Journal of Math-for-industry, Vol.2(2010A-5) , pp.39-56

Decay estimates on solutions of the linearized compressible Navier-Stokes equation around a Poiseuille type flow

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Received on March 3, 2010

Abstract. Decay estimates on solutions to the linearized compressible Navier-Stokes equation around a Poiseuille type flow are established. It is shown that if the Reynolds and Mach numbers are sufficiently small, solutions of the linearized problem decay in L^2 norm as an $n-1$ dimensional heat kernel. Furthermore, it is proved that the asymptotic leading part of solutions is given by solutions of an $n-1$ dimensional linear heat equation with a convective term.

Keywords. compressible Navier-Stokes equation, decay estimates, asymptotic behavior, Poiseuille type flow

1. Introduction

This paper is concerned with the asymptotic behavior of solutions to the compressible Navier-Stokes equation around a Poiseuille type flow.

We consider the system of equations

$$\partial_t \rho + \mathrm{div}\,(\rho v) = 0, \tag{1.1}$$

$$\rho(\partial_t v + v\cdot\nabla v) - \mu\Delta v - (\mu+\mu')\nabla \mathrm{div}\, v + \nabla P(\rho) = g\rho \boldsymbol{f}_\alpha \tag{1.2}$$

in an n dimensional infinite layer $\Omega_\ell = \mathbf{R}^{n-1}\times(0,\ell)$:

$$\Omega_\ell = \{x=(x',x_n)\,; \; x'=(x_1,\cdots,x_{n-1})\in\mathbf{R}^{n-1},\, 0<x_n<\ell\}.$$

Here $n\ge 2$; $\rho=\rho(x,t)$ and $v=(v^1(x,t),\cdots,v^n(x,t))$ denote the unknown density and velocity at time $t\ge 0$ and position $x\in\Omega_\ell$, respectively ; $P(\rho)=K\rho^a$ is the pressure, where $K>0$ and $a\ge 1$ are constants; μ and μ' are the viscosity coefficients that are assumed to be constants satisfying $\mu>0$, $\frac{2}{n}\mu+\mu'\ge 0$; g is the gravity constant; and $\boldsymbol{f}_\alpha$ is an external force of the form $\boldsymbol{f}_\alpha = (\sin\alpha, 0,\cdots,0,-\cos\alpha)$ with a constant α.

The system (1.1)–(1.2) is considered under the boundary condition

$$v|_{x_n=0,\ell} = 0. \tag{1.3}$$

It is not difficult to see that the problem (1.1)–(1.3) has the stationary solution $\overline{u}_s = (\overline{\rho}_s, \overline{v}_s)$:

$$\overline{\rho}_s = \begin{cases} \left(\rho_*^{a-1} + \frac{(a-1)g\cos\alpha}{aK}(\ell - x_n)\right)^{\frac{1}{a-1}} & \text{if } a>1,\\ \rho_* e^{\frac{g\cos\alpha}{K}(\ell-x_n)} & \text{if } a=1,\end{cases}$$

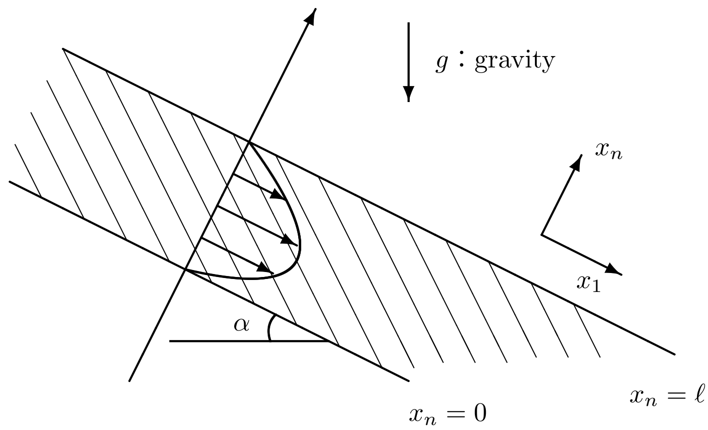
$$\overline{v}_s = (\overline{v}_s^1(x_n)\,,\,0\,,\,\cdots,0),$$

$$\overline{v}_s^1(x_n) = \frac{g\sin\alpha}{\mu}\int_0^l G_\ell(x_n,y_n)\overline{\rho_s}(y_n)\,dy_n,$$

with

$$G_\ell(x_n,y_n) = \begin{cases} \frac{1}{\ell}(\ell-x_n)y_n & (0<y_n<x_n)\\ \frac{1}{\ell}x_n(\ell-y_n) & (x_n<y_n<\ell)\,.\end{cases}$$

Here ρ_* is a given positive constant. See the figure below.



We are interested in the large time behavior of solutions to problem (1.1)–(1.3) when the initial value (ρ_0, v_0) is sufficiently close to the stationary solution $\overline{u}_s = (\overline{\rho}_s, \overline{v}_s)$.

The stability problem of flows in an infinite layer or cylindrical domains has been widely studied as a good subject to study pattern formation phenomena and turbulent flows.

Basic analysis with mathematical rigor has been well established for incompressible flows in these domains. For example, one can systematically analyze qualitative properties of solutions of the governing equations such as the stability and bifurcation, since the incompressible Navier-Stokes equation can be treated in a category of semilinear parabolic equations (e.g., [1, 2, 3, 4, 5, 6, 18, 24, 25]). On the other hand, it seems that there has been not so much mathematically rigorous analysis for compressible flows in these domains. This is because the governing equations for compressible flows are hyperbolic-parabolic systems of quasilinear equations; and, thus, the extension of the analysis for the incompressible case to the compressible one is not straightforward. As a first step of the mathematical analysis for compressible flows, we have begun to investigate the dynamics around simple flows such as the motionless state and parallel flows. The stability of the motionless state was investigated in [9, 10, 11] and it was shown that the motionless state is stable for sufficiently small initial disturbances and the disturbances behave in large time as solutions of an $n-1$ dimensional linear heat equation. A similar result also holds for the case of cylindrical domains ([8, 14]). As for parallel flows, it was proved in [12] that the plane Couette flow is asymptotically stable for sufficiently small initial disturbances if the Reynolds number and Mach number are sufficiently small. Furthermore, the disturbances behave in large time as solutions of an $n-1$ dimensional linear heat equation with a convective term. This kind of diffusive behavior is different from the case of unbounded domains such as the whole space, half space and exterior domains, where hyperbolic aspect of the system also appears in the asymptotic leading part of solutions in large time (e.g., [7, 13, 16, 17, 19, 20, 21, 22]).

Our next step is to extend the analysis of the plane Couette flow to the case of a Poiseuille type flow. In this paper we will consider the linearized problem around such a flow and establish decay estimates on solutions similar to those in the case of the plane Couette flow. Based on the analysis in this paper, the nonlinear problem will be treated elsewhere.

Our main results of this paper are summarized as follows. A non-dimensional form of the linearized system is written as

$$\partial_t u + Lu = 0 \tag{1.4}$$

on the domain $\Omega = \mathbf{R}^{n-1} \times (0,1)$:

$$\Omega = \{x=(x',x_n)\,;\ x'=(x_1,\cdots,x_{n-1})\in\mathbf{R}^{n-1},\ 0<x_n<1\}$$

under the initial and boundary conditions

$$u|_{t=0} = u_0, \quad w|_{x_n=0,1} = 0. \tag{1.5}$$

Here $u = {}^T(\phi, w)$ and

$$L = \begin{pmatrix} v_s^1\partial_{x_1} & \gamma^2 \mathrm{div}\,(\rho_s\,\cdot) \\ \nabla\left(\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\,\cdot\right) & -\frac{\nu}{\rho_s}\Delta I_n - \frac{(\nu+\nu')}{\rho_s}\nabla\mathrm{div} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\frac{\nu}{\gamma^2\rho_s}\boldsymbol{e}_1 & v_s^1\partial_{x_1}I_n + (\partial_{x_n}v_s^1)\boldsymbol{e}_1{}^T\boldsymbol{e}_n \end{pmatrix},$$

where ρ_s, v_s^1 and $\tilde{P}'(\rho_s)$ are the non-dimensional form of $\overline{\rho}_s$, $\overline{v}_s^1$ and $P'(\overline{\rho}_s)$ respectively; I_n denotes the $n\times n$ identity matrix; $\boldsymbol{e}_j$ denotes the unit vector in x_j direction; $\boldsymbol{e}_1{}^T\boldsymbol{e}_n$ is the matrix with (i,j) components given by $\delta_{i1}\delta_{nj}$; ν, ν' and γ^2 are some positive constants. Here and in what follows the superscript ${}^T\cdot$ means transposition. The Reynolds number Re and Mach number Ma are given by $Re = 1/\nu = \mu/(\rho_*\ell V)$ and $Ma = 1/\gamma = V/\sqrt{P'(\rho_*)}$, respectively, with $V = (\rho_* g\ell^2\sin\alpha)/\mu$. We also introduce the parameter $\omega = ((a-1)g\ell\cos\alpha)/(a\rho_*^{a-1}K)$ if $a>1$ and $\omega = (g\ell\cos\alpha)/K$ if $a=1$. We will prove that there exist positive numbers Re_0, Ma_0 and ω_0 such that if $Re\le Re_0$, $Ma\le Ma_0$ and $|\omega|\le\omega_0$, then the solution $u(t) = (\phi(t), w(t))$ of the linearized problem (1.4)–(1.5) satisfies

$$\|\partial_{x'}^k\partial_{x_n}^l u(t)\|_2 \le C\{t^{-\frac{n-1}{4}-\frac{k}{2}}\|u_0\|_{L^1} + e^{-dt}(\|u_0\|_{H^1\times L^2} + \|\partial_{x'}w_0\|_{L^2})\} \tag{1.6}$$

for $t\ge1$, $k+l\le1$, and

$$\|u(t) - G_t *_{x'} \mathit{\Pi}^{(0)}u_0\|_{L^2} \le C\{t^{-\frac{n-1}{4}-\frac{1}{2}}\|u_0\|_{L^1} + e^{-dt}(\|u_0\|_{H^1\times L^2} + \|\partial_{x'}w_0\|_{L^2})\} \tag{1.7}$$

for $t\ge1$ with a function $G_t *_{x'} \mathit{\Pi}^{(0)}u_0$ whose Fourier transform in x' is given by

$$\mathscr{F}\left(G_t *_{x'} \mathit{\Pi}^{(0)}u_0\right) = e^{-(i\kappa_0\xi_1+\kappa_1\xi_1^2+\kappa_2|\xi''|^2)t}[\widehat{\phi}_0(\xi')]u^{(0)}.$$

Here $\xi' = (\xi_1,\xi'')\in\mathbf{R}^{n-1}$, $\xi''=(\xi_2,\cdots,\xi_{n-1})\in\mathbf{R}^{n-2}$; $u^{(0)}$ is a function of x_n only; $[\widehat{\phi}_0(\xi')]$ is a quantity given by

$$[\widehat{\phi}_0(\xi')] = \int_0^1 \widehat{\phi}_0(\xi',x_n)\,dx_n;$$

with $\widehat{\phi}_0$ being the Fourier transform of ϕ_0 in x'; and κ_j $(j=0,1,2)$ are positive constants depending on ρ_*, ℓ, V, μ, μ' and $P'(\rho_*)$. Precise statements of the results will be given in section 3.

As in the case of the plane Couette flow [12], these decay estimates will be useful for the nonlinear problem. In contrast to the case of the plane Couette flow, besides the decay estimates themselves, a decomposition argument in the proof will also play an important role in the study of the nonlinear problem.

To obtain the decay estimates, as in [12], we consider the Fourier transform of (1.4) in $x'\in\mathbf{R}^{n-1}$:

$$\partial_t\widehat{u} + \widehat{L}_{\xi'}\widehat{u} = 0, \quad \widehat{u}|_{t=0} = \widehat{u}_0,$$

where $\xi' = (\xi_1, \cdots, \xi_{n-1}) \in \mathbf{R}^{n-1}$ denotes the dual variable.

The operator $\widehat{L}_{\xi'}$ has different characters between the cases $|\xi'| \ll 1$ and $|\xi'| \gg 1$. We thus decompose the semigroup e^{-tL} associated with (1.4) into two parts: $e^{-tL} = \mathscr{F}^{-1}(e^{-t\widehat{L}_{\xi'}}|_{|\xi'|\le R}) + \mathscr{F}^{-1}(e^{-t\widehat{L}_{\xi'}}|_{|\xi'|\ge R})$ for some $R > 0$, where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform. To analyze the high frequency part $|\xi'| \ge R$, we employ the Fourier transformed version of Matsumura-Nishida's energy method and derive the exponential decay property of the corresponding part of the semigroup e^{-tL}. This can be done exactly in the same way as in [12]. To investigate the part of bounded frequencies $|\xi'| \le R$, we make use of a certain decomposition of the corresponding part of the semigroup based on the spectrum of the operator of the zero frequency $\xi' = 0$. Concerning this part we need more precise argument than that in [12]. The decomposition argument for the bounded frequency part will be also useful in the study of the nonlinear problem. Once the necessary estimates for the bounded frequency part are obtained, then the asymptotic behavior (1.7) follows from the analysis of the low frequency part $|\xi'| \ll 1$ in a similar manner to that in [12].

This paper is organized as follows. In section 2 we rewrite the problem into the system of equations for the disturbance in a non-dimensional form. Our main results in this paper are then stated in section 3. In section 4 we prove the decay estimates, and, in the proof, we employ a certain decomposition argument for the bounded frequency part. A proof of the asymptotic behavior (1.7) is then outlined in section 5.

2. Stationary Solution and Formulation of the Problem

In this section we rewrite the problem into the one for the disturbance in a non-dimensional form.

Let ρ_* be a given positive number. We will look for a stationary flow whose density $\overline{\rho}_s$ satisfies

$$\overline{\rho}_s|_{x_n=\ell} = \rho_*. \tag{2.1}$$

With this in mind we introduce the following dimensionless variables:

$$x = \ell\widetilde{x}, \quad t = \frac{\ell}{V}\widetilde{t}, \quad v = V\widetilde{v}, \quad \rho = \rho_*\widetilde{\rho}, \quad P = \rho_* V^2 \widetilde{P}$$

with

$$V = \frac{\rho_* g \ell^2 \sin\alpha}{\mu}.$$

Then the problem (1.1)–(1.3) is transformed into the following dimensionless problem on the layer $\Omega = \mathbf{R}^{n-1} \times (0,1)$:

$$\partial_t\widetilde{\rho} + \operatorname{div}(\widetilde{\rho}\widetilde{v}) = 0, \tag{2.2}$$

$$\widetilde{\rho}(\partial_t\widetilde{v} + \widetilde{v}\cdot\nabla\widetilde{v}) - \nu\Delta\widetilde{v} - (\nu+\nu')\nabla\operatorname{div}\widetilde{v} + \widetilde{P}'(\widetilde{\rho})\nabla\widetilde{\rho} = \beta^2\rho\boldsymbol{f}_\alpha \tag{2.3}$$

$$\widetilde{v}|_{x_n=0,1} = 0. \tag{2.4}$$

Here ν, ν' and β are the non-dimensional parameters:

$$\nu = \frac{\mu}{\rho_*\ell V}, \quad \nu' = \frac{\mu'}{\rho_*\ell V}, \quad \beta = \frac{\sqrt{g\ell}}{V}.$$

We also introduce a parameter γ:

$$\gamma = \sqrt{\widetilde{P}'(1)} = \frac{\sqrt{P'(\rho_*)}}{V}$$

and a parameter ω:

$$\omega = \begin{cases} \frac{(a-1)g\ell\cos\alpha}{a\rho_*^{a-1}K} & \text{if } a > 1, \\ \frac{g\ell\cos\alpha}{K} & \text{if } a = 1. \end{cases}$$

We state the existence of a stationary solution of Poiseuille type flow.

Proposition 2.1. *Assume that ω satisfies $\omega > -1$ if $a > 1$. Then problem (2.2)–(2.4) has a smooth stationary solution $u_s = (\rho_s, v_s)$ of the form:*

$$\rho_s(x_n) = \begin{cases} (1+\omega(1-x_n))^{\frac{1}{a-1}} & \text{if } a > 1, \\ e^{\omega(1-x_n)} & \text{if } a = 1, \end{cases}$$

$$v_s(x) = (v_s^1(x_n), 0, \cdots, 0)$$

with

$$v_s^1(x_n) = \int_0^1 G(x_n, y_n)\rho_s(y_n)dy_n.$$

Here

$$G(x_n, y_n) = \begin{cases} (1-x_n)y_n & (0 < y_n < x_n) \\ x_n(1-y_n) & (x_n < y_n < 1). \end{cases}$$

Furthermore, $u_s = (\rho_s, v_s)$ satisfies the following estimates:

$$\rho_s(x_n) \ge \begin{cases} (1+\min\{0,\omega\})^{\frac{1}{a-1}} > 0 & \text{if } a > 1, \\ e^{\min\{0,\omega\}} > 0 & \text{if } a = 1, \end{cases}$$

$$\rho_s(x_n) = 1 + O(\omega), \quad |\partial_{x_n}^k \rho_s(x_n)| \le C_k|\omega|^k,$$

$$v_s^1(x_n) = \frac{1}{2}x_n(1-x_n) + O(\omega), \quad |\partial_{x_n}^k v_s^1(x_n)| \le C_k,$$

$$\tilde{P}'(\rho_s(x_n)) = \gamma^2(1+O(\omega)), \quad |\partial_{x_n}^k \tilde{P}'(\rho_s(x_n))| \le C_k\gamma^2|\omega|^k,$$

uniformly in $x_n \in [0,1]$.

Proof. In view of (2.1) we look for a stationary solution of the form

$$\rho_s = \rho_s(x_n), \quad v_s = (v_s^1(x_n), 0, \cdots, 0), \quad \rho_s(1) = 1.$$

Then we see that (ρ_s, v_s^1) is a solution of

$$-\nu\partial_{x_n}^2 v_s^1 = \beta^2\rho_s\sin\alpha, \quad v_s^1|_{x_n=0,1} = 0, \tag{2.5}$$

(2.6) $$\tilde{P}'(\rho_s)\partial_{x_n}\rho_s = -\beta^2\rho_s\cos\alpha.$$

Since $\tilde{P}'(\rho_s) = \frac{aK\rho_*^{a-1}}{V^2}\rho_s^{a-1}$, it is easy to integrate (2.6). By using $\rho_s(1) = 1$, we obtain the desired ρ_s, and, then, v_s is obtained by inverting (2.5). It is straightforward to obtain the estimates on ρ_s and v_s. Furthermore, by noting $\tilde{P}'(1) = \gamma^2$, we have the desired estimates on $\tilde{P}'(\rho_s)$. This completes the proof. □

Remark. We note that the Reynolds number Re and Mach number Ma is given by $Re = \nu^{-1}$ and $Ma = \gamma^{-1}$, respectively.

Setting $\widetilde{\rho} = \rho_s + \gamma^{-2}\phi$ and $\widetilde{v} = v_s + w$ in (2.2)–(2.4), we arrive at the initial boundary value problem for the disturbance $u = (\phi, w)$:

(2.7) $$\partial_t\phi + v_s^1\partial_{x_1}\phi + \gamma^2\mathrm{div}\,(\rho_s w) = f^0(\phi, w),$$

(2.8)
$$\partial_t w - \tfrac{\nu}{\rho_s}\Delta w - \tfrac{(\nu+\nu')}{\rho_s}\nabla\mathrm{div}\,w + \nabla\left(\tfrac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\phi\right) + v_s^1\partial_{x_1}w + (\partial_{x_n}v_s^1)w^n\boldsymbol{e}_1 - \tfrac{\nu}{\gamma^2\rho_s}\phi\boldsymbol{e}_1 = \boldsymbol{g}(\phi, w),$$

(2.9) $$w|_{x_n=0,1} = 0,$$

(2.10) $$(\phi, w)|_{t=0} = (\phi_0, w_0).$$

Here
$$f^0(\phi, w) = -\mathrm{div}\,(\phi w),$$

$$\begin{aligned}\boldsymbol{g}(\phi, w) &= -w\cdot\nabla w + \tfrac{\nu\phi}{(\gamma^2\rho_s+\phi)\rho_s}\left\{-\Delta w + \tfrac{\phi}{\gamma^2\rho_s}\Delta v_s\right\}\\ &\quad - \tfrac{(\nu+\nu')\phi}{(\gamma^2\rho_s+\phi)\rho_s}\nabla\mathrm{div}\,w + \tfrac{\gamma^2}{(\gamma^2\rho_s+\phi)}\left\{\tfrac{\phi}{\gamma^2\rho_s}\nabla\left(\tfrac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\phi\right)\right.\\ &\quad \left. -\nabla\left(\gamma^{-4}\phi^2\tilde{P}_2(\gamma^{-2}\phi, \rho_s)\right)\right\}\end{aligned}$$

with
$$\tilde{P}_2(\phi, \rho_s) = \int_0^1 (1-\theta)\tilde{P}''(\rho_s + \theta\phi)\,d\theta.$$

3. Main Results

Our main concern in this paper is the estimates of solutions to the linearized problem, i.e., problem (2.7)–(2.10) with $f^0(\phi, w) = 0$ and $\boldsymbol{g}(\phi, w) = 0$.

We first introduce some notation which will be used throughout the paper. For a domain D and $1 \le p \le \infty$ we denote by $L^p(D)$ the usual Lebesgue space on D and its norm is denoted by $\|\cdot\|_{L^p(D)}$. Let m be a nonnegative integer. $W^{m,p}(D)$ denotes the m th order L^p Sobolev space on D with norm $\|\cdot\|_{W^{m,p}(D)}$. When $p = 2$, the space $W^{m,2}(D)$ is denoted by $H^m(D)$ and its norm is denoted by $\|\cdot\|_{H^m(D)}$. $C_0^m(D)$ stands for the set of all C^m functions which have compact support in D. We denote by $W_0^{1,p}(D)$ the completion of $C_0^1(D)$ in $W^{1,p}(D)$. In particular, $W_0^{1,2}(D)$ is denoted by $H_0^1(D)$.

We simply denote by $L^p(D)$ (resp., $W^{m,p}(D)$, $H^m(D)$) the set of all vector fields $w = {}^T(w^1, \cdots, w^n)$ on D with $w^j \in L^p(D)$ (resp., $W^{m,p}(D)$, $H^m(D)$), $j = 1, \cdots, n$, and its norm is also denoted by $\|\cdot\|_{L^p(D)}$ (resp., $\|\cdot\|_{W^{m,p}(D)}$, $\|\cdot\|_{H^m(D)}$). For $u = {}^T(\phi, w)$ with $\phi \in W^{k,p}(D)$ and $w = {}^T(w^1, \cdots, w^n) \in W^{m,q}(D)$, we define $\|u\|_{W^{k,p}(D)\times W^{m,q}(D)}$ by $\|u\|_{W^{k,p}(D)\times W^{m,q}(D)} = \|\phi\|_{W^{k,p}(D)} + \|w\|_{W^{m,q}(D)}$. When $k = m$ and $p = q$, we simply write $\|u\|_{W^{k,p}(D)\times W^{k,p}(D)} = \|u\|_{W^{k,p}(D)}$.

In case $D = \Omega$ we abbreviate $L^p(\Omega)$ (resp., $W^{m,p}(\Omega)$, $H^m(\Omega)$) as L^p (resp., $W^{m,p}$, H^m). In particular, the norm $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_{L^p}$ is denoted by $\|\cdot\|_p$.

In case $D = (0, 1)$ we denote the norm of $L^p(0, 1)$ by $|\cdot|_p$. The inner product of $L^2(0, 1)$ is denoted by

$$(f, g) = \int_0^1 f(x_n)\overline{g(x_n)}\,dx_n, \quad f, g \in L^2(0, 1).$$

Here $\overline{g}$ denotes the complex conjugate of g. For $u_j = {}^T(\phi_j, w_j) \in L^2(0, 1)$ with $w_j = {}^T(w_j^1, \cdots, w_j^n)$ $(j = 1, 2)$, we also define a weighted inner product $\langle u_1, u_2\rangle$ by

$$\langle u_1, u_2\rangle = \int_0^1 \phi_1\overline{\phi}_2\tfrac{\tilde{P}'(\rho_s)}{\gamma^4\rho_s}\,dx_n + \int_0^1 w_1\overline{w}_2\rho_s\,dx_n.$$

Furthermore, for $f \in L^1(0, 1)$ we denote the mean value of f in $(0, 1)$ by $[f]$:

$$[f] = (f, 1) = \int_0^1 f(x_n)\,dx_n.$$

For $u = {}^T(\phi, w) \in L^1(0, 1)$ with $w = {}^T(w^1, \cdots, w^n)$ we define $[u]$ by

$$[u] = [\phi] + [w^1] + \cdots + [w^n].$$

The norms of $W^{m,p}(0, 1)$ and $H^m(0, 1)$ are denoted by $|\cdot|_{W^{m,p}}$ and $|\cdot|_{H^m}$, respectively.

We often write $x \in \Omega$ as

$$x = {}^T(x', x_n), \quad x' = {}^T(x_1, \cdots, x_{n-1}) \in \mathbf{R}^{n-1}.$$

Partial derivatives of a function u in x, x', x_n and t are denoted by $\partial_x u$, $\partial_{x'}u$, $\partial_{x_n}u$ and $\partial_t u$, respectively. We also write higher order partial derivatives of u in x as $\partial_x^k u = (\partial_x^\alpha u;\, |\alpha| = k)$.

We denote the $k \times k$ identity matrix by I_k. In particular, when $k = n+1$, we simply write I for I_{n+1}. We also define $(n+1)\times(n+1)$ diagonal matrices Q_0, Q_n and $\widetilde{Q}$ by

$$Q_0 = \mathrm{diag}\,(1, 0, \cdots, 0), \quad Q_n = \mathrm{diag}\,(0, \cdots, 0, 1)$$

and
$$\widetilde{Q} = \mathrm{diag}\,(0, 1, \cdots, 1).$$

We then have, for $u = {}^T(\phi, w) \in \mathbf{R}^{n+1}$, $w = {}^T(w^1, \cdots, w^n)$,

$$Q_0 u = \begin{pmatrix}\phi\\0\end{pmatrix}, \quad Q_n u = \begin{pmatrix}0\\0\\w^n\end{pmatrix}, \quad \widetilde{Q}u = \begin{pmatrix}0\\w\end{pmatrix}.$$

We note that

$$[Q_0 u] = [\phi] \quad \text{for } u = {}^T(\phi, w).$$

We next introduce some notation about integral operators. For a function $f = f(x')$ ($x' \in \mathbf{R}^{n-1}$), we denote its Fourier transform by $\widehat{f}$ or $\mathscr{F} f$:

$$\widehat{f}(\xi') = (\mathscr{F} f)(\xi') = \int_{\mathbf{R}^{n-1}} f(x') e^{-i\xi'\cdot x'}\, dx'.$$

The inverse Fourier transform is denoted by $\mathscr{F}^{-1}$:

$$(\mathscr{F}^{-1} f)(x) = (2\pi)^{-(n-1)} \int_{\mathbf{R}^{n-1}} f(\xi') e^{i\xi'\cdot x'}\, d\xi'.$$

For a function $K(x_n, y_n)$ on $(0,1)\times(0,1)$ we will denote by Kf the integral operator $\int_0^1 K(x_n, y_n) f(y_n)\, dy_n$.

We will denote the resolvent set of a closed operator A by $\rho(A)$ and the spectrum of A by $\sigma(A)$. For $\Lambda \in \mathbf{R}$ and $\theta \in (\frac{\pi}{2}, \pi)$ we denote the set $\{\lambda \in \mathbf{C};\ |\arg(\lambda - \Lambda)| \le \theta\}$ by $\Sigma(\Lambda, \theta)$:

$$\Sigma(\Lambda, \theta) = \{\lambda \in \mathbf{C};\ |\arg(\lambda - \Lambda)| \le \theta\}.$$

We now state our main results.

Let us consider the linearized problem

$$\partial_t u + Lu = 0, \quad w|_{x_n=0,1} = 0, \quad u|_{t=0} = u_0. \tag{3.1}$$

Here $u = {}^T(\phi, w)$ and L is the operator of the form

$$L = \begin{pmatrix} v_s^1 \partial_{x_1} & \gamma^2 \mathrm{div}\,(\rho_s\, \cdot) \\ \nabla\left(\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s}\, \cdot\right) & -\frac{\nu}{\rho_s}\Delta I_n - \frac{(\nu+\nu')}{\rho_s}\nabla \mathrm{div} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\frac{\nu}{\gamma^2\rho_s}\boldsymbol{e}_1 & v_s^1\partial_{x_1} I_n + (\partial_{x_n} v_s^1)\boldsymbol{e}_1{}^T\boldsymbol{e}_n \end{pmatrix}.$$

We denote the solution operator for (3.1) by $\mathscr{U}(t)$.

Theorem 3.1. *Suppose that $u_0 = {}^T(\phi_0, w_0) \in H^1 \times L^2$ and that $\partial_{x'} w_0 \in L^2$. Then (3.1) has a unique solution $u(t) = \mathscr{U}(t)u_0$; and it satisfies the estimates*

$$\|\mathscr{U}(t)u_0\|_2 \le C\|u_0\|_2$$

and

$$\|\partial_x \mathscr{U}(t)u_0\|_2 \le C\{t^{-\frac{1}{2}}\|u_0\|_{H^1\times L^2} + \|\partial_{x'} w_0\|_2\}$$

for $0 < t \le 1$.

Theorem 3.1 can be proved exactly in the same way as in [12]. We omit the proof.

As for the estimates in large time, we have the following result.

Theorem 3.2. *There exist constants $\nu_0 > 0$, $\gamma_0 > 0$ and ω_0 such that if $\nu \ge \nu_0$, $\gamma^2/(2\nu + \nu') \ge \gamma_0^2$ and $|\omega| \le \omega_0$, then the estimates*

$$\begin{aligned} &\|\partial_{x'}^k \partial_{x_n}^l \mathscr{U}(t)u_0\|_2 \\ &\quad \le C\{t^{-\frac{n-1}{4}-\frac{k}{2}}\|u_0\|_{L^1(\mathbf{R}^{n-1};L^2(0,1))} \\ &\qquad + e^{-dt}(\|u_0\|_{H^1\times L^2} + \|\partial_{x'} w_0\|_2)\} \quad (k + l \le 1) \end{aligned}$$

hold uniformly for $t \ge 1$, $u_0 = {}^T(\phi_0, w_0) \in H^1 \times L^2$ with $\partial_{x'} w_0 \in L^2$ and $u_0 \in L^1(\mathbf{R}^{n-1}; L^2(0,1))$. Here d is a positive constant.

We will prove Theorem 3.2 in section 4.

The decay rate in Theorem 3.2 is the same one as that of an $n-1$ dimensional heat kernel. The next result shows that this is an optimal decay rate, and, in fact, the asymptotic leading part of solutions is given by an $n-1$ dimensional heat kernel which moves in x_1 direction with a constant speed.

Theorem 3.3. *There exist constants $\nu_0 > 0$, $\gamma_0 > 0$ and ω_0 such that if $\nu \ge \nu_0$, $\gamma^2/(2\nu+\nu') \ge \gamma_0^2$ and $|\omega| \le \omega_0$, then for any $u_0 = {}^T(\phi_0, w_0) \in (H^1 \times L^2) \cap L^1$ with $\partial_{x'} w_0 \in L^2$ the solution $u(t) = \mathscr{U}(t)u_0$ of problem (3.1) is decomposed as*

$$\mathscr{U}(t)u_0 = \mathscr{U}^{(0)}(t)u_0 + \mathscr{U}^{(\infty)}(t)u_0,$$

where each term on the right-hand side has the following properties.

(i) *The function $\mathscr{U}^{(0)}(t)u_0$ satisfies the following estimates (3.2) and (3.3) uniformly for $t \ge 1$:*

$$\|\partial_{x'}^k \partial_{x_n}^l \mathscr{U}^{(0)}(t)u_0\|_2 \le Ct^{-\frac{n-1}{4}-\frac{k}{2}}\|u_0\|_1 \quad (k+l \le 1), \tag{3.2}$$

$$\|\mathscr{U}^{(0)}(t)u_0 - G_t *_{x'} \varPi^{(0)}u_0\|_2 \le Ct^{-\frac{n-1}{4}-\frac{1}{2}}\|u_0\|_1, \tag{3.3}$$

Here

$$G_t *_{x'} \varPi^{(0)}u_0 = \mathscr{F}^{-1}\left(e^{-(i\kappa_0\xi_1 + \kappa_1\xi_1^2 + \kappa_2|\xi''|^2)t}\widehat{\varPi}^{(0)}\widehat{u}_0\right)$$

with $\widehat{\varPi}^{(0)}\widehat{u}_0 = [Q_0\widehat{u}_0]u^{(0)} = [\widehat{\phi}_0]u^{(0)}$, where $u^{(0)} = u^{(0)}(x_n)$ is the function given in Lemma 4.3 below; and κ_j ($j = 0,1,2$) are some positive constants satisfying

$$\kappa_0 = \frac{1}{6} + O(\omega),$$

$$\kappa_1 = \left(\frac{\gamma^2}{12\nu} + O\left(\frac{1}{\nu}\right) + O\left(\frac{2\nu+\nu'}{\gamma^2}\right)\right)(1 + O(\omega)),$$

$$\kappa_2 = \frac{\gamma^2}{12\nu}(1 + O(\omega)).$$

Furthermore, if $u_0 = {}^T(\mathrm{div}\,\Psi_0, \partial_x w_0)$ with $\Psi_0^n|_{x_n=0,1} = 0$ then it holds that

$$\|\mathscr{U}^{(0)}(t)u_0\|_2 \le Ct^{-\frac{n-1}{4}-\frac{1}{2}}(\|\Psi_0\|_1 + \|w_0\|_1) \tag{3.4}$$

for all $t \ge 1$.

(ii) *There exists a constant $d > 0$ such that $\mathscr{U}^{(\infty)}(t)u_0$ satisfies*

$$\|\partial_x^l \mathscr{U}^{(\infty)}(t)u_0\|_2 \le Ce^{-dt}(\|u_0\|_{H^1\times L^2} + \|\partial_{x'} w_0\|_2) \quad (l = 0,1) \tag{3.5}$$

for all $t \ge 1$.

Remark. We easily see that $\|G_t *_{x'} \varPi^{(0)}u_0\|_2 = O(t^{-\frac{n-1}{4}})$ if $u_0 \in L^1$. We also see that the function $G_t *_{x'} \varPi^{(0)}u_0$

is written in the form $G_t *_{x'} \Pi^{(0)} u_0 = (\phi^{(0)}(x',t), 0)$ with $\phi^{(0)}(x',t)$ satisfying

$$\partial_t \phi^{(0)} - \kappa_1 \partial_{x_1}^2 \phi^{(0)} - \kappa_2 \Delta'' \phi^{(0)} + \kappa_0 \partial_{x_1} \phi^{(0)} = 0,$$

$$\phi^{(0)}|_{t=0} = \int_0^1 \phi_0(x', x_n)\, dx_n,$$

where $\Delta'' = \partial_{x_2}^2 + \cdots + \partial_{x_{n-1}}^2$.

A proof of Theorem 3.3 will be outlined in section 5.

4. Proof of Theorem 3.2

In this section we prove Theorem 3.2. From now on we simply denote $\nu + \nu'$ by $\tilde{\nu}$:

$$\tilde{\nu} = \nu + \nu'.$$

To prove Theorem 3.2, we consider the Fourier transform of (3.1) in x' variable. The Fourier transform of (3.1) is written as

$$\text{(4.1)} \quad \partial_t \widehat{\phi} + i\xi_1 v_s^1 \widehat{\phi} + i\gamma^2 \xi' \cdot (\rho_s \widehat{w}') + \gamma^2 \partial_{x_n}(\rho_s \widehat{w}^n) = 0,$$

(4.2)
$$\partial_t \widehat{w}' + \nu(|\xi'|^2 - \partial_{x_n}^2)\widehat{w}' - i\tilde{\nu}\xi'(i\xi' \cdot \widehat{w}' + \partial_{x_n}\widehat{w}^n)$$
$$+ i\xi'\left(\tfrac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\widehat{\phi}\right) + i\xi_1 v_s^1 \widehat{w}' + (\partial_{x_n} v_s^1)\widehat{w}^n \boldsymbol{e}_1' - \tfrac{\nu}{\gamma^2\rho_s}\widehat{\phi}\boldsymbol{e}_1' = 0,$$

(4.3)
$$\partial_t \widehat{w}^n + \nu(|\xi'|^2 - \partial_{x_n}^2)\widehat{w}^n - \tilde{\nu}\partial_{x_n}(i\xi' \cdot \widehat{w}' + \partial_{x_n}\widehat{w}^n)$$
$$+ \partial_{x_n}\left(\tfrac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\widehat{\phi}\right) + i\xi_1 v_s^1 \widehat{w}^n = 0,$$

$$\text{(4.4)} \qquad \widehat{w}|_{x_n=0,1} = 0$$

for $t > 0$, and

$$\text{(4.5)} \qquad \widehat{u}|_{t=0} = \widehat{u}_0 = {}^T(\widehat{\phi}_0, \widehat{w}_0).$$

Here $\widehat{\phi} = \widehat{\phi}(\xi', x_n, t)$ and $\widehat{w} = \widehat{w}(\xi', x_n, t)$ are the Fourier transform of $\phi = \phi(x', x_n, t)$ and $w = w(x', x_n, t)$ in $x' \in \mathbf{R}^{n-1}$ with $\xi' \in \mathbf{R}^{n-1}$ being the dual variable. We thus arrive at the following problem

$$\text{(4.6)} \qquad \frac{d}{dt}u + \widehat{L}_{\xi'} u = 0, \quad w|_{x_n=0,1} = 0, \quad u|_{t=0} = u_0.$$

with a parameter $\xi' \in \mathbf{R}^{n-1}$. Here $u = {}^T(\phi(x_n,t), w(x_n,t))$ $(x_n \in [0,1], t \ge 0)$ and $\widehat{L}_{\xi'}$ is the operator of the form

$$\widehat{L}_{\xi'} = \widehat{A}_{\xi'} + \widehat{B}_{\xi'} + \widehat{C}_0,$$

where

$$\widehat{A}_{\xi'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\nu}{\rho_s}\mathscr{A}_{\xi'} I_{n-1} + \frac{\tilde{\nu}}{\rho_s}\xi'^T\xi' & -i\frac{\tilde{\nu}}{\rho_s}\xi'\partial_{x_n} \\ 0 & -i\frac{\tilde{\nu}}{\rho_s}{}^T\xi'\partial_{x_n} & \frac{\nu}{\rho_s}\mathscr{A}_{\xi'} - \frac{\tilde{\nu}}{\rho_s}\partial_{x_n}^2 \end{pmatrix}$$

with

$$\mathscr{A}_{\xi'} = |\xi'|^2 - \partial_{x_n}^2,$$

$$\widehat{B}_{\xi'} = \begin{pmatrix} i\xi_1 v_s^1 & i\gamma^2\rho_s{}^T\xi' & \gamma^2\partial_{x_n}(\rho_s\,\cdot) \\ i\xi'\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s} & i\xi_1 v_s^1 I_{n-1} & 0 \\ \partial_{x_n}\left(\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\,\cdot\right) & 0 & i\xi_1 v_s^1 \end{pmatrix},$$

$$\widehat{C}_0 = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{\nu}{\gamma^2\rho_s}\boldsymbol{e}_1' & 0 & (\partial_{x_n} v_s^1)\boldsymbol{e}_1' \\ 0 & 0 & 0 \end{pmatrix}.$$

We note that, for each fixed ξ', $\widehat{L}_{\xi'}$ is a closed operator on $H^1(0,1) \times L^2(0,1)$ with domain of definition $D(\widehat{L}_{\xi'}) = H^1(0,1) \times (H^2(0,1) \cap H_0^1(0,1))$; and $-\widehat{L}_{\xi'}$ generates an analytic semigroup $e^{-t\widehat{L}_{\xi'}}$ in $H^1(0,1) \times L^2(0,1)$. Therefore, for each fixed ξ', problem (4.1)–(4.5) has a unique solution $\widehat{u}(t) = e^{-t\widehat{L}_{\xi'}}\widehat{u}_0 \in C([0,\infty); H^1(0,1) \times L^2(0,1))$ with $\widehat{u}(t) \in C((0,\infty); D(\widehat{L}_{\xi'}))$. The solution $\mathscr{U}(t)u_0$ of (3.1) is then given by $\mathscr{U}(t)u_0 = \mathscr{F}^{-1}(e^{-t\widehat{L}_{\xi'}}\widehat{u}_0)$.

To prove Theorem 3.2 we decompose $\mathscr{U}(t)u_0$ in the following way. Let $R > 0$. Define $\chi^{(1)}(\xi')$ and $\chi^{(\infty)}(\xi')$ by $\chi^{(1)}(\xi') = 1$ if $|\xi'| \le R$, $\chi^{(1)}(\xi') = 0$ if $|\xi'| \ge R$, and $\chi^{(\infty)} = 1 - \chi^{(1)}$.

We decompose $\mathscr{U}(t)u_0$ as

$$\mathscr{U}(t)u_0 = U_1(t)u_0 + U_\infty(t)u_0,$$

where

$$U_j(t)u_0 = \mathscr{F}^{-1}\left(\chi^{(j)} e^{-t\widehat{L}_{\xi'}}\widehat{u}_0\right), \quad j = 1, \infty.$$

In a similar manner to the proof of [12, Proposition 6.1], one can obtain the following decay estimate for $U_\infty(t)u_0$ by using the Fourier transformed version of the Matsumura-Nishida energy method (cf., [23]).

Proposition 4.1. *There exist constants $R_0 > 0$, $\nu_0 > 0$, $\gamma_0 > 0$ and $\omega_0 > 0$ such that if $R \ge R_0$, $\nu \ge \nu_0$, $\gamma^2/(\nu + \tilde{\nu}) \ge \gamma_0^2$ and $|\omega| \le \omega_0$, then the estimate*

$$\|U_\infty(t)u_0\|_{H^1} \le Ce^{-d(t-1)}(\|u_0\|_{H^1 \times L^2} + \|\partial_{x'} w_0\|_2)$$

holds for $t \ge 1$ with a positive constant d.

To complete the proof of Theorem 3.2 we will show the following estimate for $U_1(t)u_0$.

Proposition 4.2. *For each $R > 0$, there exist $\nu_0 > 0$, $\gamma_0 > 0$ and $\omega_0 > 0$ such that if $\nu \ge \nu_0$, $\gamma^2/(\nu + \tilde{\nu}) \ge \gamma_0^2$ and $|\omega| \le \omega_0$, then the estimate*

$$\|\partial_{x'}^k \partial_{x_n}^l U_1(t)u_0\|_2 \le Ct^{-\frac{n-1}{4} - \frac{k}{2}}\|u_0\|_{L^1(\mathbf{R}^{n-1}; L^2(0,1))}$$

holds for $t \ge 1$, $k, l = 0, 1$.

Theorem 3.2 now follows from Proposition 4.1 and Proposition 4.2 with $R = R_0$.

Proof of Proposition 4.2. To prove Proposition 4.2 we decompose $u(t)$ based on a spectral property of $\widehat{L}_{\xi'}$ with $\xi'=0$, namely, $\widehat{L}_0$.

We introduce the adjoint operator of $\widehat{L}_{\xi'}$ with respect to the weighted inner product $\langle\cdot,\cdot\rangle$. We define an operator $\widehat{L}_{\xi'}$ by

$$\widehat{L}^*_{\xi'} = \widehat{A}^*_{\xi'} + \widehat{B}^*_{\xi'} + \widehat{C}^*_0$$

with domain of definition $D(\widehat{L}^*_{\xi'}) = D(\widehat{L}_{\xi'})$, where

$$\widehat{A}^*_{\xi'} = \widehat{A}_{\xi'}, \quad \widehat{B}^*_{\xi'} = -\widehat{B}_{\xi'}$$

and

$$\widehat{C}^*_0 = \begin{pmatrix} 0 & -\frac{\nu\gamma^2\rho_s}{\tilde{P}'(\rho_s)}{}^T\boldsymbol{e}'_1 & 0 \\ 0 & 0 & 0 \\ 0 & (\partial_{x_n}v^1_s)^T\boldsymbol{e}'_1 & 0 \end{pmatrix}.$$

We then have

$$\langle \widehat{A}_{\xi'}u, v\rangle = \langle u, \widehat{A}^*_{\xi'}v\rangle = \langle u, \widehat{A}_{\xi'}v\rangle,$$

$$\langle \widehat{B}_{\xi'}u, v\rangle = \langle u, \widehat{B}^*_{\xi'}v\rangle = -\langle u, \widehat{B}_{\xi'}v\rangle,$$

$$\langle \widehat{C}_0 u, v\rangle = \langle u, \widehat{C}^*_0 v\rangle$$

and

$$\langle \widehat{L}_{\xi'}u, v\rangle = \langle u, \widehat{L}^*_{\xi'}v\rangle$$

for $u, v\in D(\widehat{L}_{\xi'})$.

We begin with a lemma on the eigenvalue problem for $\widehat{L}_0$ and $\widehat{L}^*_0$.

Lemma 4.3. (i) *$\lambda=0$ is a simple eigenvalue of $\widehat{L}_0$ and $\widehat{L}^*_0$.*

(ii) *The eigenspaces for $\lambda=0$ of $\widehat{L}_0$ and $\widehat{L}^*_0$ are spanned by $u^{(0)}$ and $u^{(0)*}$ respectively, where*

$$u^{(0)} = {}^T(\phi^{(0)}, w^{(0),1}\boldsymbol{e}'_1, 0)$$

and

$$u^{(0)*} = {}^T(\phi^{(0)*}, 0, 0)$$

with

$$\phi^{(0)}(x_n) = \alpha_0 \tfrac{\gamma^2\rho_s(x_n)}{\tilde{P}'(\rho_s(x_n))}, \quad \alpha_0 = \left(\int_0^1 \tfrac{\gamma^2\rho_s}{\tilde{P}'(\rho_s)}\,dx_n\right)^{-1},$$

$$w^{(0),1}(x_n) = \tfrac{1}{\gamma^2}\int_0^1 G(x_n,y_n)\phi^{(0)}(y_n)\,dy_n,$$

$$\phi^{(0)*}(x_n) = \tfrac{\gamma^2}{\alpha_0}\phi^{(0)}(x_n).$$

(iii) *The eigenprojections $\widehat{\Pi}^{(0)}$ and $\widehat{\Pi}^{(0)*}$ for $\lambda=0$ of $\widehat{L}_0$ and $\widehat{L}^*_0$ are given by*

$$\widehat{\Pi}^{(0)}u = \langle u, u^{(0)*}\rangle u^{(0)} = [Q_0 u]u^{(0)},$$

$$\widehat{\Pi}^{(0)*}u = \langle u, u^{(0)}\rangle u^{(0)*},$$

respectively.

(iv) *Let $u^{(0)} = u^{(0)}_0 + u^{(0)}_1$, where*

$$u^{(0)}_0 = {}^T(\phi^{(0)},0,0), \quad u^{(0)}_1 = {}^T(0, w^{(0),1}\boldsymbol{e}'_1, 0).$$

Then $u^{(0)*} = \frac{\gamma^2}{\alpha_0}u^{(0)}_0$ and

$$\langle u, u^{(0)}\rangle = \frac{\alpha_0}{\gamma^2}[\phi] + (w^1, w^{(0),1}\rho_s)$$

for $u = {}^T(\phi, w', w^n)$.

Remark 4.4. We note that

$$\phi^{(0)} = O(1),\ \alpha_0 = O(1),\ w^{(0),1} = O(1/\gamma^2),$$

which will be frequently used in the argument below without mentioning.

Proof of Lemma 4.3. If $\widehat{L}_0 u = 0$, we have

$$\begin{cases} \gamma^2\partial_{x_n}(\rho_s w^n) = 0, \\ -\frac{\nu}{\rho_s}\partial^2_{x_n}w' - \frac{\nu}{\gamma^2\rho_s}\phi\boldsymbol{e}'_1 + (\partial_{x_n}v^1_s)w^n\boldsymbol{e}'_1 = 0, \\ -\frac{\nu+\tilde{\nu}}{\rho_s}\partial^2_{x_n}w^n + \partial_{x_n}\left(\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\phi\right) = 0, \\ w|_{x_n=0,1} = 0. \end{cases}$$

The first equation, together with the boundary condition, gives $w^n=0$. Then, by the third equation, $\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\phi$ is a constant, and therefore, $\phi = c\frac{\gamma^2\rho_s}{\tilde{P}'(\rho_s)}$. Furthermore, the second equation, together with the boundary condition, gives $w' = w^1\boldsymbol{e}'_1$ with $w^1 = \frac{1}{\gamma^2}\int_0^1 G(x_n,y_n)\phi(y_n)\,dy_n$. Taking $c=\alpha_0$ we obtain the eigenfunction $u^{(0)}$.

Consider next $\widehat{L}_0 u = u^{(0)}$. Then we have

$$\gamma^2\partial_{x_n}(\rho_s w^n) = \phi^{(0)}, \quad w^n|_{x_n=0,1} = 0.$$

It follows, by integration by parts, that

$$[\phi^{(0)}] = \gamma^2[\partial_{x_n}(\rho_s w^n)] = 0.$$

But, clearly, $[\phi^{(0)}]\neq 0$, and so, we conclude that there is no u such that $\widehat{L}_0 u = u^{(0)}$. This shows that the eigenvalue 0 of $\widehat{L}_0$ is simple. Similarly one can prove assertion (i) for $\widehat{L}^*_0$. The remaining assertions now follow from (i). We omit the details. This completes the proof. □

Based on Lemma 4.3 we decompose $u(t)$ into the parts of the eigenspace for $\lambda=0$ and its complementary space. Let $u = {}^T(\phi, w)\in H^1(0,1)\times L^2(0,1)$. We decompose u as

$$u = \sigma u^{(0)} + u_1,$$

where

$$\sigma = [Q_0 u] = \langle u, u^{(0)*}\rangle,$$

$$u_1 = (I - \widehat{\Pi}^{(0)})u.$$

Remark 4.5. Due to the boundary condition $w|_{x_n=0,1} = 0$, the Poincaré inequality holds for the velocity part: $|w|_2 \leq$

$|\partial_{x_n} w|_2$. Concerning the density part ϕ, the Poincaré inequality does not necessarily hold. However, if $u_1 = (I - \widehat{\Pi}^{(0)})u = {}^T(\phi_1, w_1)$, then $[Q_0 u_1] = [\phi_1] = 0$, which implies that the Poincaré inequality also holds for ϕ_1. Therefore, $|\phi_1|_2$ can also be controlled by $|\partial_{x_n}\phi_1|_2$. This simple observation will be useful in the argument below.

Using the decomposition introduced above, we rewrite problem (4.6). To do so, we define some notation. We write

$$\widetilde{M}_{\xi'} = \widehat{L}_{\xi'} - \widehat{L}_0 = \widetilde{A}_{\xi'} + \widetilde{B}_{\xi'},$$

where

$$\begin{aligned}\widetilde{A}_{\xi'} &= \widehat{A}_{\xi'} - \widehat{A}_0 \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\nu}{\rho_s}|\xi'|^2 I_{n-1} + \frac{\tilde{\nu}}{\rho_s}\xi'^T\xi' & -i\frac{\tilde{\nu}}{\rho_s}\xi'\partial_{x_n} \\ 0 & -i\frac{\tilde{\nu}}{\rho_s}{}^T\xi'\partial_{x_n} & \frac{\nu}{\rho_s}|\xi'|^2 \end{pmatrix},\end{aligned}$$

$$\widetilde{B}_{\xi'} = \widehat{B}_{\xi'} - \widehat{B}_0 = \begin{pmatrix} i\xi_1 v_s^1 & i\gamma^2\rho_s\,{}^T\xi' & 0 \\ i\xi'\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s} & i\xi_1 v_s^1 I_{n-1} & 0 \\ 0 & 0 & i\xi_1 v_s^1 \end{pmatrix}.$$

Decomposing $u(t)$ in (4.6) as $u(t) = \sigma(t)u^{(0)} + u_1(t)$, we have

$$\frac{d}{dt}(\sigma u^{(0)} + u_1) + \widehat{L}_0 u_1 + \widetilde{M}_{\xi'}(\sigma u^{(0)} + u_1) = 0.$$

Applying $\widehat{\Pi}^{(0)}$ and $I - \widehat{\Pi}^{(0)}$ to this equation, we have

$$\begin{cases} \frac{d}{dt}\sigma u^{(0)} + \widehat{\Pi}^{(0)}\widetilde{M}_{\xi'}(\sigma u^{(0)} + u_1) = 0, \\ \frac{d}{dt}u_1 + \widehat{L}_0 u_1 + (I - \widehat{\Pi}^{(0)})\widetilde{M}_{\xi'}(\sigma u^{(0)} + u_1) = 0. \end{cases}$$

Since $\widehat{\Pi}^{(0)}\widetilde{M}_{\xi'}u = [Q_0\widetilde{M}_{\xi'}u]u^{(0)}$ and $Q_0\widetilde{M}_{\xi'} = Q_0\widetilde{B}_{\xi'}$, we arrive at

$$\frac{d}{dt}\sigma + [Q_0\widetilde{B}_{\xi'}(\sigma u^{(0)} + u_1)] = 0, \tag{4.7}$$

$$\frac{d}{dt}u_1 + \widehat{L}_0 u_1 + \widetilde{M}_{\xi'}(\sigma u^{(0)} + u_1) - [Q_0\widetilde{B}_{\xi'}(\sigma u^{(0)} + u_1)]u^{(0)} = 0. \tag{4.8}$$

Proposition 4.2 can be proved by estimating solutions of (4.7)–(4.8). We will frequently use the following lemma.

Lemma 4.6. (i) $\langle u^{(0)}, u_1\rangle = \langle u_1^{(0)}, u_1\rangle$ *holds for* $u_1 \in R(I - \widehat{\Pi}^{(0)})$.

(ii) $\left|[Q_0\widetilde{B}_{\xi'}u_0^{(0)}]\right| + \left|[Q_0\widetilde{B}_{\xi'}u_1^{(0)}]\right| \le C|\xi'|$.

(iii) $\left|[Q_0\widetilde{B}_{\xi'}u_1]\right| \le C|\xi'|(|\phi_1|_2 + \gamma^2|w_1'|)$ *holds for* $u_1 = {}^T(\phi_1, w_1', w_1^n) \in R(I - \widehat{\Pi}^{(0)})$.

The proof of Lemma 4.6 is straightforward, so we omit it.

We will employ an energy method to obtain the necessary estimates on solutions of (4.7)–(4.8).

We introduce some notations. For $u = {}^T(\phi, w)$ we define $E_0(u)$ by

$$E_0(u) = \frac{1}{\gamma^2}\left|\sqrt{\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}}\phi\right|_2^2 + \left|\sqrt{\rho_s}w\right|_2^2.$$

For $v = \phi$, $v = w = {}^T(w_1, \cdots, w^n)$ or $v = {}^T(\phi, w)$, we define $D_{\xi'}(v)$ by

$$D_{\xi'}(v) = |\xi'|^2|v|_2^2 + |\partial_{x_n}v|_2^2,$$

and, for $w = {}^T(w_1, \cdots, w^n)$, we define $\widetilde{D}_{\xi'}(w)$ by

$$\widetilde{D}_{\xi'}(w) = \nu D_{\xi'}(w) + \tilde{\nu}|i\xi'\cdot w' + \partial_{x_n}w^n|_2^2.$$

Proposition 4.7. *There exists* $\nu_0 > 0$ *such that if* $\nu \ge \nu_0$, *then*

$$\begin{aligned}&\tfrac{d}{dt}\left(\tfrac{\alpha_0}{\gamma^2}|\sigma|^2 + E_0(u_1)\right) + \widetilde{D}_{\xi'}(w_1) \\ &\quad \le C\left\{\left(\tfrac{\nu}{\gamma^4} + \tfrac{1}{\gamma^2}\right)|\phi_1|_2^2 + \left(\tfrac{\nu+\tilde{\nu}}{\gamma^4} + \tfrac{1}{\gamma^2}\right)|\xi'|^2|\sigma|^2\right\}.\end{aligned} \tag{4.9}$$

Proof. Multiplying (4.7) by $\overline{\sigma}(t)$ and taking the real part of the resulting equation, we have

$$\frac{1}{2}\frac{d}{dt}|\sigma|^2 + \mathrm{Re}\,\{[Q_0\widetilde{B}_{\xi'}(\sigma u^{(0)} + u_1)]\overline{\sigma}\} = 0.$$

Since $\widetilde{B}_{\xi'}^* = -\widetilde{B}_{\xi'}$, $u^{(0)} = u_0^{(0)} + u_1^{(0)}$ and $u^{(0)*} = \frac{\gamma^2}{\alpha_0}u_0^{(0)}$, we have

$$\begin{aligned}[Q_0\widetilde{B}_{\xi'}u_1]\overline{\sigma} &= \langle\widetilde{B}_{\xi'}u_1, \sigma u^{(0)*}\rangle = -\langle u_1, \widetilde{B}_{\xi'}(\sigma u^{(0)*})\rangle \\ &= -\tfrac{\gamma^2}{\alpha_0}\langle u_1, \widetilde{B}_{\xi'}(\sigma u_0^{(0)})\rangle.\end{aligned}$$

Furthermore, we have

$$[Q_0\widetilde{B}_{\xi'}(\sigma u^{(0)})]\overline{\sigma} = i\xi_1|\sigma|^2\left\{[v_s^1\phi^{(0)}] + \gamma^2[\rho_s w^{(0),1}]\right\},$$

and, thus, $\mathrm{Re}\left\{[Q_0\widetilde{B}_{\xi'}(\sigma u^{(0)})]\overline{\sigma}\right\} = 0$. It then follows that

$$\frac{1}{2}\frac{d}{dt}|\sigma|^2 - \frac{\gamma^2}{\alpha_0}\mathrm{Re}\,\langle u_1, \widetilde{B}_{\xi'}(\sigma u_0^{(0)})\rangle = 0. \tag{4.10}$$

We next take the inner product of (4.8) with u_1. Then the real part of the resulting equation gives

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}E_0(u_1) + \mathrm{Re}\,\langle\widehat{L}_0u_1, u_1\rangle + \mathrm{Re}\,\langle\widetilde{M}_{\xi'}(\sigma u^{(0)} + u_1), u_1\rangle \\ &\quad -\mathrm{Re}\left\{[Q_0\widetilde{B}_{\xi'}(\sigma u^{(0)} + u_1)]\langle u^{(0)}, u_1\rangle\right\} = 0.\end{aligned} \tag{4.11}$$

Since $\widehat{B}_{\xi'}^* = -\widehat{B}_{\xi'}$, we have $\mathrm{Re}\,\langle\widehat{B}_{\xi'}u_1, u_1\rangle = 0$. It then follows that

$$\begin{aligned}
&\mathrm{Re}\,\langle \widehat{L}_0 u_1, u_1\rangle + \mathrm{Re}\,\langle \widetilde{M}_{\xi'}(\sigma u^{(0)} + u_1), u_1\rangle \\
&= \mathrm{Re}\,\langle \widehat{A}_{\xi'} u_1, u_1\rangle + \mathrm{Re}\,\langle \widehat{C}_0 u_1, u_1\rangle + \mathrm{Re}\,\langle \widetilde{A}_{\xi'}(\sigma u^{(0)}), u_1\rangle \\
&\quad + \mathrm{Re}\,\langle \widetilde{B}_{\xi'}(\sigma u^{(0)}), u_1\rangle \\
&= \widetilde{D}_{\xi'}(w_1) + \mathrm{Re}\,\langle \widehat{C}_0 u_1, u_1\rangle + \mathrm{Re}\,\langle \widetilde{A}_{\xi'}(\sigma u^{(0)}), u_1\rangle \\
&\quad + \mathrm{Re}\,\langle \widetilde{B}_{\xi'}(\sigma u^{(0)}), u_1\rangle.
\end{aligned}$$

This, together with (4.11), yields

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt} E_0(u_1) + \widetilde{D}_{\xi'}(w_1) + \mathrm{Re}\,\langle \widehat{C}_0 u_1, u_1\rangle \\
&\quad + \mathrm{Re}\,\langle \widetilde{A}_{\xi'}(\sigma u^{(0)}), u_1\rangle + \mathrm{Re}\,\langle \widetilde{B}_{\xi'}(\sigma u^{(0)}), u_1\rangle \\
&\quad - \mathrm{Re}\,\left\{[Q_0 \widetilde{B}_{\xi'}(\sigma u^{(0)} + u_1)]\langle u^{(0)}, u_1\rangle\right\} = 0.
\end{aligned} \tag{4.12}$$

We add $\frac{\alpha_0}{\gamma^2}\times$(4.10) to (4.12) to eliminate $\mathrm{Re}\,\langle \widetilde{B}_{\xi'}(\sigma u_0^{(0)}), u_1\rangle$ and obtain

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\left(\tfrac{\alpha_0}{\gamma^2}|\sigma|^2 + E_0(u_1)\right) + \widetilde{D}_{\xi'}(w_1) \\
&= -\mathrm{Re}\,\langle \widehat{C}_0 u_1, u_1\rangle - \mathrm{Re}\,\langle \widetilde{A}_{\xi'}(\sigma u^{(0)}), u_1\rangle \\
&\quad - \mathrm{Re}\,\langle \widetilde{B}_{\xi'}(\sigma u_1^{(0)}), u_1\rangle \\
&\quad + \mathrm{Re}\,\left\{[Q_0 \widetilde{B}_{\xi'}(\sigma u^{(0)} + u_1)]\langle u^{(0)}, u_1\rangle\right\}.
\end{aligned} \tag{4.13}$$

Applying Lemma 4.6, and the Poincaré and Hölder inequalities to the right side of (4.13), we have the desired estimate. This completes the proof. □

Before proceeding further we introduce some quantities. We define $J(u)$ by

$$J(u) = -2\mathrm{Re}\,\langle \sigma u^{(0)} + u_1, \widehat{B}_{\xi'}\widetilde{Q} u_1\rangle \quad \text{for} \quad u = \sigma u^{(0)} + u_1.$$

We note that if $\gamma^2 \geq 1$, then there exists a constant $b_0 > 0$ such that

$$|J(u)| \leq \frac{b_0\gamma^2}{2\nu}\left(\frac{\alpha_0}{\gamma^2}|\sigma|^2 + E_0(u_1)\right) + \frac{1}{2}\widetilde{D}_{\xi'}(w_1).$$

Let b_1 be a positive constant (to be determined in Proposition 4.8 below) and define $E_1(u)$ by

$$E_1(u) = \left(1 + \tfrac{b_1\gamma^2}{\nu}\right)\left(\tfrac{\alpha_0}{\gamma^2}|\sigma|^2 + E_0(u_1)\right) + \widetilde{D}_{\xi'}(w_1) + J(u).$$

Taking b_1 suitably large, we have the following estimate for $E_1(u)$.

Proposition 4.8. *There exist constants $b_1 \geq b_0$ and $\nu_0 > 0$ such that if $\nu \geq \nu_0$ and $\gamma^2 \geq 1$, then*

$$\begin{aligned}
&\tfrac{d}{dt}E_1(u) + \left(1 + \tfrac{b_1\gamma^2}{\nu}\right)\widetilde{D}_{\xi'}(w_1) + \left|\sqrt{\rho_s}\partial_t w_1\right|_2^2 \\
&\leq C\left\{\left(\tfrac{1}{\gamma^2} + \tfrac{1}{\nu} + \tfrac{\nu^2}{\gamma^4}\right)|\phi_1|_2^2 + \tfrac{1}{\gamma^2}|\xi'|^2|\phi_1|_2^2\right. \\
&\quad \left. + \left(\tfrac{\nu+\tilde{\nu}}{\gamma^2} + \tfrac{1}{\nu} + \tfrac{1}{\gamma^2}\right)|\xi'|^2|\sigma|^2 + \tfrac{(\nu+\tilde{\nu})^2}{\gamma^4}|\xi'|^4|\sigma|^2\right\}.
\end{aligned} \tag{4.14}$$

In what follows we fix b_1 in $E_1(u)$ as the number given in Proposition 4.8.

Proof of Proposition 4.8. We recall that $u = \sigma u^{(0)} + u_1$ satisfies

$$\frac{d}{dt}u + \widehat{L}_{\xi'}u = 0,$$

which implies

$$\langle \partial_t u, \partial_t \widetilde{Q} u_1\rangle + \langle \widehat{L}_{\xi'}u, \partial_t \widetilde{Q} u_1\rangle = 0 \tag{4.15}$$

with $u = \sigma u^{(0)} + u_1$. Since

$$\partial_t \sigma = -[Q_0 \widetilde{B}_{\xi'}(\sigma u^{(0)} + u_1)]$$

and

$$\langle u^{(0)}, \partial_t \widetilde{Q} u_1\rangle = \langle u_1^{(0)}, \partial_t \widetilde{Q} u_1\rangle,$$

by using Remark 4.4 and Lemma 4.6, we have

$$\begin{aligned}
&\mathrm{Re}\,\langle \partial_t u, \partial_t \widetilde{Q} u_1\rangle \\
&= \mathrm{Re}\,\left\{\langle \partial_t \sigma u^{(0)}, \partial_t \widetilde{Q} u_1\rangle + \langle \partial_t u_1, \partial_t \widetilde{Q} u_1\rangle\right\} \\
&= \mathrm{Re}\,\left\{-[Q_0 \widetilde{B}_{\xi'}(\sigma u^{(0)} + u_1)]\langle u_1^{(0)}, \partial_t \widetilde{Q} u_1\rangle + \left|\sqrt{\rho_s}\partial_t w_1\right|_2^2\right\} \\
&\geq \tfrac{3}{4}\left|\sqrt{\rho_s}\partial_t w_1\right|_2^2 - C\left\{\tfrac{|\xi'|^2}{\gamma^4}(|\sigma|^2 + |\phi_1|_2^2) + \tfrac{1}{\nu}\widetilde{D}_{\xi'}(w_1)\right\}.
\end{aligned} \tag{4.16}$$

Since $\widehat{L}_0 u^{(0)} = 0$ and $\widehat{B}_0 u^{(0)} = 0$, we see that

$$\begin{aligned}
&\langle \widehat{L}_{\xi'}u, \partial_t \widetilde{Q} u_1\rangle \\
&= \langle \widetilde{M}_{\xi'}(\sigma u^{(0)}), \partial_t \widetilde{Q} u_1\rangle + \langle \widehat{L}_{\xi'}u_1, \partial_t \widetilde{Q} u_1\rangle \\
&= \langle \widetilde{A}_{\xi'}(\sigma u^{(0)}), \partial_t \widetilde{Q} u_1\rangle + \langle \widetilde{B}_{\xi'}(\sigma u^{(0)}), \partial_t \widetilde{Q} u_1\rangle \\
&\quad + \langle \widehat{A}_{\xi'}u_1, \partial_t \widetilde{Q} u_1\rangle + \langle \widehat{B}_{\xi'}u_1, \partial_t \widetilde{Q} u_1\rangle + \langle \widehat{C}_0 u_1, \partial_t \widetilde{Q} u_1\rangle \\
&= \langle \widetilde{A}_{\xi'}(\sigma u^{(0)}), \partial_t \widetilde{Q} u_1\rangle + \langle \widehat{B}_{\xi'}(\sigma u^{(0)} + u_1), \partial_t \widetilde{Q} u_1\rangle \\
&\quad + \langle \widehat{A}_{\xi'}u_1, \partial_t \widetilde{Q} u_1\rangle + \langle \widehat{C}_0 u_1, \partial_t \widetilde{Q} u_1\rangle.
\end{aligned} \tag{4.17}$$

We will show

$$\begin{aligned}
&\mathrm{Re}\,\left\{\langle \widehat{B}_{\xi'}(\sigma u^{(0)} + u_1), \partial_t \widetilde{Q} u_1\rangle + \langle \widehat{A}_{\xi'}u_1, \partial_t \widetilde{Q} u_1\rangle\right\} \\
&\geq \tfrac{1}{2}\tfrac{d}{dt}\left(\widetilde{D}_{\xi'}(w_1) + J(u)\right) - \varepsilon\left|\sqrt{\rho_s}\partial_t w_1\right|_2^2 \\
&\quad - C\left\{\tfrac{|\xi'|^2}{\gamma^2}(|\sigma|^2 + |\phi_1|_2^2) + \tfrac{\gamma^2}{\nu}\widetilde{D}_{\xi'}(w_1)\right. \\
&\quad \left. + \tfrac{1}{\varepsilon\nu}\widetilde{D}_{\xi'}(w_1) + \tfrac{|\xi'|^2}{\varepsilon\gamma^4}|\sigma|^2\right\}
\end{aligned} \tag{4.18}$$

for any $\varepsilon > 0$.

It is easy to see that

$$\mathrm{Re}\,\langle \widehat{A}_{\xi'}u_1, \partial_t \widetilde{Q} u_1\rangle = \frac{1}{2}\frac{d}{dt}\widetilde{D}_{\xi'}(w_1). \tag{4.19}$$

Since $\widehat{B}^*_{\xi'} = -\widehat{B}_{\xi'}$, we have

$$(4.20)\quad \begin{aligned} &\langle \widehat{B}_{\xi'}(\sigma u_0^{(0)}), \partial_t \widetilde{Q} u_1\rangle \\ &= -\tfrac{d}{dt}\langle \sigma u_0^{(0)}, \widehat{B}_{\xi'}\widetilde{Q}u_1\rangle + \langle \partial_t \sigma u_0^{(0)}, \widehat{B}_{\xi'}\widetilde{Q}u_1\rangle. \end{aligned}$$

By (4.7), we have

$$\partial_t \sigma = -[Q_0 \widetilde{B}_{\xi'}(\sigma u^{(0)} + u_1)].$$

Lemma 4.6 then implies

$$(4.21)\quad \begin{aligned} &\left|\langle \partial_t \sigma u_0^{(0)}, \widehat{B}_{\xi'}\widetilde{Q}u_1\rangle\right| \\ &\le \left|[Q_0 \widetilde{B}_{\xi'}(\sigma u^{(0)} + u_1)]\right| \left|\langle u_0^{(0)}, \widehat{B}_{\xi'}\widetilde{Q}u_1\rangle\right| \\ &\le C\left\{\tfrac{|\xi'|^2}{\gamma^2}(|\sigma|^2 + |\phi_1|_2^2) + \tfrac{\gamma^2}{\nu}\widetilde{D}_{\xi'}(w_1)\right\}. \end{aligned}$$

Similarly,

$$(4.22)\quad \begin{aligned} &\langle \widehat{B}_{\xi'}u_1, \partial_t \widetilde{Q}u_1\rangle \\ &= -\tfrac{d}{dt}\langle u_1, \widehat{B}_{\xi'}\widetilde{Q}u_1\rangle + \langle \partial_t u_1, \widehat{B}_{\xi'}\widetilde{Q}u_1\rangle \\ &= -\tfrac{d}{dt}\langle u_1, \widehat{B}_{\xi'}\widetilde{Q}u_1\rangle + \langle \partial_t Q_0 u_1, \widehat{B}_{\xi'}\widetilde{Q}u_1\rangle \\ &\quad + \langle \partial_t \widetilde{Q}u_1, \widehat{B}_{\xi'}\widetilde{Q}u_1\rangle. \end{aligned}$$

We first estimate the second term on the right of (4.22). By (4.8), we have

$$\begin{aligned} \partial_t Q_0 u_1 &= -Q_0\left\{\widehat{L}_{\xi'}u_1 + \widetilde{M}_{\xi'}(\sigma u^{(0)}) \right. \\ &\qquad \left. -[Q_0\widetilde{B}_{\xi'}(\sigma u^{(0)} + u_1)]u^{(0)}\right\} \\ &= -\left\{Q_0 \widehat{B}_{\xi'}u_1 + Q_0\widetilde{B}_{\xi'}(\sigma u^{(0)}) \right. \\ &\qquad \left. -[Q_0\widetilde{B}_{\xi'}(\sigma u^{(0)} + u_1)]u_0^{(0)}\right\}. \end{aligned}$$

Since $\langle \partial_t Q_0 u_1, \widehat{B}_{\xi'}\widetilde{Q}u_1\rangle = \langle \partial_t Q_0 u_1, Q_0\widehat{B}_{\xi'}\widetilde{Q}u_1\rangle$, we see from Lemma 4.6 that

$$(4.23)\quad \begin{aligned} &\left|\langle \partial_t Q_0 u_1, \widehat{B}_{\xi'}\widetilde{Q}u_1\rangle\right| \\ &\le C\left\{|Q_0\widehat{B}_{\xi'}u_1|_2 + |Q_0\widetilde{B}_{\xi'}(\sigma u^{(0)})|_2 \right. \\ &\quad \left. + \left|[Q_0\widetilde{B}_{\xi'}(\sigma u^{(0)} + u_1)]\right| |u_0^{(0)}|_2\right\} \times |Q_0\widehat{B}_{\xi'}\widetilde{Q}u_1|_2 \\ &\le C\left\{\tfrac{|\xi'|^2}{\gamma^2}(|\sigma|^2 + |\phi_1|_2^2) + \tfrac{\gamma^2}{\nu}\widetilde{D}_{\xi'}(w_1)\right\}. \end{aligned}$$

The third term on the right of (4.22) is estimated as

$$\begin{aligned} \left|\langle \partial_t \widetilde{Q}u_1, \widehat{B}_{\xi'}\widetilde{Q}u_1\rangle\right| &\le C|\sqrt{\rho_s}\partial_t w_1|_2 |\xi'||w_1|_2 \\ &\le \varepsilon|\sqrt{\rho_s}\partial_t w_1|_2^2 + \tfrac{C}{\varepsilon\nu}\widetilde{D}_{\xi'}(w_1) \end{aligned}$$

for any $\varepsilon > 0$. This, together with (4.23), gives

$$(4.24)\quad \begin{aligned} &\mathrm{Re}\,\langle \widehat{B}_{\xi'}u_1, \partial_t \widetilde{Q}u_1\rangle \\ &\ge -\tfrac{d}{dt}\langle u_1, \widehat{B}_{\xi'}\widetilde{Q}u_1\rangle - \varepsilon|\sqrt{\rho_s}\partial_t w_1|_2^2 \\ &\quad - C\left\{\tfrac{|\xi'|^2}{\gamma^2}(|\sigma|^2 + |\phi_1|_2^2) + \left(\tfrac{\gamma^2}{\nu} + \tfrac{1}{\varepsilon\nu}\right)\widetilde{D}_{\xi'}(w_1)\right\} \end{aligned}$$

for any $\varepsilon > 0$.

To complete the proof of (4.18), it remains to estimate $\langle \widehat{B}_{\xi'}(\sigma u_1^{(0)}), \partial_t \widetilde{Q}u_1\rangle$. By Remark 4.4, this can be estimates as

$$(4.25)\quad \begin{aligned} \left|\langle \widehat{B}_{\xi'}(\sigma u_1^{(0)}), \partial_t \widetilde{Q}u_1\rangle\right| &\le C\tfrac{|\xi'|}{\gamma^2}|\sigma||\sqrt{\rho_s}\partial_t w_1|_2 \\ &\le \varepsilon|\sqrt{\rho_s}\partial_t w_1|_2^2 + \tfrac{C}{\varepsilon}\tfrac{|\xi'|^2}{\gamma^4}|\sigma|^2 \end{aligned}$$

for any $\varepsilon > 0$. We thus obtain (4.18) from (4.19), (4.20), (4.21), (4.24) and (4.25).

A straightforward computation gives

$$(4.26)\quad \begin{aligned} &\left|\langle \widehat{A}_{\xi'}(\sigma u^{(0)}), \partial_t \widetilde{Q}u_1\rangle\right| \\ &\le \varepsilon|\sqrt{\rho_s}\partial_t w_1|_2^2 + \tfrac{C}{\varepsilon}\tfrac{(\nu+\tilde{\nu})^2}{\gamma^4}|\xi'|^2(1 + |\xi'|^2)|\sigma|^2 \end{aligned}$$

and

$$(4.27)\quad \begin{aligned} &\left|\langle \widehat{C}_0 u_1, \partial_t \widetilde{Q}u_1\rangle\right| \\ &\le \varepsilon|\sqrt{\rho_s}\partial_t w_1|_2^2 + \tfrac{C}{\varepsilon}\left\{\tfrac{\nu^2}{\gamma^4}|\phi_1|_2^2 + |w_1|_2^2\right\} \\ &\le \varepsilon|\sqrt{\rho_s}\partial_t w_1|_2^2 + \tfrac{C}{\varepsilon}\left\{\tfrac{\nu^2}{\gamma^4}|\phi_1|_2^2 + \tfrac{1}{\nu}\widetilde{D}_{\xi'}(w_1)\right\} \end{aligned}$$

for any $\varepsilon > 0$.

Taking $\varepsilon > 0$ suitably small, we see from (4.15)–(4.18), (4.26) and (4.27) that

$$(4.28)\quad \begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}(\widetilde{D}_{\xi'}(w_1) + J(u)) + \tfrac{1}{2}|\sqrt{\rho_s}\partial_t w_1|_2^2 \\ &\le C\left\{\tfrac{\nu^2}{\gamma^4}|\phi_1|_2^2 + \tfrac{|\xi'|^2}{\gamma^2}(|\sigma|^2 + |\phi_1|_2^2) \right. \\ &\quad + \tfrac{(\nu+\tilde{\nu})^2}{\gamma^4}|\xi'|^2(1 + |\xi'|^2)|\sigma|^2 \\ &\quad \left. + \left(\tfrac{\gamma^2}{\nu} + \tfrac{1}{\nu}\right)\widetilde{D}_{\xi'}(w_1)\right\}. \end{aligned}$$

Adding (4.28) to $\left(1 + \frac{b_1\gamma^2}{\nu}\right) \times (4.9)$ with suitably large $b_1 \ge b_0$, we obtain the desired estimate. This completes the proof. □

We next derive a dissipative estimate for $\partial_{x_n}\phi_1$, which also controls $|\phi_1|_2$ by Poincaré's inequality as mentioned in Remark 4.5.

Proposition 4.9. *There exists a constant $\omega_0 > 0$ such that if $|\omega| \le \omega_0$, then*

$$(4.29)\quad \begin{aligned} &\tfrac{d}{dt}\tfrac{1}{\gamma^2}\left|\sqrt{\tfrac{\widetilde{P}'(\rho_s)}{\gamma^2\rho_s}}\partial_{x_n}\phi_1\right|_2^2 + \tfrac{1}{\nu+\tilde{\nu}}\left|\tfrac{\widetilde{P}'(\rho_s)}{\gamma^2}\partial_{x_n}\phi_1\right|_2^2 \\ &\le C\left\{\left(\tfrac{\nu+\tilde{\nu}}{\nu}|\omega|^2 + \tfrac{1}{\nu+\tilde{\nu}}\left(\tfrac{1}{\nu} + |\xi'|^2\right)\right)\widetilde{D}_{\xi'}(w_1) \right. \\ &\quad + \tfrac{1}{\nu+\tilde{\nu}}|\sqrt{\rho_s}\partial_t w_1|_2^2 \\ &\quad \left. + \tfrac{\nu+\tilde{\nu}}{\gamma^4}|\xi'|^2(|\sigma|^2 + |\phi_1|_2^2)\right\}. \end{aligned}$$

Proof. The first row of equation (4.8) is written as

$$\partial_t\phi_1 + i\xi_1 v_s^1\phi_1 + \gamma^2\rho_s i\xi'\cdot w_1' + \gamma^2\partial_{x_n}(\rho_s w_1^n) + i\xi_1 v_s^1\sigma\phi^{(0)} + \gamma^2\rho_s i\xi_1\sigma w^{(0),1} - [Q_0\widetilde{B}_{\xi'}(\sigma u^{(0)}+u_1)]\phi^{(0)} = 0.$$

Differentiating this in x_n we have

$$\partial_t\partial_{x_n}\phi_1 + i\xi_1 v_s^1\partial_{x_n}\phi_1 + \gamma^2\rho_s\partial_{x_n}^2 w_1^n = H^0, \tag{4.30}$$

where

$$\begin{aligned} H^0 \;=\; -\{ & i\xi_1(\partial_{x_n}v_s^1)\phi_1 + \gamma^2\partial_{x_n}(\rho_s i\xi'\cdot w_1') \\ & + \gamma^2\partial_{x_n}\rho_s\partial_{x_n}w_1^n + \gamma^2\partial_{x_n}(w_1^n\partial_{x_n}\rho_s) \\ & + i\xi_1\sigma(\partial_{x_n}(v_s^1\phi^{(0)}) + \gamma^2\partial_{x_n}(\rho_s w^{(0),1})) \\ & - [Q_0\widetilde{B}_{\xi'}(\sigma u^{(0)}+u_1)]\partial_{x_n}\phi^{(0)}\}. \end{aligned}$$

We next rewrite the n th row of equation (4.8) as

$$-\frac{\nu+\tilde{\nu}}{\rho_s}\partial_{x_n}^2 w_1^n + \frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\partial_{x_n}\phi_1 = \tilde{g}_1^n, \tag{4.31}$$

where

$$\begin{aligned} \tilde{g}_1^n \;=\; -\Big\{ & \partial_t w_1^n + i\xi_1 v_s^1 w_1^n + \tfrac{\nu}{\rho_s}|\xi'|^2 w_1^n \\ & - \tfrac{\tilde{\nu}}{\rho_s} i\xi'\cdot\partial_{x_n}w_1' + \phi_1\partial_{x_n}\left(\tfrac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\right) \\ & - \tfrac{\tilde{\nu}}{\rho_s} i\xi_1\sigma\partial_{x_n}w^{(0),1}\Big\}. \end{aligned}$$

Adding $\frac{\gamma^2\rho_s^2}{\nu+\tilde{\nu}}\times$ (4.31) to (4.30), we have

$$\partial_t\partial_{x_n}\phi_1 + i\xi' v_s^1\partial_{x_n}\phi_1 + \frac{\rho_s\tilde{P}'(\rho_s)}{\nu+\tilde{\nu}}\partial_{x_n}\phi_1 = H, \tag{4.32}$$

where

$$H = H^0 + \frac{\gamma^2\rho_s^2}{\nu+\tilde{\nu}}\tilde{g}_1^n.$$

We take the inner product of (4.32) with $\frac{\tilde{P}'(\rho_s)}{\gamma^4\rho_s}\partial_{x_n}\phi_1$. Then the real part of the resulting equation gives

$$\begin{aligned} & \tfrac{1}{2}\tfrac{d}{dt}\tfrac{1}{\gamma^2}\left|\sqrt{\tfrac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}}\partial_{x_n}\phi_1\right|_2^2 + \tfrac{1}{\nu+\tilde{\nu}}\left|\tfrac{\tilde{P}'(\rho_s)}{\gamma^2}\partial_{x_n}\phi_1\right|_2^2 \\ & = \mathrm{Re}\left(H, \tfrac{\tilde{P}'(\rho_s)}{\gamma^4\rho_s}\partial_{x_n}\phi_1\right) \\ & \le \tfrac{1}{4}\tfrac{1}{\nu+\tilde{\nu}}\left|\tfrac{\tilde{P}'(\rho_s)}{\gamma^2}\partial_{x_n}\phi_1\right|_2^2 + C\tfrac{\nu+\tilde{\nu}}{\gamma^4}\left|\tfrac{1}{\rho_s}H\right|_2^2, \end{aligned}$$

and, hence,

$$\begin{aligned} & \tfrac{1}{2}\tfrac{d}{dt}\tfrac{1}{\gamma^2}\left|\sqrt{\tfrac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}}\partial_{x_n}\phi_1\right|_2^2 + \tfrac{3}{4}\tfrac{1}{\nu+\tilde{\nu}}\left|\tfrac{\tilde{P}'(\rho_s)}{\gamma^2}\partial_{x_n}\phi_1\right|_2^2 \\ & \le C\tfrac{\nu+\tilde{\nu}}{\gamma^4}\left|\tfrac{1}{\rho_s}H\right|_2^2. \end{aligned}$$

By Proposition 2.1, $\rho_s = 1+O(\omega)$ and $\frac{\tilde{P}'(\rho_s)}{\gamma^2} = 1+O(\omega)$. Using these facts and Lemma 4.6, we can estimate $\left|\frac{1}{\rho_s}H\right|_2^2$ to obtain the desired estimate. This completes the proof. □

We finally derive a dissipative estimate for σ.

Proposition 4.10. *There exist constants $\nu_0>0$ and $\gamma_0>0$ such that if $\nu\ge\nu_0$ and $\frac{\gamma^2}{\nu+\tilde{\nu}}\ge\gamma_0^2$, then*

$$\begin{aligned} & \tfrac{d}{dt}|\sigma|^2 + \tfrac{\tilde{\alpha}_0\gamma^2}{\nu}\min\{1,|\xi'|^2\}|\sigma|^2 \\ & \le C\Big\{\left(\tfrac{\nu}{\gamma^2}+\tfrac{\gamma^2}{\nu}\right)(1+|\xi'|^2)|\phi_1|_2^2 \\ & \qquad + \tfrac{\gamma^2}{\nu}|\sqrt{\rho_s}\partial_t w_1|_2^2 + \tfrac{\gamma^2}{\nu}(1+\tilde{\nu})\widetilde{D}_{\xi'}(w_1)\Big\} \end{aligned} \tag{4.33}$$

with some positive constant $\tilde{\alpha}_0$.

Proof. Since

$$[Q_0\widetilde{B}_{\xi'}(\sigma u^{(0)}+u_1)] = [Q_0\widetilde{B}_{\xi'}u^{(0)}]\sigma + i\xi_1[v_s^1\phi_1] + \gamma^2[\rho_s i\xi'\cdot w_1'],$$

we see from (4.7) that

$$\partial_t\sigma + [Q_0\widetilde{B}_{\xi'}u^{(0)}]\sigma + \gamma^2[\rho_s i\xi'\cdot w_1'] = -i\xi_1[v_s^1\phi_1]. \tag{4.34}$$

On the other hand, since $\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\phi^{(0)} = \alpha_0$, we have

$$\widetilde{B}'_{\xi'}u_0^{(0)} = i\xi'\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\phi^{(0)} = i\xi'\alpha_0.$$

Here and in what follows, $\widetilde{B}_{\xi'}$ denotes the $(n-1)\times(n+1)$ matrix operator defined by

$$\widetilde{B}'_{\xi'} = \left(\begin{array}{ccc} i\xi'\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s} & i\xi_1 v_s^1 I_{n-1} & 0\end{array}\right).$$

This, together with (4.8), gives

$$(|\xi'|^2 - \partial_{x_n}^2)w_1' = -\frac{\alpha_0}{\nu}i\xi'\sigma\rho_s + h_1'. \tag{4.35}$$

Here

$$\begin{aligned} h_1' \;=\; -\tfrac{\rho_s}{\nu}\Big\{ & \partial_t w_1' + \widehat{C}_0' u_1 - \tfrac{\tilde{\nu}}{\rho_s}i\xi'(i\xi'\cdot w_1' + \partial_{x_n}w_1^n) \\ & + \widetilde{B}'_{\xi'}u_1 + \sigma\widetilde{M}'_{\xi'}u_1^{(0)} \\ & - [Q_0\widetilde{B}_{\xi'}(\sigma u^{(0)}+u_1)]w^{(0),1}\boldsymbol{e}_1'\Big\}, \end{aligned}$$

where $\widetilde{M}'_{\xi'}$ and $\widehat{C}_0'$ are $(n-1)\times(n+1)$ matrix operators defined by

$$\widetilde{M}'_{\xi'} = \left(\begin{array}{ccc} 0 & \frac{\tilde{\nu}}{\rho_s}\xi'^T\xi' & -\frac{\tilde{\nu}}{\rho_s}i\xi'\partial_{x_n}\end{array}\right) + \widetilde{B}'_{\xi'},$$

$$\widehat{C}_0' = \left(\begin{array}{ccc} -\frac{\nu}{\gamma^2\rho_s}\boldsymbol{e}_1' & 0 & (\partial_{x_n}v_s^1)\boldsymbol{e}_1'\end{array}\right).$$

We next introduce an operator A on $L^2(0,1)$ with domain of definition $D(A)$. We define A by $A\varphi = -\partial_{x_n}^2\varphi$ for $\varphi\in D(A) = H^2(0,1)\cap H_0^1(0,1)$. By (4.35) we have

$$(|\xi'|^2 + A)w' = -\frac{\alpha_0}{\nu}i\xi'\sigma\rho_s + h_1'.$$

It then follows that

$$w' = -\frac{\alpha_0}{\nu} i\xi' \sigma(|\xi'|^2 + A)^{-1}\rho_s + (|\xi'|^2 + A)^{-1}h'_1.$$

Substituting this into (4.34) we have

(4.36)
$$\partial_t \sigma + [Q_0 \widetilde{B}_{\xi'} u^{(0)}]\sigma + \tfrac{\alpha_0\gamma^2}{\nu}[\rho_s(|\xi'|^2 + A)^{-1}\rho_s]|\xi'|^2\sigma = h^0,$$

where

$$h^0 = -i\xi_1[v_s^1\phi_1] - \gamma^2[\rho_s i\xi' \cdot (|\xi'|^2 + A)^{-1}h'_1].$$

We multiply (4.36) by $\overline{\sigma}$. Then since $[Q_0\widetilde{B}_{\xi'}u^{(0)}] \in i\mathbf{R}$, the real part of the resulting equation yields

$$\frac{1}{2}\frac{d}{dt}|\sigma|^2 + \frac{\alpha_0\gamma^2}{\nu}[\rho_s(|\xi'|^2 + A)^{-1}\rho_s]|\xi'|^2|\sigma|^2 = \mathrm{Re}\,(\overline{\sigma}h^0).$$

Since

$$|\xi'|^2(|\xi'|^2 + A)^{-1}\rho_s \to \rho_s \text{ in } L^2(0,1) \text{ as } |\xi'| \to \infty$$

and

$$\begin{aligned}&(|\xi'|^2 + A)^{-1}\\ &\quad = A^{-1} - A^{-2}|\xi'|^2 \textstyle\sum_{N=0}^{\infty}(-1)^N|\xi'|^{2N}A^{-N} \text{ for } |\xi'| < 1,\end{aligned}$$

and since $[\rho_s(|\xi'|^2 + A)^{-1}\rho_s] = |(|\xi'|^2 + A)^{-\frac{1}{2}}\rho_s|_2^2$ is continuous in ξ', we see that there exists a constant $\tilde{\alpha}_0 > 0$ such that

$$\frac{\alpha_0\gamma^2}{\nu}[\rho_s(|\xi'|^2 + A)^{-1}\rho_s]|\xi'|^2 \ge \frac{\tilde{\alpha}_0\gamma^2}{\nu}\min\{1, |\xi'|^2\}$$

for all ξ'.

Furthermore, since

$$|(|\xi'|^2 + A)^{-1}f|_2 \le \frac{1}{|\xi'|^2 + 1}|f|_2,$$

a direct calculation gives

$$\begin{aligned}&\left|\mathrm{Re}\,(\overline{\sigma}h^0)\right|\\ &\quad \le \tfrac{\tilde{\gamma}^2}{\nu}\left(\tfrac{1}{4} + \tfrac{(\nu+\tilde{\nu})^2}{\gamma^4} + \tfrac{1}{\nu}\right)\min\{1, |\xi'|^2\}\\ &\qquad + C\left\{\left(\tfrac{\nu}{\gamma^2} + \tfrac{\gamma^2}{\nu}\right)(1 + |\xi'|^2)|\phi_1|_2^2 + \tfrac{\gamma^2}{\nu}|\sqrt{\rho_s}\partial_t w_1|_2^2\right.\\ &\qquad \left. + \tfrac{\gamma^2}{\nu}|w_1|_2^2 + \tfrac{\gamma^2}{\nu}\tilde{\nu}^2|i\xi' \cdot w'_1 + \partial_{x_n}w_1^n|_2^2\right\}.\end{aligned}$$

We thus find the desired estimate provided that $\frac{(\nu+\tilde{\nu})^2}{\gamma^4} + \frac{1}{\nu} \le \frac{1}{2}$. This completes the proof. □

We are now in a position to prove Proposition 4.2.

Proof of Proposition 4.2. For a given $R > 0$ we assume $|\xi'| \le R$.

Let b_2 be a positive number (to be determined later) and define $E_2(u)$ by

$$\begin{aligned}E_2(u) \;&=\; \tfrac{\alpha_0}{\gamma^2}|\sigma|^2 + E_0(u_1) + \tfrac{b_2}{\nu+\tilde{\nu}}E_1(u)\\ &\quad + \tfrac{1}{\gamma^2}\left|\sqrt{\tfrac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}}\partial_{x_n}\phi_1\right|_2^2.\end{aligned}$$

We add (4.9) and $\frac{b_2}{\nu+\tilde{\nu}} \times$ (4.14) to (4.29). Then

$$\begin{aligned}&\tfrac{d}{dt}E_2(u) + \left(1 + \tfrac{b_2}{\nu+\tilde{\nu}} + \tfrac{b_1b_2\gamma^2}{\nu(\nu+\tilde{\nu})}\right)\widetilde{D}_{\xi'}(w_1)\\ &\quad + \tfrac{b_2}{\nu+\tilde{\nu}}\left|\sqrt{\rho_s}\partial_t w_1\right|_2^2 + \tfrac{1}{\nu+\tilde{\nu}}\left|\tfrac{\tilde{P}'(\rho_s)}{\gamma^2}\partial_{x_n}\phi_1\right|_2^2\\ &\quad \le C\left\{\left(\tfrac{\nu+\tilde{\nu}}{\nu}|\omega|^2 + \tfrac{1}{\nu+\tilde{\nu}}\left(\tfrac{1}{\nu} + |\xi'|^2\right)\right)\widetilde{D}_{\xi'}(w_1)\right.\\ &\qquad + \tfrac{1}{\nu+\tilde{\nu}}|\sqrt{\rho_s}\partial_t w_1|_2^2\\ &\qquad \left. + \left(\tfrac{\nu+\tilde{\nu}}{\gamma^4} + \tfrac{1}{\gamma^2}\right)\left((1 + |\xi'|^2)|\phi_1|_2^2 + |\xi'|^2|\sigma|^2\right)\right\}\\ &\quad + C\tfrac{b_2}{\nu+\tilde{\nu}}\left\{\left(\tfrac{1}{\gamma^2} + \tfrac{1}{\nu} + \tfrac{\nu^2}{\gamma^4}\right)|\phi_1|_2^2 + \tfrac{1}{\gamma^2}|\xi'|^2|\phi_1|_2^2\right.\\ &\qquad \left. + \left(\tfrac{\nu+\tilde{\nu}}{\gamma^2} + \tfrac{1}{\nu} + \tfrac{1}{\gamma^2}\right)|\xi'|^2|\sigma|^2 + \tfrac{(\nu+\tilde{\nu})^2}{\gamma^4}|\xi'|^4|\sigma|^2\right\}.\end{aligned}$$

Take $b_2 > 0$ so that $C \le \frac{b_2}{2}$. Then $|\omega|$ is taken so small that $C\frac{\nu+\tilde{\nu}}{\nu}|\omega|^2 \le \frac{1}{4}$; and ν and $\frac{\gamma^2}{\nu}$ are taken so large that $\frac{C}{\nu} \le \frac{b_2}{4}$ and $CR^2 \le \frac{b_1b_2}{4}\frac{\gamma^2}{\nu}$. Then the terms $\widetilde{D}_{\xi'}(w_1)$ and $|\sqrt{\rho_s}\partial_t w_1|_2^2$ on the right are absorbed into the left. Since $|\phi_1|_2^2 \le C_1\left|\frac{\tilde{P}'(\rho_s)}{\gamma^2}\partial_{x_n}\phi_1\right|_2^2$ for some $C_1 > 0$, we take γ^2, ν and $\frac{\gamma^2}{\nu+\tilde{\nu}}$ so large that $CC_1\left(\frac{\nu+\tilde{\nu}}{\gamma^4} + \frac{1}{\gamma^2}\right)(1 + R^2) \le \frac{1}{4}\frac{1}{\nu+\tilde{\nu}}$ and $CC_1b_2\left\{\left(\frac{1}{\gamma^2} + \frac{1}{\nu} + \frac{\nu^2}{\gamma^4}\right) + \frac{R^2}{\gamma^2}\right\} \le \frac{1}{4}$. Then the terms $|\phi_1|_2^2$ on the right are absorbed into the left. We thus arrive at

(4.37)
$$\begin{aligned}&\tfrac{d}{dt}E_2(u) + \tfrac{1}{2}\left(1 + \tfrac{b_2}{\nu+\tilde{\nu}} + \tfrac{b_1b_2\gamma^2}{\nu(\nu+\tilde{\nu})}\right)\widetilde{D}_{\xi'}(w_1)\\ &\quad + \tfrac{1}{2}\tfrac{b_2}{\nu+\tilde{\nu}}\left|\sqrt{\rho_s}\partial_t w_1\right|_2^2 + \tfrac{1}{2}\tfrac{1}{\nu+\tilde{\nu}}\left|\tfrac{\tilde{P}'(\rho_s)}{\gamma^2}\partial_{x_n}\phi_1\right|_2^2\\ &\quad \le C\left(\tfrac{\nu+\tilde{\nu}}{\gamma^4} + \tfrac{1}{\nu(\nu+\tilde{\nu})} + \tfrac{1}{\gamma^2}\right)|\xi'|^2(1 + |\xi'|^2)|\sigma|^2.\end{aligned}$$

Let b_3 be a positive number (to be determined later) and define $E_3(u)$ by

$$E_3(u) = E_2(u) + \tfrac{b_3\nu}{\gamma^2(\nu+\tilde{\nu})}|\sigma|^2$$

We add $\frac{b_3\nu}{\gamma^2(\nu+\tilde{\nu})} \times$ (4.33) to (4.37). Then

$$\begin{aligned}&\tfrac{d}{dt}E_3(u) + \tfrac{1}{2}\left(1 + \tfrac{b_2}{\nu+\tilde{\nu}} + \tfrac{b_1b_2\gamma^2}{\nu(\nu+\tilde{\nu})}\right)\widetilde{D}_{\xi'}(w_1)\\ &\quad + \tfrac{1}{2}\tfrac{b_2}{\nu+\tilde{\nu}}\left|\sqrt{\rho_s}\partial_t w_1\right|_2^2\\ &\quad + \tfrac{1}{2}\tfrac{1}{\nu+\tilde{\nu}}\left|\tfrac{\tilde{P}'(\rho_s)}{\gamma^2}\partial_{x_n}\phi_1\right|_2^2 + \tfrac{b_3\tilde{\alpha}_0}{\nu+\tilde{\nu}}\min\{1, |\xi'|^2\}|\sigma|^2\\ &\quad \le C\left(\tfrac{\nu+\tilde{\nu}}{\gamma^4} + \tfrac{1}{\nu(\nu+\tilde{\nu})} + \tfrac{1}{\gamma^2}\right)|\xi'|^2(1 + |\xi'|^2)|\sigma|^2\\ &\qquad + C\tfrac{b_3\nu}{\gamma^2(\nu+\tilde{\nu})}\left\{\left(\tfrac{\nu}{\gamma^2} + \tfrac{\gamma^2}{\nu}\right)(1 + |\xi'|^2)|\phi_1|_2^2\right.\\ &\qquad \left. + \tfrac{\gamma^2}{\nu}|\sqrt{\rho_s}\partial_t w_1|_2^2 + \tfrac{\gamma^2}{\nu}(1 + \tilde{\nu})\widetilde{D}_{\xi'}(w_1)\right\}.\end{aligned}$$

We may assume $\nu \ge 1$ and $\frac{\nu}{\gamma^2} \le 1$. We take b_3 so small that $Cb_3 \le \frac{1}{4}$, $Cb_3 \le \frac{b_2}{4}$ and $2CC_1b_3(1 + R^2) \le \frac{1}{4}$. Then the terms $\widetilde{D}_{\xi'}(w_1)$, $|\sqrt{\rho_s}\partial_t w_1|_2^2$ and $|\phi_1|_2^2$ on the right are absorbed into the left. We finally take $\frac{\gamma^2}{\nu+\tilde{\nu}}$ and ν so large that

$C\left(\frac{\nu+\tilde{\nu}}{\gamma^4}+\frac{1}{\nu(\nu+\tilde{\nu})}+\frac{1}{\gamma^2}\right)(1+R^2)\max\{1,R^2\}\le\frac{b_3\tilde{\alpha}_0}{2(\nu+\tilde{\nu})}$. We then arrive at

$$\begin{aligned}&\tfrac{d}{dt}E_3(u)+\tfrac{1}{4}\left(1+\tfrac{b_2}{\nu+\tilde{\nu}}+\tfrac{b_1b_2\gamma^2}{\nu(\nu+\tilde{\nu})}\right)\widetilde{D}_{\xi'}(w_1)\\&+\tfrac{1}{4}\tfrac{b_2}{\nu+\tilde{\nu}}\left|\sqrt{\rho_s}\partial_t w_1\right|_2^2+\tfrac{1}{4}\tfrac{1}{\nu+\tilde{\nu}}\left|\tfrac{\tilde{P}'(\rho_s)}{\gamma^2}\partial_{x_n}\phi_1\right|_2^2\\&+\tfrac{b_3\tilde{\alpha}_0}{2(\nu+\tilde{\nu})}\min\{1,|\xi'|^2\}|\sigma|^2\\&\le 0\end{aligned}\tag{4.38}$$

for all ξ' with $|\xi'|\le R$.

Since

$$\widetilde{D}_{\xi'}(w_1)+\left|\tfrac{\tilde{P}'(\rho_s)}{\gamma^2}\partial_{x_n}\phi_1\right|_2^2+|\sigma|^2\ge c_0E_3(u)$$

for some constant $c_0>0$, we see that there exists a constant $\tilde{c}_0>$ such that

$$\frac{d}{dt}E_3(u)+\tilde{c}_0\min\{1,|\xi'|^2\}E_3(u)\le 0.$$

It then follows

$$E_3(u)(t)\le e^{-\tilde{c}_0\min\{1,|\xi'|^2\}(t-1)}E_3(u)(1).\tag{4.39}$$

Let $\widehat{u}={}^T(\widehat{\phi},\widehat{w})$ be the solution of problem (4.1)–(4.5). Since $E_3(u)\ge c_1|u|_{H^1}$ for some constant $c_1>0$, we see from (4.39) that

$$\begin{aligned}&|\xi'|^{2k}|\widehat{u}(\xi',\cdot,t)|_{H^1}^2\\&\le C|\xi'|^{2k}e^{-\tilde{c}_0\min\{1,|\xi'|^2\}(t-1)}E_3(\widehat{u}(\xi',\cdot,1))\end{aligned}\tag{4.40}$$

for ξ' with $|\xi'|\le R$. Integrating this over $|\xi'|\le R$, we obtain

$$\begin{aligned}&\|\partial_{x'}^k U_1(t)u_0\|_{L^2(\mathbf{R}^{n-1};H^1(0,1))}^2\\&\le C(t-1)^{-\frac{n-1}{2}-k}\sup_{|\xi'|\le R}E_3(\widehat{u}(\xi',\cdot,1)).\end{aligned}\tag{4.41}$$

But, similarly to the proof of Theorem 3.1 (see [12]), we can show that

$$\sup_{|\xi'|\le R}E_3(\widehat{u}(\xi',\cdot,1))\le C\|u_0\|_{L^1(\mathbf{R}^{n-1};L^2(0,1))}^2.$$

This, together with (4.41), gives the desired estimate in Proposition 4.2. This completes the proof. □

5. Outline of Proof of Theorem 3.3

Once the estimate (4.40) is established for the bounded frequency part, Theorem 3.3 can be proved in a similar manner to the case of the plane Couette flow [12]. So we here give an outline of the proof.

To prove Theorem 3.3 we further decompose $U_1(t)u_0$ into two parts. Let $0<r<R$. We define $\chi^{(0)}(\xi')$ by

$$\chi^{(0)}(\xi')=1\text{ if }|\xi'|\le r,\quad\chi^{(0)}(\xi')=0\text{ if }|\xi'|\ge r.$$

We decompose $U_1(t)u_0$ as

$$U_1(t)u_0=U_0(t)u_0+U_{1,\infty}(t)u_0,$$

where

$$U_0(t)u_0=\mathscr{F}^{-1}\left(\chi^{(0)}\chi^{(1)}e^{-t\widehat{L}_{\xi'}}\widehat{u}_0\right)$$

and

$$U_{1,\infty}(t)u_0=\mathscr{F}^{-1}\left((1-\chi^{(0)})\chi^{(1)}e^{-t\widehat{L}_{\xi'}}\widehat{u}_0\right).$$

We have the following estimates.

Proposition 5.1. (i) *For each r and R satisfying $0<r<R$ there exist $\nu_0>0$, $\gamma_0>0$ and $\omega_0>0$ such that if $\nu\ge\nu_0$, $\gamma^2/(\nu+\tilde{\nu})\ge\gamma_0^2$ and $|\omega|\le\omega_0$, then $U_{1,\infty}(t)u_0$ satisfies the estimate* (3.5) *in Theorem* 3.3 (ii) *with $\mathscr{U}^{(\infty)}(t)u_0$ replaced by $U_{1,\infty}(t)u_0$ for a constant $d=d(r)>0$.*

(ii) *There exist $r_0>0$, $\nu_0>0$, $\gamma_0>0$ and $\omega_0>0$ such that if $r\le r_0$, $\nu\ge\nu_0$, $\gamma^2/(\nu+\tilde{\nu})\ge\gamma_0^2$ and $|\omega|\le\omega_0$, then $U_0(t)u_0$ is written as*

$$U_0(t)u_0=\mathscr{U}^{(0)}(t)u_0+\mathscr{R}^{(0)}(t)u_0,$$

where $\mathscr{U}^{(0)}(t)u_0$ has the properties in Theorem 3.3 (i) *and $\mathscr{R}^{(0)}(t)u_0$ satisfies the estimate* (3.5) *in Theorem* 3.3 (ii) *with $\mathscr{U}^{(\infty)}(t)u_0$ replaced by $\mathscr{R}^{(0)}(t)u_0$.*

Theorem 3.3 immediately follows from Proposition 4.1 and Proposition 5.1 with $r=r_0$ and $R=R_0$ by setting $\mathscr{U}^{(\infty)}(t)u_0=\mathscr{R}^{(0)}(t)u_0+U_{1,\infty}(t)u_0+U_\infty(t)u_0$. So we will give an outline of the proof of Proposition 5.1.

Proposition 5.1 (i) follows from (4.40); if we integrate (4.40) over $0<r\le|\xi'|\le R$, then

$$\|U_0(t)u_0\|_{H^1}\le Ce^{-d(t-1)}\|\mathscr{U}(1)u_0\|_{H^1}$$

with $d=\frac{1}{2}\tilde{c}_0\min\{1,r^2\}>0$. Applying Theorem 3.1 to estimate $\|\mathscr{U}(1)u_0\|_{H^1}$, we have the desired estimate.

Proposition 5.1 (ii) follows from Lemma 5.4, Theorems 5.5 and 5.6 below. To prove Proposition 5.1 (ii) we first investigate the spectrum of $-\widehat{L}_0$, and, then, by perturbation argument, we analyze the spectrum of $-\widehat{L}_{\xi'}$ for $|\xi'|\ll 1$.

We consider the resolvent problem

$$\lambda u+\widehat{L}_{\xi'}u=f,\tag{5.1}$$

where $\lambda\in\mathbf{C}$ is the resolvent parameter; $u={}^T(\phi,w',w^n)$ is an unknown function of $x_n\in[0,1]$; and $f={}^T(f^0,g',g^n)$ is a given function of $x_n\in[0,1]$. To investigate problem (5.1) we write $\widehat{L}_{\xi'}$ in the following form:

$$\widehat{L}_{\xi'}=\widehat{L}_0+\sum_{j=1}^{n-1}\xi_j\widehat{L}_j^{(1)}+\sum_{j,k=1}^{n-1}\xi_j\xi_k\widehat{L}_{jk}^{(2)},$$

where $\xi'={}^T(\xi_1,\cdots,\xi_{n-1})$, and

$$\widehat{L}_0=\begin{pmatrix}0&0&\gamma^2\partial_{x_n}(\rho_s\,\cdot)\\-\frac{\nu}{\gamma^2\rho_s}\boldsymbol{e}_1'&-\frac{\nu}{\rho_s}\partial_{x_n}^2I_{n-1}&(\partial_{x_n}v_s^1)\boldsymbol{e}_1'\\\partial_{x_n}\left(\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\,\cdot\right)&0&-\frac{\nu+\tilde{\nu}}{\rho_s}\partial_{x_n}^2\end{pmatrix},$$

$$\widehat{L}_j^{(1)} = \begin{pmatrix} iv_s^1\delta_{1j} & i\gamma^2\rho_s\,{}^T\boldsymbol{e}_j' & 0 \\ i\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\boldsymbol{e}_j' & iv_s^1\delta_{1j}I_{n-1} & -i\frac{\tilde{\nu}}{\rho_s}\boldsymbol{e}_j'\partial_{x_n} \\ 0 & -i\frac{\tilde{\nu}}{\rho_s}{}^T\boldsymbol{e}_j'\partial_{x_n} & iv_s^1\delta_{1j} \end{pmatrix},$$

$$\widehat{L}_{jk}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\nu}{\rho_s}\delta_{jk}I_{n-1} + \frac{\tilde{\nu}}{\rho_s}\boldsymbol{e}_j'\,{}^T\boldsymbol{e}_k' & 0 \\ 0 & 0 & \frac{\nu}{\rho_s}\delta_{jk} \end{pmatrix}.$$

Here $\boldsymbol{e}_j'$ denotes the unit vector of $\mathbf{R}^{n-1}$ in ξ_j-direction.

We will treat $\widehat{L}_{\xi'}$ as a perturbation from $\widehat{L}_0$. As for the case $\xi' = 0$ we have the following result.

Lemma 5.2. *There exists a constant $\omega_0 > 0$ such that if $|\omega| \le \omega_0$, then the following assertions hold.*

(i) *There exist positive numbers η_0 and θ_0 with $\theta_0 \in (\frac{\pi}{2}, \pi)$ such that $\Sigma(-\eta_0, \theta_0) \setminus \{0\} \subset \rho(-\widehat{L}_0)$. Furthermore, the following estimates hold uniformly for $\lambda \in \rho(-\widehat{L}_0) \cap \Sigma(-\eta_0, \theta_0) \setminus \{0\}$:*

$$\left|(\lambda + \widehat{L}_0)^{-1} f\right|_{H^1\times L^2} \le \frac{C}{|\lambda|}|f|_{H^1\times L^2},$$

$$\left|\partial_{x_n}^l \widetilde{Q}(\lambda + \widehat{L}_0)^{-1} f\right|_2 \le C\left(\frac{1}{|\lambda|} + \frac{1}{(|\lambda|+1)^{1-\frac{l}{2}}}\right)|f|_{H^1\times L^2}$$

for $l = 1, 2$,

$$\left|\partial_{x_n}^2 Q_0(\lambda + \widehat{L}_0)^{-1} f\right|_2 \le C\left(\frac{1}{|\lambda|} + \frac{1}{(|\lambda|+1)^{\frac{1}{2}}}\right)|f|_{H^2\times H^1}.$$

The same assertion also holds for $-\widehat{L}_0^*$.

(ii) $\lambda = 0$ *is a simple eigenvalue of $-\widehat{L}_0$ and $-\widehat{L}_0^*$; and the associated eigenprojections are given by the ones given in Lemma* 4.3 (iii).

Proof. Lemma 5.2 can be proved by the energy method as in section 4 and the proof of [14, Proposition 3.15], or by a perturbation argument from the case $\omega = 0$. We here give an outline of the latter argument.

Since $\widehat{L}_0$ depends on ω through ρ_s (recall that $\rho_s = 1 + O(\omega)$), we introduce the notation $\widehat{\mathcal{L}}_\omega$:

$$\widehat{\mathcal{L}}_\omega = \widehat{A}_0 + \widehat{B}_0.$$

Then we have

$$\widehat{L}_0 = \widehat{A}_0 + \widehat{B}_0 + \widehat{C}_0 = \widehat{\mathcal{L}}_\omega + \widehat{C}_0.$$

Although $\widehat{C}_0$ also depends on ω, we keep the same notation for it. In terms of $\widehat{\mathcal{L}}_\omega$, problem (5.1) is written as

$$\lambda u + \widehat{\mathcal{L}}_\omega u = f - \widehat{C}_0 u,$$

namely,

$$\lambda\phi + \gamma^2\partial_{x_n}(\rho_s w^n) = f^0, \tag{5.2}$$

$$\lambda w' - \frac{\nu}{\rho_s}\partial_{x_n}^2 w' = g' + \frac{\nu}{\gamma^2\rho_s}\phi\boldsymbol{e}_1' - (\partial_{x_n} v_s^1)w^n\boldsymbol{e}_1', \tag{5.3}$$

$$\lambda w^n - \frac{\nu+\tilde{\nu}}{\rho_s}\partial_{x_n}^2 w^n + \partial_{x_n}\left(\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\phi\right) = g^n \tag{5.4}$$

with the boundary condition

$$w|_{x_n=0,1} = 0.$$

We see from (5.2) and (5.4) that ϕ and w^n do not couple with w' and they are determined by f^0 and g^n. Once ϕ and w^n are known, then w' is obtained by inverting (5.3). In view of this observation, we have

$$u = (\lambda + \widehat{\mathcal{L}}_\omega)^{-1} f - \widehat{C}_0(\lambda + \widehat{\mathcal{L}}_\omega)^{-1} f. \tag{5.5}$$

Here we have used $Q'(\lambda + \widehat{\mathcal{L}}_\omega)^{-1} = (\lambda + \widehat{\mathcal{L}}_\omega)^{-1} Q'$ and $\widehat{C}_0 = Q'\widehat{C}_0 = \widehat{C}_0(Q_0 + Q_n)$, where $Q' = I - (Q_0 + Q_n)$. Therefore, estimates on $(\lambda + \widehat{\mathcal{L}}_\omega)^{-1}$ will lead to the desired estimates on $u = (\lambda + \widehat{L}_0)^{-1} f$.

We regard $\widehat{\mathcal{L}}_\omega$ as a perturbation from $\widehat{\mathcal{L}}_0$ to estimate $(\lambda + \widehat{\mathcal{L}}_\omega)^{-1}$. As for $\widehat{\mathcal{L}}_0$ we have the following result.

Lemma 5.3. (i) *There exist positive numbers η_0 and θ_0 with $\theta_0 \in (\frac{\pi}{2}, \pi)$ such that $\Sigma(-\eta_0, \theta_0) \setminus \{0\} \subset \rho(-\widehat{\mathcal{L}}_0)$. Furthermore, the following estimates hold uniformly for $\lambda \in \rho(-\widehat{\mathcal{L}}_0) \cap \Sigma(-\eta_0, \theta_0) \setminus \{0\}$:*

$$\left|(\lambda + \widehat{\mathcal{L}}_0)^{-1} f\right|_{H^l\times L^2} \le \frac{C}{|\lambda|}|f|_{H^l\times L^2} \quad (l = 0, 1),$$

$$\left|\partial_{x_n}^l \widetilde{Q}(\lambda + \widehat{\mathcal{L}}_0)^{-1} f\right|_2 \le \frac{C}{(|\lambda|+1)^{1-\frac{l}{2}}}\left(1 + \frac{1}{|\lambda|}\right)|f|_{H^{l-1}\times L^2} \quad (l = 1, 2).$$

(ii) $\lambda = 0$ *is a simple eigenvalue of $-\widehat{\mathcal{L}}_0$.*

Lemma 5.3 was given in [12], which can be proved in a similar manner to the proof of [10, Lemmas 3.1 and 3.2].

We continue the proof of Lemma 5.2. Since ρ_s is smooth, strictly positive, and $\rho_s = 1 + O(\omega)$, we have

$$\left|(\widehat{\mathcal{L}}_\omega - \widehat{\mathcal{L}}_0)u\right|_{H^1\times L^2} \le C|\omega||u|_{H^1\times H^2}.$$

This, together with Lemma 5.3, implies that if $|\omega| \ll 1$, then $\Sigma(-\frac{\eta_0}{2}, \theta_0) \cap \{|\lambda| \ge \frac{\eta_0}{2}\} \subset \rho(-\widehat{\mathcal{L}}_\omega)$ and

$$\left|(\lambda + \widehat{\mathcal{L}}_\omega)^{-1} f\right|_{H^1\times L^2} \le \frac{C}{|\lambda|}|f|_{H^1\times L^2},$$

$$\left|\partial_{x_n}^l \widetilde{Q}(\lambda + \widehat{\mathcal{L}}_\omega)^{-1} f\right|_2 \le C\left(\frac{1}{|\lambda|} + \frac{1}{(|\lambda|+1)^{1-\frac{l}{2}}}\right)|f|_{H^1\times L^2}$$

for $l = 1, 2$. Furthermore, in view of the proof of Lemma 4.3, one can find that $\lambda = 0$ is a simple eigenvalue; and $\sigma(-\widehat{\mathcal{L}}_\omega) \cap \{|\lambda| < \frac{\eta_0}{2}\} = \{0\}$. Lemma 5.2 now follows from (5.5), except the estimate on $\partial_{x_n}^2 Q_0(\lambda + \widehat{L}_0)^{-1} f$.

Let us estimate $\partial_{x_n}^2 Q_0(\lambda+\widehat{L}_0)^{-1}f$. As in the proof of (4.32), one can obtain

$$\lambda\partial_{x_n}\phi+\frac{\rho_s\tilde{P}'(\rho_s)}{\nu+\tilde{\nu}}\partial_{x_n}\phi=H,$$

where

$$\begin{aligned} H &= -\gamma^2\partial_{x_n}\rho_s\partial_{x_n}w^n-\gamma^2\partial_{x_n}(w^n\partial_{x_n}\rho_s)+\partial_{x_n}f^0 \\ &\quad -\tfrac{\gamma^2\rho_s^2}{\nu+\tilde{\nu}}\left\{\lambda w^n+\phi\partial_{x_n}\left(\tfrac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\right)-g^n\right\}. \end{aligned}$$

It follows that

$$\left(\lambda+\tfrac{\gamma^2}{\nu+\tilde{\nu}}\right)\partial_{x_n}\phi=\tfrac{\gamma^2}{\nu+\tilde{\nu}}\left(1-\tfrac{\rho_s\tilde{P}'(\rho_s)}{\gamma^2}\right)\partial_{x_n}\phi+H.$$

Since $\rho_s=1+O(\omega)$ and $\gamma^2=\tilde{P}'(1)$, we obtain, for $|\omega|\ll 1$,

$$\begin{aligned} &|\partial_{x_n}^2\phi|_2 \\ &\le \tfrac{C}{|\lambda|+1}\left\{|\omega||\partial_{x_n}^2\phi|_2+|\phi|_{H^1}+|\partial_{x_n}H|_2\right\} \\ &\le \tfrac{1}{2}|\partial_{x_n}^2\phi|_2 \\ &\qquad +\tfrac{C}{|\lambda|+1}\left\{|\phi|_{H^1}+|w|_{H^2}+|\lambda||w|_{H^1}+|f|_{H^2\times H^1}\right\} \\ &\le \tfrac{1}{2}|\partial_{x_n}^2\phi|_2+C\left(\tfrac{1}{|\lambda|}+\tfrac{1}{(|\lambda|+1)^{\frac{1}{2}}}\right)|f|_{H^2\times H^1}, \end{aligned}$$

from which the desired estimate on $\partial_{x_n}^2 Q_0(\lambda+\widehat{L}_0)^{-1}f$ follows. This completes the proof. □

Based on Lemma 5.2, we can obtain the following estimates on $(\lambda+\widehat{L}_{\xi'})^{-1}$ for $|\xi'|\ll 1$, by changing η_0 and θ_0 suitably if necessary.

Lemma 5.4. *Let ω_0 be the number given in Lemma 5.2. Then if $|\omega|\le\omega_0$, then there exists a positive number $r_0=r_0(\eta_0,\theta_0)$ such that the set $\Sigma(-\eta_0,\theta_0)\cap\{\lambda;\ |\lambda|\ge\frac{\eta_0}{2}\}$ is in $\rho(-\widehat{L}_{\xi'})$ for $|\xi'|\le r_0$. Furthermore, the following estimates hold uniformly in $\lambda\in\Sigma(-\eta_0,\theta_0)\cap\{\lambda;\ |\lambda|\ge\frac{\eta_0}{2}\}$ and ξ' with $|\xi'|\le r_0$:*

$$\left|(\lambda+\widehat{L}_{\xi'})^{-1}f\right|_{H^1\times L^2}\le\tfrac{C}{|\lambda|}|f|_{H^1\times L^2},$$

$$\left|\partial_{x_n}^l\widetilde{Q}(\lambda+\widehat{L}_{\xi'})^{-1}f\right|_2\le C\left(\tfrac{1}{|\lambda|}+\tfrac{1}{(|\lambda|+1)^{1-\frac{l}{2}}}\right)|f|_{H^1\times L^2}$$

for $l=1,2$,

$$\left|\partial_{x_n}^2 Q_0(\lambda+\widehat{L}_{\xi'})^{-1}f\right|_2\le C\left(\tfrac{1}{|\lambda|}+\tfrac{1}{(|\lambda|+1)^{\frac{1}{2}}}\right)|f|_{H^2\times H^1}.$$

The same assertion also holds for $-\widehat{L}_{\xi'}^*$.

The proof of Lemma 5.4 is the same as that of [12, Theorem 5.2].

As for the spectrum of $-\widehat{L}_{\xi'}$ near $\lambda=0$, we have the following result.

Theorem 5.5. *There exist positive numbers ν_0, γ_0, ω_0 and r_0 such that if $\nu\ge\nu_0$, $\gamma^2/(\nu+\tilde{\nu})\ge\gamma_0^2$ and $|\omega|\le\omega_0$, then for each ξ' with $|\xi'|\le r_0$ it holds that*

$$\sigma(-\widehat{L}_{\xi'})\cap\{\lambda;|\lambda|<\frac{\eta_0}{2}\}=\{\lambda_0(\xi')\},$$

where $\lambda_0(\xi')$ is a simple eigenvalue of $-\widehat{L}_{\xi'}$ that has the form

$$\lambda_0(\xi')=-i\kappa_0-\kappa_1\xi_1^2-\kappa_2|\xi''|^2+O(|\xi'|^3)$$

as $|\xi'|\to 0$. Here $\xi''=(\xi_2,\cdots,\xi_{n-1})$; and κ_j $(j=0,1,2)$ are some positive numbers having the properties given in Theorem 3.3 (i).

Proof. Theorem 5.5 is proved by applying the analytic perturbation theory [15]. We here derive the asymptotics of $\lambda_0(\xi')$ only.

We proceed as in [12]. Since $\lambda_0(\xi')$ is simple, we can see that $\lambda_0(\xi')$ and $\widehat{\Pi}(\xi')$ are expanded as

$$\lambda_0(\xi')=\lambda^{(0)}+\sum_{j=1}^{n-1}\xi_j\lambda_j^{(1)}+\sum_{j,k=1}^{n-1}\xi_j\xi_k\lambda_{jk}^{(2)}+O(|\xi'|^3),$$

$$\widehat{\Pi}(\xi')=\widehat{\Pi}^{(0)}+\sum_{j=1}^{n-1}\xi_j\widehat{\Pi}_j^{(1)}+O(|\xi'|^2)$$

with $\lambda^{(0)}=0$ and

$$\begin{aligned} \lambda_j^{(1)} &= -\langle\widehat{L}_j^{(1)}u^{(0)},u^{(0)*}\rangle=-[Q_0\widehat{L}_j^{(1)}u^{(0)}], \\ \lambda_{jk}^{(2)} &= -\langle\widehat{L}_{jk}^{(2)}u^{(0)},u^{(0)*}\rangle+\langle\widehat{L}_j^{(1)}\widehat{S}\widehat{L}_k^{(1)}u^{(0)},u^{(0)*}\rangle \\ &= -[Q_0\widehat{L}_{jk}^{(2)}u^{(0)}]+[Q_0\widehat{L}_j^{(1)}\widehat{S}\widehat{L}_k^{(1)}u^{(0)}], \\ \widehat{\Pi}_j^{(1)} &= -\widehat{\Pi}^{(0)}\widehat{L}_j^{(1)}\widehat{S}-\widehat{S}\widehat{L}_j^{(1)}\widehat{\Pi}^{(0)}, \end{aligned}$$

where $\widehat{S}=\left((I-\widehat{\Pi}^{(0)})\widehat{L}_0(I-\widehat{\Pi}^{(0)})\right)^{-1}$.

Since $\frac{\tilde{P}'(\rho_s)}{\gamma^2\rho_s}\phi^{(0)}=\alpha_0$, we have

$$\widehat{L}_j^{(1)}u^{(0)}=i\begin{pmatrix} v_s^1\phi^{(0)}\delta_{1j}+\gamma^2\rho_s w^{(0),1}\delta_{1j} \\ \alpha_0\boldsymbol{e}'_j+v_s^1 w^{(0),1}\delta_{1j}\boldsymbol{e}'_j \\ -\frac{\tilde{\nu}}{\rho_s}\partial_{x_n}w^{(0),1}\delta_{1j}\end{pmatrix}. \tag{5.6}$$

It follows that

$$\begin{aligned} \lambda_j^{(1)}=-[Q_0\widehat{L}_j^{(1)}u^{(0)}] &= -i[v_s^1\phi^{(0)}\delta_{1j}+\gamma^2\rho_s w^{(0),1}\delta_{1j}] \\ &= -i\left(\tfrac{1}{6}+O(\omega)\right)\delta_{1j}. \end{aligned}$$

We next compute $\lambda_{jk}^{(2)}$. Since $Q_0\widehat{L}_{jk}^{(2)}=0$, we have

$$\langle\widehat{L}_{jk}^{(2)}u^{(0)},u^{(0)*}\rangle=[Q_0\widehat{L}_{jk}^{(2)}u^{(0)}]=0.$$

Let $j\ge 2$. Then it follows from (5.6) that

$$\widehat{S}\widehat{L}_j^{(1)}u^{(0)}=\begin{pmatrix}0 \\ i\frac{\alpha_0}{\nu}(-\partial_{x_n}^2)^{-1}\rho_s\boldsymbol{e}'_j \\ 0\end{pmatrix}.$$

We thus find that if $k \neq j$, then

$$\widehat{L}_k^{(1)} \widehat{S} \widehat{L}_j^{(1)} u^{(0)} = \begin{pmatrix} 0 \\ * \\ * \end{pmatrix},$$

which implies that

$$\langle \widehat{L}_k^{(1)} \widehat{S} \widehat{L}_j^{(1)} u^{(0)}, u^{(0)*} \rangle = [Q_0 \widehat{L}_k^{(1)} \widehat{S} \widehat{L}_j^{(1)} u^{(0)}] = 0.$$

Therefore, $\lambda_{kj}^{(2)} = 0$ for $j \geq 2$ and $k \neq j$.

Furthermore, we have

$$\widehat{L}_j^{(1)} \widehat{S} \widehat{L}_j^{(1)} u^{(0)} = \begin{pmatrix} -\frac{\alpha_0 \gamma^2}{\nu} \rho_s (-\partial_{x_n}^2)^{-1} \rho_s \\ * \\ * \end{pmatrix},$$

and, hence,

$$\begin{aligned} \langle \widehat{L}_j^{(1)} \widehat{S} \widehat{L}_j^{(1)} u^{(0)}, u^{(0)*} \rangle &= [Q_0 \widehat{L}_j^{(1)} \widehat{S} \widehat{L}_j^{(1)} u^{(0)}] \\ &= -[\tfrac{\alpha_0 \gamma^2}{\nu} \rho_s (-\partial_{x_n}^2)^{-1} \rho_s] \\ &= -\tfrac{\alpha_0 \gamma^2}{\nu} |(-\partial_{x_n}^2)^{-\frac{1}{2}} \rho_s|_2^2 < 0. \end{aligned}$$

Since $-\frac{\alpha_0 \gamma^2}{\nu} |(-\partial_{x_n}^2)^{-\frac{1}{2}} \rho_s|_2^2 = -\frac{\gamma^2}{12\nu}(1 + O(\omega))$, we obtain

$$\lambda_{jj}^{(2)} = -\kappa_2, \quad \kappa_2 = \frac{\gamma^2}{12\nu}(1 + O(\omega)) > 0$$

for $j \geq 2$.

We next consider λ_{k1}. Let us compute $\widehat{S} \widehat{L}_1^{(1)} u^{(0)}$. We first note that $(I - \widehat{\Pi}^{(0)}) \widehat{L}_1^{(1)} u^{(0)} = \lambda_1^{(1)} u^{(0)} + \widehat{L}_1^{(1)} u^{(0)}$. Let $\tilde{u}^{(1)} = {}^T(\tilde{\phi}^{(1)}, \tilde{w}^{(1)'}, \tilde{w}^{(1),n})$ be a solution of

$$\widehat{L}^{(0)} \tilde{u}^{(1)} = (I - \widehat{\Pi}^{(0)}) \widehat{L}_1^{(1)} u^{(0)} = \lambda_1^{(1)} u^{(0)} + \widehat{L}_1^{(1)} u^{(0)}.$$

Then

$$\begin{cases} \gamma^2 \partial_{x_n} (\rho_s \tilde{w}^{(1),n}) = \lambda_1^{(1)} \phi^{(0)} + i v_s^1 \phi^{(0)} + i \gamma^2 \rho_s w^{(0),1}, \\ -\frac{\nu}{\rho_s} \partial_{x_n}^2 \tilde{w}^{(1)'} - \frac{\nu}{\gamma^2 \rho_s} \tilde{\phi}^{(1)} \boldsymbol{e}_1' + (\partial_{x_n} v_s^1) \tilde{w}^{(1),n} \boldsymbol{e}_1' \\ \qquad = (\lambda_1^{(1)} w^{(0),1} + i \frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} \phi^{(0)} + i v_s^1 w^{(0),1}) \boldsymbol{e}_1', \\ -\frac{\nu + \tilde{\nu}}{\rho_s} \partial_{x_n}^2 \tilde{w}^{(1),n} + \partial_{x_n} \left(\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} \tilde{\phi}^{(1)} \right) = -i \frac{\tilde{\nu}}{\rho_s} \partial_{x_n} w^{(0),1}, \\ \tilde{w}^{(1)}|_{x_n = 0,1} = 0. \end{cases}$$

From the first equation we see that

$$\tilde{w}^{(1),n} = O(\tfrac{1}{\gamma^2})(1 + O(\omega)),$$

and, then, by the third equation,

$$\tilde{\phi}^{(1)} = \left(O(\tfrac{\nu + \tilde{\nu}}{\gamma^2}) + O(\tfrac{\tilde{\nu}}{\gamma^2}) \right) (1 + O(\omega)).$$

Since $\frac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s} \phi^{(0)} = \alpha_0$, it follows from the second equation that

$$\tilde{w}^{(1)'} = i \tfrac{\alpha_0}{\nu} (-\partial_{x_n}^2)^{-1} \rho_s \boldsymbol{e}_1' + \left(O(\tfrac{\nu + \tilde{\nu}}{\gamma^4}) + O(\tfrac{1}{\nu \gamma^2}) \right) (1 + O(\omega)).$$

In terms of $\tilde{u}^{(1)}$, $\widehat{S} \widehat{L}_1^{(1)} u^{(0)}$ is given by

$$\widehat{S} \widehat{L}_1^{(1)} u^{(0)} = (I - \Pi^{(0)}) \tilde{u}^{(1)} = \tilde{u}^{(1)} - [\tilde{\phi}^{(1)}] u^{(0)}.$$

Therefore,

$$\begin{aligned} &\langle \widehat{L}_1^{(1)} \widehat{S} \widehat{L}_1^{(1)} u^{(0)}, u^{(0)*} \rangle \\ &\quad = [Q_0 \widehat{L}_1^{(1)} \widehat{S} \widehat{L}_1^{(1)} u^{(0)}] \\ &\quad = -\tfrac{\alpha_0 \gamma^2}{\nu} |(-\partial_{x_n}^2)^{-\frac{1}{2}} \rho_s|_2^2 \\ &\qquad + \left(O(\tfrac{\nu + \tilde{\nu}}{\gamma^2}) + O(\tfrac{1}{\nu}) \right) (1 + O(\omega)) \\ &\quad = -\left(\tfrac{\gamma^2}{12\nu} + O(\tfrac{\nu + \tilde{\nu}}{\gamma^2}) + O(\tfrac{1}{\nu}) \right) (1 + O(\omega)). \end{aligned}$$

We thus conclude

$$\lambda_{11}^{(2)} = -\kappa_1, \quad \kappa_1 = \left(\tfrac{\gamma^2}{12\nu} + O(\tfrac{\nu + \tilde{\nu}}{\gamma^2}) + O(\tfrac{1}{\nu}) \right) (1 + O(\omega)).$$

One can see that $\kappa_1 > 0$ if $\frac{\nu + \tilde{\nu}}{\gamma^2}$, $\frac{1}{\nu}$ and ω are sufficiently small, and the desired asymptotics of $\lambda_0(\xi')$ is obtained. This completes the proof. □

We finally give estimates on the eigenprojection $\widehat{\Pi}(\xi')$ for $\lambda_0(\xi')$, which can be obtained exactly in a similar manner to the proof of [12, Theorem 5.4].

Theorem 5.6. (i) *Let $\widehat{\Pi}(\xi')$ be the eigenprojection associated with $\lambda_0(\xi')$. Then there exists a positive number r_0 such that for any ξ' with $|\xi'| \leq r_0$ the projection $\widehat{\Pi}(\xi')$ is written in the form*

$$\widehat{\Pi}(\xi') u = \int_0^1 \widehat{\Pi}(\xi', x_n, y_n) u(y_n) \, dy_n$$

with

$$\widehat{\Pi}(\xi', x_n, y_n) = \widehat{\Pi}^{(0)} + \sum_{j=1}^{n-1} \xi_j \widehat{\Pi}_j^{(1)}(x_n, y_n) + \widehat{\Pi}^{(2)}(\xi', x_n, y_n).$$

Here $\widehat{\Pi}^{(0)} = Q_0$; and $\widehat{\Pi}_j^{(1)}(x_n, y_n)$ $(j = 1, \cdots, n-1)$ and $\widehat{\Pi}^{(2)}(\xi', x_n, y_n)$ satisfy

$$\left| \partial_{x_n}^k \partial_{y_n}^l \widehat{\Pi}_j^{(1)}(\cdot, \cdot) \right|_{L^\infty((0,1) \times (0,1))} \leq C,$$

$$\left| \partial_{x_n}^k \partial_{y_n}^l \partial_{\xi'}^{\alpha'} \widehat{\Pi}^{(2)}(\xi', \cdot, \cdot) \right|_{L^\infty((0,1) \times (0,1))} \leq C |\xi'|^2$$

for $0 \leq k, l \leq 1$ and any multi-index α' uniformly in ξ' with $|\xi'| \leq r_0$.

(ii) *If $Q_0 u|_{x_n = 0,1} = 0$, then $\widehat{\Pi}(\xi')$ satisfies*

$$\widehat{\Pi}(\xi') \, (\partial_{x_n} u) = -\left(\partial_{y_n} \widehat{\Pi}(\xi') \right) u,$$

$$\widehat{\Pi}^{(0)}\left(\partial_{x_n}u\right)=0,$$
$$\widehat{\Pi}_j^{(1)}\left(\partial_{x_n}u\right)=-\left(\partial_{y_n}\widehat{\Pi}_j^{(1)}\right)u,$$
$$\widehat{\Pi}^{(2)}(\xi')\left(\partial_{x_n}u\right)=-\left(\partial_{y_n}\widehat{\Pi}^{(2)}(\xi')\right)u.$$

Proposition 5.1 (ii) now follows from Lemma 5.4, Theorems 5.5 and 5.6 in the following way.

By Lemma 5.4, $U_0(t)u_0$ is written as

$$U_0(t)u_0=\mathscr{F}^{-1}\left(\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\chi^{(0)}(\xi')(\lambda+\widehat{L}_{\xi'})^{-1}\,d\lambda\right),$$

where $\Gamma=\{\lambda=\eta+se^{\pm i\theta};\ s\ge 0\}$ with some $\eta>0$ and $\theta\in(\frac{\pi}{2},\pi)$.

By Lemma 5.4 and Theorem 5.5, we can deform the contour Γ into $\Gamma_0\cup\widetilde{\Gamma}$ and a suitable circle around 0, where

$$\Gamma_0=\{\lambda=-\eta_0+is;\ |s|\le s_0\},\quad \widetilde{\Gamma}=\{\lambda=\eta+se^{\pm i\theta};\ s\ge\widetilde{s}_0\}.$$

Here we choose positive numbers s_0 and $\widetilde{s}_0$ so that Γ_0 connects with $\widetilde{\Gamma}$ at the end points of Γ_0. It then follows from Theorems 5.5, 5.6 and the residue theorem that $U_0(t)u_0$ is written as

$$U_0(t)u_0=\mathscr{U}^{(0)}(t)u_0+\mathscr{R}^{(0)}(t)u_0, \tag{5.7}$$

where

$$\mathscr{U}^{(0)}(t)u_0=\mathscr{F}^{-1}\left(\chi^{(0)}e^{\lambda_0(\xi')t}\widehat{\Pi}(\xi')\widehat{u}_0\right)$$

and

$$\mathscr{R}^{(0)}(t)u_0=\mathscr{F}^{-1}\left(\frac{1}{2\pi i}\int_{\Gamma_0\cup\widetilde{\Gamma}}e^{\lambda t}\chi^{(0)}(\lambda+\widehat{L}_{\xi'})^{-1}\widehat{u}_0\,d\lambda\right).$$

The desired estimate for $\mathscr{R}^{(0)}(t)u_0$ follows from Lemma 5.4.

As for $\mathscr{U}^{(0)}(t)u_0$, we further write it as

$$\begin{aligned}\mathscr{U}^{(0)}(t)u_0 &= G_t*_{x'}\Pi^{(0)}u_0+\mathscr{U}_1^{(0)}(t)u_0+\mathscr{U}_2^{(0)}(t)u_0\\ &\quad+\mathscr{U}_3^{(0)}(t)u_0+\mathscr{U}_4^{(0)}(t)u_0,\end{aligned} \tag{5.8}$$

where

$$G_t*_{x'}\Pi^{(0)}u_0=\mathscr{F}^{-1}\left(e^{-(i\kappa_0\xi_1+\kappa_1\xi_1^2+\kappa_2|\xi''|^2)t}\widehat{\Pi}^{(0)}\widehat{u}_0\right),$$

$$\mathscr{U}_1^{(0)}(t)u_0=\mathscr{F}^{-1}\left((\chi^{(0)}-1)e^{-(i\kappa_0\xi_1+\kappa_1\xi_1^2+\kappa_2|\xi''|^2)t}\widehat{\Pi}^{(0)}\widehat{u}_0\right),$$

$$\mathscr{U}_2^{(0)}(t)u_0=\mathscr{F}^{-1}\left(\chi^{(0)}e^{-(i\kappa_0\xi_1+\kappa_1\xi_1^2+\kappa_2|\xi''|^2)t}\widehat{\Pi}^{(1)}(\xi')\widehat{u}_0\right),$$

$$\mathscr{U}_3^{(0)}(t)u_0=\mathscr{F}^{-1}\left(\chi^{(0)}e^{-(i\kappa_0\xi_1+\kappa_1\xi_1^2+\kappa_2|\xi''|^2)t}\widehat{\Pi}^{(2)}(\xi')\widehat{u}_0\right)$$

and

$$\mathscr{U}_4^{(0)}(t)u_0=\mathscr{F}^{-1}\left(\chi^{(0)}(e^{\lambda_0(\xi')t}-e^{-(i\kappa_0\xi_1+\kappa_1\xi_1^2+\kappa_2|\xi''|^2)t})\widehat{\Pi}(\xi')\widehat{u}_0\right)$$

with

$$\widehat{\Pi}^{(1)}(\xi')=\sum_{j=1}^{n-1}\xi_j\widehat{\Pi}_j^{(1)}.$$

Applying Theorem 5.5 and Theorem 5.6 to (5.8), we see that $\mathscr{U}^{(0)}(t)u_0$ has the desired properties. We thus conclude that (5.7) gives the desired decomposition of $U_0(t)$ as described in Proposition 5.1.

Acknowledgments

This work was partially supported by Grant-in Aid for Scientific Research (B) No.19340033 from the Japan Society for the Promotion of Science.

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