification theory for separable extensions of K ([De], [He]), we show the equality $m_{L/K} = u_{L/K}$ to prove Theorem 1.1. In Section 4, we begin with a review of the ramification theory of Abbes and Saito ([AS1], [AS2]). Then we consider the property (P_m) in the imperfect residue field case, and translate our results in Section 3 to the language of their theory. In the Appendix, we prove a Galois theoretic property on filtrations of the absolute Galois group of an arbitrary field.

Notation. If L is a finite extension of K in K^{alg} , then we denote by $e_{L/K}$ the ramification index of L/K and by \mathcal{O}_L the integral closure of \mathcal{O}_K in L . We extend the valuation v_K of K to K^{alg} uniquely and also denote it by v_K .

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2 Ramification theory via (P_m)

In this section, we study the property (P_m) . Let m be a real number. For a finite separable extension L of K , we put

$$m_{L/K} := \inf\{m \in \mathbb{R} \mid (P_m) \text{ is true for } \tilde{L}/K\},$$

where \tilde{L} is the Galois closure of L over K . If $L = K$, the property (P_m) holds for all real number m , so that we have $m_{L/K} = -\infty$. The following proposition is a basic property of the number $m_{L/K}$:

Proposition 2.1. *Let L be a finite Galois extension of K . Then the number $m_{L/K}$ is non-negative and finite if $[L : K] \geq 2$.*

Proof. If $[L : K] \geq 2$, it is clear that (P_m) is not true for L/K and any real number $m \leq 0$. Thus we have $m_{L/K} \geq 0$. Hence we show the number $m_{L/K}$ is finite. Choose an element α of \mathcal{O}_L such that $\mathcal{O}_L = \mathcal{O}_K[\alpha]$. Let P be the minimal polynomial of α over K and $\alpha = \alpha_1, \dots, \alpha_n$ the zeros of P in \bar{K} . Suppose there exists an \mathcal{O}_K -algebra homomorphism $\eta : \mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K}^m$ for an algebraic extension E of K and $m > n \sup_{i \neq 1} v_K(\alpha - \alpha_i)$. Then we have $v_K(P(\beta)) \geq m$, where β is a lift of $\eta(\alpha)$ in \mathcal{O}_E . By the inequalities

$$n \sup_i v_K(\beta - \alpha_i) \geq v_K(P(\beta)) > n \sup_{i \neq 1} v_K(\alpha - \alpha_i),$$

we have $v_K(\beta - \alpha_{i_0}) > \sup_{i \neq 1} v_K(\alpha - \alpha_i)$ for some i_0 . By Krasner's lemma, we have $K(\alpha_{i_0}) \subset K(\beta)$. Thus we obtain a K -embedding $L = K(\alpha) \xrightarrow{\sim} K(\alpha_{i_0}) \subset K(\beta) \subset E$. Hence (P_m) is true for $m > n \sup_{i \neq 1} v_K(\alpha - \alpha_i)$. Therefore, we have $m_{L/K} \leq n \sup_{i \neq 1} v_K(\alpha - \alpha_i) < \infty$. \square

We can easily check the equality $m_{L/K} = m_{L/K'}$ for any unramified sub-extension K'/K of L/K . This often allows us to assume that L/K is totally ramified. More generally, the number $m_{L/K}$ is stable under unramified base change as follows:

Proposition 2.2. *Let L be a finite Galois extension of K . Then we have the inequality $e_{K'/K} m_{L/K} \geq m_{LK'/K'}$ for any finite separable extension K' of K , with equality if K'/K is an unramified Galois extension.*

Proof. Put $L' := LK'$ and $e' := e_{K'/K}$. Suppose there exists an $\mathcal{O}_{K'}$ -algebra homomorphism $\eta : \mathcal{O}_{L'} \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K'}^m$ for an algebraic extension E of K' and $m > e' m_{L/K}$. Then the composite map defined by

$$\eta' : \mathcal{O}_L \hookrightarrow \mathcal{O}_{L'} \xrightarrow{\eta} \mathcal{O}_E/\mathfrak{a}_{E/K'}^m = \mathcal{O}_E/\mathfrak{a}_{E/K}^{e'^{-1}m}$$

is an $\mathcal{O}_{K'}$ -algebra homomorphism. In particular, η' is an \mathcal{O}_K -algebra homomorphism. By the property (P_m) , there exists a K -embedding $L \hookrightarrow E$ corresponding to η' . Since L/K is a Galois extension, there exists a K' -embedding $L' = LK' \hookrightarrow E$. Hence (P_m) is true for L'/K' and $m > e' m_{L/K}$. Thus we have the inequality $e' m_{L/K} \geq m_{L'/K'}$. Next, assume K'/K is an unramified Galois extension. Then we show the inequality $m_{L/K} \leq m_{L'/K'}$. We may assume L/K

is totally ramified. Note that $L \cap K' = K$. Suppose there exists an \mathcal{O}_K -algebra homomorphism $\eta : \mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K}^m$ for an algebraic extension E of K and $m > m_{L'/K'}$. Choose a uniformizer α of \mathcal{O}_L . Let β be a lift of $\eta(\alpha)$ in \mathcal{O}_E . Since L/K is totally ramified and K'/K is unramified, α is also a uniformizer of $\mathcal{O}_{L'}$. Hence the map $\mathcal{O}_{L'} \rightarrow \mathcal{O}_{EK'}/\mathfrak{a}_{EK'/K'}^m$ defined by $\alpha \mapsto \beta$ is an $\mathcal{O}_{K'}$ -algebra homomorphism. By the property (P_m) , there exists a K' -embedding $L' \hookrightarrow EK'$. Since both L and K' are Galois extensions of K and $L \cap K' = K$, the image of the composite map $L \hookrightarrow L' \hookrightarrow EK'$ is contained in E . Therefore, (P_m) is true for L/K and $m > m_{L'/K'}$. Hence the result follows. \square

To define a filtration of G_K , we show that the property (P_m) is stable under composition of finite Galois extensions of K as follows:

Proposition 2.3. *Let L and K' be finite Galois extensions of K . For a real number m , if (P_m) is true for both L/K and K'/K , then (P_m) is also true for the composite extension LK'/K . In other words, we have $m_{LK'/K} \leq \max\{m_{L/K}, m_{K'/K}\}$.*

Proof. Put $L' := LK'$. Assume (P_m) is true for L/K and K'/K . Suppose there exists an \mathcal{O}_K -algebra homomorphism $\eta : \mathcal{O}_{L'} \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K}^m$ for an algebraic extension E of K . Then the composite maps defined by

$$\eta' : \mathcal{O}_L \hookrightarrow \mathcal{O}_{L'} \xrightarrow{\eta} \mathcal{O}_E/\mathfrak{a}_{E/K}^m, \quad \eta'' : \mathcal{O}_{K'} \hookrightarrow \mathcal{O}_{L'} \xrightarrow{\eta} \mathcal{O}_E/\mathfrak{a}_{E/K}^m$$

are also \mathcal{O}_K -algebra homomorphisms. By the property (P_m) , this implies the existence of K -embeddings $L \hookrightarrow E$ and $K' \hookrightarrow E$ by the assumption on m . Since L/K and K'/K are Galois extensions, we obtain a K -embedding $L' \hookrightarrow E$. Therefore, (P_m) is true for L'/K . \square

By this proposition, we can define two increasing filtrations $(\bar{K}_{< m})_{m \in \mathbb{R}}$ and $(\bar{K}_{\leq m})_{m \in \mathbb{R}}$ of \bar{K} as follows: For any real number m , $\bar{K}_{< m}$ (resp. $\bar{K}_{\leq m}$) is defined by the composite field of all finite Galois extensions of K such that $m_{L/K} < m$ (resp. $m_{L/K} \leq m$). Then we put

$$G_K^{\geq m} := \text{Gal}(\bar{K}/\bar{K}_{< m}), \quad G_K^{> m} := \text{Gal}(\bar{K}/\bar{K}_{\leq m}),$$

that are closed normal subgroups of G_K . Clearly, these filtrations form decreasing filtrations of G_K .

Remark 2.4. The definition of $G_K^{\geq m}$ and $G_K^{> m}$ does not require the assumption that k is perfect, so that these filtrations can be defined even when the residue field of K is imperfect.

Let L be a finite separable extension of K . Choose an element $\alpha \in \mathcal{O}_L$ such that $\mathcal{O}_L = \mathcal{O}_K[\alpha]$. Let $P(T) \in \mathcal{O}_K[T]$ be the minimal polynomial of α over K and $\alpha = \alpha_1, \dots, \alpha_n$ the zeros of P in \bar{K} .

Proposition 2.5. *Let L be a finite Galois extension of K and m a real number. Then the following conditions are equivalent:*

- (i) L/K is unramified.
- (ii) $m_{L/K} \leq 0$.
- (iii) $m_{L/K} < 1$.

Proof. First, assume L/K is unramified. Suppose there exists an \mathcal{O}_K -algebra homomorphism $\eta : \mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K}^m$ for an algebraic extension E of K and $m > 0$. Since \mathcal{O}_L is formally étale as an \mathcal{O}_K -algebra, we see η lifts uniquely to an \mathcal{O}_K -algebra homomorphism $\mathcal{O}_L \rightarrow \mathcal{O}_E$ (cf. [GD], 0_{IV}.19.10.2). Thus (i) implies (ii). Since it is clear that (ii) implies (iii), it is enough to verify that (iii) implies (i). Assume $[L : K] \geq 2$. Take $E = K$ and choose a uniformizer β of K . Then the map $\mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K}^1$ defined by $\alpha \mapsto \beta$ is an \mathcal{O}_K -algebra homomorphism since $v_K(P(\beta)) = 1$. However, there is no K -embedding $L \hookrightarrow E$. Therefore, (P_m) is not true for L/K and $m = 1$. Hence the result follows. \square

By the properties of the number $m_{L/K}$, our filtration $(G_K^{[m]})_{m \in \mathbb{R}}$ has the following properties:

Theorem 2.6. *Let m be a real number. Then we have:*

- (i) $G_K^{\geq m} = G_K$ if $m \leq 0$. Moreover, we have $\bigcap_m G_K^{\geq m} = 1$ and $\bigcup_m G_K^{\geq m} = G_K$.
- (ii) Let K' be a finite separable extension of K , of ramification index e' . We identify the Galois group $G_{K'} := \text{Gal}(\bar{K}/K')$ with a subgroup of G_K . Then, for a real number $m > 0$, we have $G_{K'}^{\geq e'm} \subset G_K^{\geq m}$, with equality if K'/K is unramified.
- (iii) For a real number $0 < m \leq 1$, $G_K^{\geq m}$ is the inertia subgroup of G_K .

Proof. The assertion (i) follows from Proposition 2.2. (ii) follows from Proposition 2.2. (iii) follows from Proposition 2.5. \square

Remark 2.7. We can prove that $G_K^{>1}$ is the wild inertia subgroup of G_K by using the property (P_m) together with the classical theory of Herbrand functions in a similar way to the proof of Proposition 1.5, (ii), of [Fo]. However, we restricted ourselves here to showing what can be proved rather directly from (P_m) .

3 Ramification breaks

In this section, we compare our ramification filtration with the classical one. First, we recall the classical ramification theory for separable extensions of K studied in [De] and [He]. Let L be a finite separable extension of K . Put $H_K(L) := \text{Hom}_K(L, \bar{K})$. The order function $\mathbf{i}_{L/K}$ is defined on $H_K(L)$ by

$$\mathbf{i}_{L/K}(\sigma) := \inf_{a \in \mathcal{O}_L} v_K(\sigma(a) - a), \quad \sigma \in H_K(L).$$

Then the *ramification sets* $H_{K,i}(L)$ in the lower numbering are defined for a real number i by

$$H_{K,i}(L) := \{\sigma \in H_K(L) \mid \mathbf{i}_{L/K}(\sigma) \geq i\}.$$

The transition function $\tilde{\varphi}_{L/K} : \mathbb{R} \rightarrow \mathbb{R}$ of L/K is defined by

$$\tilde{\varphi}_{L/K}(u) := \int_0^u \text{card}(H_{K,t}(L)) dt$$

where $\text{card}(H_{K,t}(L))$ is the cardinality of $H_{K,t}(L)$. We also define the order function $\mathbf{u}_{L/K}$ by

$$\mathbf{u}_{L/K}(\sigma) := \tilde{\varphi}_{L/K}(\mathbf{i}_{L/K}(\sigma)).$$

Then the *ramification sets* $H_K^u(L)$ in the upper numbering are defined for a real number u by

$$H_K^u(L) := \{\sigma \in H_K(L) \mid \mathbf{u}_{L/K}(\sigma) \geq u\}.$$

If L is a Galois extension of K with Galois group G , then we have $H_K^u(L) = G^{(u)}$ by definition, where the latter is the upper numbering ramification group in the sense of [Fo]. Namely, we put $G^{(u)} := G^{u-1}$, where G^u is the upper numbering ramification group defined in [Se].

Proposition 3.1 ([De], Prop. 6.1). *Let u be a real number and L a finite separable extension of K . We denote by 1 the identity map of L into \bar{K} . Then the following conditions are equivalent:*

- (i) $H_K^u(\bar{L}) = 1$, where \bar{L} is the Galois closure of L over K .
- (ii) $H_K^u(L) = 1$.

Let L be a finite separable extension of K . We denote by $u_{L/K}$ the greatest upper ramification break of L/K defined by

$$u_{L/K} := \inf\{u \in \mathbb{R} \mid H_K^u(L) = 1\}.$$

We put $u_{K/K} = -\infty$ by convention. The next lemma is a basic property of the number $u_{L/K}$:

Lemma 3.2. *For finite separable extensions $K \subset M \subset L$, we have $u_{M/K} \leq u_{L/K}$.*

Proof. We may assume L and M are Galois extensions of K by Lemma 3.1. By the compatibility of the quotient, $\text{Gal}(L/K)^{(u)} = 1$ implies $\text{Gal}(M/K)^{(u)} = 1$ for a real number u (cf. Def. 5.1). Thus we obtain the inequality. \square

Fontaine proved the following proposition:

Proposition 3.3 ([Fo], Prop. 1.5). *Let L be a finite Galois extension of K and m a real number. Then there are the following relations:*

- (i) *If we have $m > u_{L/K}$, then (P_m) is true.*
- (ii) *If (P_m) is true, then we have $m > u_{L/K} - e_{L/K}^{-1}$.*

By this proposition, we have the inequalities

$$u_{L/K} - e_{L/K}^{-1} \leq m_{L/K} \leq u_{L/K},$$

for a finite separable extension L of K . More precisely, we have the following equality:

Proposition 3.4. *For a finite separable extension L of K , we have $m_{L/K} = u_{L/K}$.*

Proof. We may assume L/K is a Galois extension by Lemma 3.1. It is enough to show that (P_m) is not true for L/K and $m < u_{L/K}$. The two numbers $m_{L/K}$ and $u_{L/K}$ are stable under unramified base change. Thus we may assume L/K is a totally ramified extension. Moreover, we may assume $L \neq K$ because if $L = K$ then $m_{L/K} = u_{L/K} = -\infty$. If L/K is a tamely ramified extension, (P_m) is not true even for $m = u_{L/K} = 1$ because we can find a counter-example to (P_m) for $m = u_{L/K}$ as follows: Take $E = K$ and choose a uniformizer β of K . Then the map $\mathcal{O}_L \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K}^m$ defined by $\alpha \mapsto \beta$ is an \mathcal{O}_K -algebra homomorphism. However, there is no K -embedding $L \hookrightarrow E$. Therefore, we may assume L/K is a wildly ramified extension. To prove this proposition, we shall find a counter-example to (P_m) for L/K and $m = u_{L/K} - e'^{-1}$, where e' can be taken an arbitrarily large number. Take a finite tamely ramified Galois extension K' of K . Put $L' := LK'$ and $e' := e_{L'/K}$. If we apply (ii) of Proposition 3.3 to L'/K , then there exists an algebraic extension E of K such that there exists an \mathcal{O}_K -algebra homomorphism $\eta : \mathcal{O}_{L'} \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K}^{m_0}$, but there is no K -embedding $L' \hookrightarrow E$, where $m_0 := u_{L'/K} - e'^{-1}$. By Lemma 3.2, we have $m_0 \geq m_1$, where $m_1 := u_{L/K} - e'^{-1}$. Consider the two \mathcal{O}_K -algebra homomorphisms defined by composite maps:

$$\eta' : \mathcal{O}_L \hookrightarrow \mathcal{O}_{L'} \xrightarrow{\eta} \mathcal{O}_E/\mathfrak{a}_{E/K}^{m_0} \rightarrow \mathcal{O}_E/\mathfrak{a}_{E/K}^{m_1}, \quad \eta'' : \mathcal{O}_{K'} \hookrightarrow \mathcal{O}_{L'} \xrightarrow{\eta} \mathcal{O}_E/\mathfrak{a}_{E/K}^{m_0}.$$

Since K'/K is a tamely ramified extension, we have $u_{K'/K} \leq 1$. On the other hand, since L'/K is a wildly ramified extension, we have $e'm_0 > e'$ as shown in the proof of [Fo], Proposition 1.5, (ii), hence we deduce $m_0 > 1$. Thus we have $m_0 > u_{K'/K}$. According to (i) of Proposition 3.3 for K'/K , there exists a K -embedding $K' \hookrightarrow E$ corresponding to η'' . If we suppose there exists a K -embedding $L \hookrightarrow E$, then there exists a K -embedding $L' = LK' \hookrightarrow E$ since L/K and K'/K are Galois extensions. This is a contradiction. Therefore, (P_m) is not true for L/K and $m = m_1$. Hence the result follows. \square

Remark 3.5. By Proposition 2.3, Lemma 3.2 and Proposition 3.4, we deduce the equality $u_{LK'/K} = \max\{u_{L/K}, u_{K'/K}\}$ for any finite separable extensions L and K' of K .

Theorem 1.1 follows from Proposition 3.4 and Proposition 5.4.

4 Abbes and Saito's ramification theory

First, we recall the ramification theory of Abbes and Saito ([AS1], [AS2]). In Subsection 4.1, we generalize the property (P_m) to the imperfect residue field case. In Subsection 4.2, we translate our results in Section 3 to the language of Abbes and Saito's ramification theory. Let K be a complete discrete valuation field whose residue field may not be perfect and G_K the absolute Galois group of K . Abbes and Saito defined a decreasing filtration $(G_K^m)_{m \geq 0}$ by closed normal subgroups G_K^m of G_K indexed with rational numbers $m \geq 0$, in such a way that $\bigcap_{m \geq 0} G_K^m = 1$, $G_K^0 = G_K$ and G_K^1 is the inertia subgroup of G_K . It is defined by using certain functors F and F^m from the category \mathcal{FE}_K of finite étale K -algebras to the category \mathcal{S}_K of finite G_K -sets. We recall here the definition of F and F^m assuming for simplicity that m is a positive integer. Let

L be a finite étale K -algebra, and let \mathcal{O}_L be the integral closure of \mathcal{O}_K in L . We define $F(L) := \text{Hom}_K(L, \bar{K}) = \text{Hom}_{\mathcal{O}_K}(\mathcal{O}_L, \mathcal{O}_{\bar{K}})$. The functor F gives an anti-equivalence of \mathcal{FE}_K with \mathcal{S}_K , thereby making \mathcal{FE}_K a Galois category. To define F^m , we proceed as follows: An *embedding* of \mathcal{O}_L is a pair $(\mathbb{B}, \mathbb{B} \rightarrow \mathcal{O}_L)$ consisting of an \mathcal{O}_K -algebra \mathbb{B} which is formally of finite type and formally smooth over \mathcal{O}_K and a surjection $\mathbb{B} \rightarrow \mathcal{O}_L$ of \mathcal{O}_K -algebras which induces an isomorphism $\mathbb{B}/\mathfrak{m}_{\mathbb{B}} \rightarrow \mathcal{O}_L/\mathfrak{m}_L$, where $\mathfrak{m}_{\mathbb{B}}$ and \mathfrak{m}_L are respectively the radicals of \mathbb{B} and \mathcal{O}_L (cf. [AS2], Def. 1.1). Let I be the kernel of the surjection $\mathbb{B} \rightarrow \mathcal{O}_L$. Define an affinoid algebra \mathcal{B}^m over K by $\mathcal{B}^m := \mathbb{B}[I/\pi_K^m]^\wedge \otimes_{\mathcal{O}_K} K$, where π_K is a uniformizer of K and \wedge means the π_K -adic completion. Let $X^m(\mathbb{B} \rightarrow \mathcal{O}_L)$ be the affinoid variety $\text{Sp}(\mathcal{B}^m)$ associated with \mathcal{B}^m . For any affinoid variety X over K , let $\pi_0(X_{\bar{K}})$ denote the set $\varprojlim_{K'} \pi_0(X \otimes_K K')$ of geometric connected components, where K' runs through the finite separable extensions of K . Then we define the functor F^m by

$$F^m(L) := \varprojlim_{(\mathbb{B} \rightarrow \mathcal{O}_L)} \pi_0(X^m(\mathbb{B} \rightarrow \mathcal{O}_L)_{\bar{K}}),$$

where $(\mathbb{B} \rightarrow \mathcal{O}_L)$ runs through the category of embeddings of \mathcal{O}_L (cf. [AS2], Def. 1.1). The projective system in the right-hand side is constant ([AS1], Lem. 3.1). The finite set $F(L)$ can be identified with a subset of $X^m(\mathbb{B} \rightarrow \mathcal{O}_L)_{\bar{K}}$, and this causes a natural surjective map $F(L) \rightarrow F^m(L)$. The m -th ramification subgroup G_K^m is characterized by the property that $F(L)/G_K^m = F^m(L)$ for all L . If the residue field of K is perfect, this filtration $(G_K^m)_m$ defined as above coincides with the classical one $(G_K^{(m)})_m$ defined in Section 3 (cf. [AS1], Ex. 6.1).

4.1 Generalization of Fontaine's proposition

In this subsection, we generalize Fontaine's proposition to the imperfect residue field case. Hence we assume the residue field of K may not be perfect once more. Let L be a finite separable extension of K and m a rational number. If X is an affinoid variety over K and x is a point of X , we denote by $K(x)$ the definition field of X over K and by X_x the geometric connected component of X which contains x . The ring \mathcal{O}_L is a complete intersection over \mathcal{O}_K . Namely, we have $\mathcal{O}_L \simeq \mathcal{O}_K[T_1, \dots, T_n]/(f_1, \dots, f_n)$ ([AS1], Lem. 7.1). We denote by z_1, \dots, z_d the common zeros of f_1, \dots, f_n in \bar{K}^n . Let $I := (f_1, \dots, f_n)$ be the ideal of \mathcal{O}_L generated by f_1, \dots, f_n . Define a surjection $\mathcal{O}_K[T_1, \dots, T_n] \rightarrow \mathcal{O}_L$ by $T_i \mapsto z_i$. Then the formal completion $\mathbb{B} \rightarrow \mathcal{O}_L$ of $\mathcal{O}_K[T_1, \dots, T_n] \rightarrow \mathcal{O}_L$, where $\mathbb{B} := \varprojlim_r \mathcal{O}_K[T_1, \dots, T_n]/I^r$, is an embedding of \mathcal{O}_L . Let $X^m := X^m(\mathbb{B} \rightarrow \mathcal{O}_L)$ be the affinoid variety over K associated with this embedding. Then we have

$$X^m(K^{\text{alg}}) = \{x \in \mathcal{O}_{K^{\text{alg}}}^n \mid v_K(f_i(x)) \geq m \ (i = 1, \dots, n)\}.$$

Remark 4.1. If m is not a rational number, then X^m does not form a K -affinoid variety.

Lemma 4.2. *Let m be a rational number and E/K an algebraic extension. Then, the map $X^m(E) \rightarrow \text{Hom}_{\mathcal{O}_K}(\mathcal{O}_L, \mathcal{O}_E/\mathfrak{a}_{E/K}^m)$ sending (x_1, \dots, x_n) to the homomorphism defined by $z_i \mapsto x_i \pmod{\mathfrak{a}_{E/K}^m}$, is surjective.*

Proof. Obvious. □

Consider the following property for L/K and m :

(P' _{m}) *For any $x \in X^m(K^{\text{alg}})$, there exists a common zero z of f_1, \dots, f_n in \bar{K}^n which is $K(x)$ -rational.*

We can easily check that (P' _{m}) is equivalent to (P _{m}) if m is a non-negative rational number by Lemma 4.2. On the other hand, we consider the following property for L/K and m :

(Q' _{m}) *For any $x \in X^m(K^{\text{alg}})$, there exists a common zero z of f_1, \dots, f_n in \bar{K}^n such that $x \notin X_{z_i}^m$ for any z_i except z .*

By definition, the property (Q' _{m}) is equivalent to the bijectivity of $F(L) \rightarrow F^m(L)$. Let $c_{L/K}$ be the conductor of L/K , which is defined by $c_{L/K} := \inf \{m \in \mathbb{Q}_{\geq 0} \mid (Q'_m) \text{ is true for } L/K.\}$ ([AS1], Def. 6.3). Then we can show the following proposition which is a generalization of (i) of Proposition 3.3 to the imperfect residue field case:

Proposition 4.3. *If (Q' _{m}) is true, then (P' _{m}) is true. In particular, we have the inequality $m_{L/K} \leq c_{L/K}$ for a finite separable extension L of K .*

Proof. We need the following lemma which is a version of Krasner's lemma. This is due to Hiranouchi and Taguchi.

Lemma 4.4. *Let X be an affinoid variety over K , and let x, y be two points of $X(K^{\text{alg}})$. Assume the G_K -conjugates of x are contained in different geometric connected components of X each other and that y is in the geometric connected component X_x which contains x . Then $K(x) \subset K(y)$.*

This lemma is proved in the same way as the classical one.

Proof. If $\sigma \in \text{Hom}_{K(y)}(K(x, y), K^{\text{alg}})$, we have $X_{x^\sigma} = X_x$ and hence σ fixes x by the assumption on x . Thus we have $K(x) \subset K(y)$. □

Now we can prove Proposition 4.3 as follows: Let m be a non-negative rational number and x a point of $X^m(K^{\text{alg}})$. By the property (Q' _{m}), there exists a zero z of f_1, \dots, f_n such that $x \in X_z^m$ but $x \notin X_{z_i}^m$ for any $z_i \neq z$. Then $K(z) \subset K(x)$ by Lemma 4.4. Hence (P' _{m}) is true. □

Remark 4.5. The author does not know whether the equality $m_{L/K} = c_{L/K}$ remains true in the case where the residue field of K is imperfect.

4.2 Comparison with Abbes and Saito's ramification theory

In this subsection, we translate our results in Section 3 to the language of Abbes and Saito's ramification theory. Let K be a complete discrete valuation field with perfect residue field and L a finite separable extension of K . We define an ultra-metric norm on \bar{K} by $|y| = \theta^{v_K(y)}$, where $0 < \theta < 1$ is a real number. Fix a generator z of \mathcal{O}_L as an \mathcal{O}_K -algebra. Let P be the minimal polynomial of z over K , $z = z_1, \dots, z_d$ the zeros of P in \bar{K} . Let X^m be the affinoid variety over K as defined in the previous subsection, so that $X^m(K^{\text{alg}}) = \{x \in \mathcal{O}_{K^{\text{alg}}} \mid v_K(P(x)) \geq m\}$. If the residue field of K is perfect, we can rewrite (P'_m) for L/K and a non-negative rational number m as follows:

(P''_m) For any $x \in X^m(K^{\text{alg}})$, there exists a zero z of P in \bar{K} which is $K(x)$ -rational.

For a zero z of P in \bar{K} and rational number $r > 0$, the set $\{x \in \bar{K} \mid |z - x| < r\}$ is connected space which contains z . Hence (Q'_m) is rewritten as follows:

(Q''_m) For any $x \in X^m(K^{\text{alg}})$, there exists a zero z of P in \bar{K} such that $|z - x| = \min_i |x - z_i|$ and $|z - x| < \min_{i \neq j} |z_i - z_j|$.

Since Abbes and Saito's filtration coincides with the classical one, we have $u_{L/K} = c_{L/K}$. Then we have the following proposition:

Proposition 4.6. *Let L be a finite Galois extension of K and m a rational number. Then we have the following relations:*

- (i) *If (Q''_m) is true, then (P''_m) is true.*
- (ii) *If (P''_m) is true, then $(Q''_{m+\varepsilon})$ is true for any $\varepsilon > 0$.*

In particular, we have the equality $m_{L/K} = c_{L/K}$.

Proof. The above (i) is the special case of Proposition 4.3. (ii) follows from Proposition 3.4 and the equality $u_{L/K} = c_{L/K}$. \square

5 Appendix

In this section, we prove a Galois theoretic property on filtration of the absolute Galois group of an arbitrary field. This section is independent from the other sections. Let K be a field, \bar{K} a fixed separable closure of K , $G_K := \text{Gal}(\bar{K}/K)$ the absolute Galois group of K and R a totally ordered set. We denote by $-\infty$ a fixed element which satisfies $-\infty \leq u$ for any $u \in R$ by convention.

Definition 5.1. Let L be a finite Galois extension of K with Galois group G and $(G^u)_{u \in R}$ a decreasing filtration on G . Then we say that the filtration $(G^u)_{u \in R}$ has the *compatibility with the quotient* if, for any $u \in R$ and a finite Galois sub-extension M/K of L/K with Galois group H , we have

$$G^u H/H = (G/H)^u.$$

Proposition 5.2. *The following conditions are equivalent:*

- (i) *There exists a decreasing filtration $(G_K^u)_{u \in R}$ on G_K .*
- (ii) *There exists a decreasing filtration $(G^u)_{u \in R}$ on $G := \text{Gal}(L/K)$ which has the compatibility with the quotient for any finite Galois extension L/K .*

Proof. We assume (i). Let L be a finite Galois extension of K with Galois group G . Then a decreasing filtration G^u can be defined by the image of G_K^u by the restriction map $G_K \rightarrow G$. Conversely, we suppose (ii). For any sequence of finite Galois extensions $K \subset L' \subset L$, the compatibility with the quotient induces a natural projection $\text{Gal}(L/K)^u \rightarrow \text{Gal}(L'/K)^u$ by the restriction map. Hence we can define a decreasing filtration G_K^u on G_K by

$$G_K^u := \varprojlim \text{Gal}(L/K)^u,$$

where L runs through the set of all finite Galois extensions of K . \square

Definition 5.3. Let G be a set and $(G^u)_{u \in R}$ a decreasing filtration on G . Then we say that $(G^u)_{u \in R}$ is *separating* if $\bigcap_u G^u = 1$ and $(G^u)_{u \in R}$ is *left continuous* if $G^u = \bigcap_{m < u} G^m$.

Let $(G_K^u)_{u \in R}$ be a decreasing filtration on G_K which is separating and left continuous, and L a finite Galois extension of K with Galois group G . Then we denote by $\bar{K}_{(u)}$ (resp. $\bar{K}_{(u+)}$) the fixed field of \bar{K} by G_K^u . Put $G_K^{u+} := \bigcup_{u' > u} G_K^{u'}$. Define G^u (resp. G^{u+}) as the image of G_K^u (resp. G_K^{u+}) by the restriction map $\pi : G_K \rightarrow G$. Put

$$u_{L/K} := \inf\{u \in R \mid G^u = 1\}.$$

We denote by $\bar{K}_{<u}$ (resp. $\bar{K}_{\leq u}$) the union of all finite Galois extension L of K in \bar{K} such that $u_{L/K} < u$ (resp. $u_{L/K} \leq u$).

Proposition 5.4. *We have $\bar{K}_{<u} = \bar{K}_{(u)}$ and $\bar{K}_{\leq u} = \bar{K}_{(u+)}$ for any $u \in R$.*

Proof. If L is a finite Galois extension of K with Galois group G , then the left continuousness makes $G^{u_{L/K}} \neq 1$. Hence $u_{L/K} < u$ (resp. $u_{L/K} \leq u$) is equivalent to $G^u = 1$ (resp. $G^{u+} = 1$). This is equivalent to $G_K^u \subset \text{Ker}(\pi) = G_L$ (resp. $G_K^{u+} \subset G_L$). The result follows it. \square

References

- [AS1] A. Abbes and T. Saito, *Ramification of local fields with imperfect residue fields I*, Amer. J. Math. **124** (2002), 879–920.
- [AS2] _____, *Ramification of local fields with imperfect residue fields II*, Documenta Math. Extra volume: Kazuya Kato’s Fiftieth Birthday (2003), 5–27.
- [CL] X. Caruso and T. Liu, *Some bound for ramification of p^n -torsion semi-stable representations*, arXiv:0805.4227v2 [math.NT] (2008).
- [Co] P. Colmez, *Conducteur d’Artin d’une représentation de de Rham*, Astérisque **319** (2008), 187–212.
- [De] P. Deligne, *Les corps locaux de caractéristique p , limites de corps locaux de caractéristique 0*, Representations of reductive groups over a local field, Travaux en Cours, Hermann, Paris, 1984, pp. 119–157.
- [Fo] J.-M. Fontaine, *Il n’y a pas de variété abélienne sur \mathbf{Z}* , Invent. Math. (1985), no. 3, 515–538.

- [GD] A. Grothendieck, J. Dieudonné, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas*, Inst. Hautes Études Sc. Publ. Math., **20**, **24**, **28**, **32** (1964–67).
- [Ha1] S. Hattori, *On a ramification bound of semi-stable mod p^n representations over a local field*, arXiv:0801.2149v3 [math.NT] (2008).
- [Ha2] _____, *Ramification of a finite flat group scheme over a local field*, J. Number Theory **118** (2006), 145–154.
- [He] C. Helou, *On the ramification breaks*, Comm. Algebra **19** (1991), no. 8, 2267–2279.
- [Se] J.-P. Serre, *Local Fields*, Graduate Texts in Mathematics **67**, Springer-Verlag (1979).

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