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Hypergeometric $\boldsymbol{\tau}$-functions of the q-Painlevé system of type $E_8^{(1)}$

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Abstract

We present the τ-functions for the hypergeometric solutions to the q-Painlevé system of type $E_8^{(1)}$ in a determinant formula whose entries are given by Rahman's q-hypergeometric integrals. By using the symmetry of the q-hypergeometric integral, we can construct fifty-six solutions and describe the action of $W(E_7^{(1)})$ on the solutions.

1 Introduction

Discrete Painlevé equations and their solutions have been studied from various viewpoints. In particular, Sakai [17] gave a natural framework for discrete Painlevé equations by means of the geometry of rational surfaces. Each equation is defined by the group of Cremona transformations on a family of surfaces obtained by blowing-up at nine points on $\mathbb{P}^2$. According to the types of rational surfaces, those discrete Painlevé equations are classified in terms of affine root systems. Also, their symmetries are described by means of affine Weyl groups, the lattice part of which gives rise to difference equations.

The elliptic difference Painlevé equation, the master equation of all the continuous and discrete Painlevé equations, is a discrete dynamical system defined on a family of rational surfaces parameterized by nine-point configurations in general position on $\mathbb{P}^2$. In the previous papers [7, 10], we presented an algebraic formulation for this system in terms of τ-functions and showed the equivalence to the formulation proposed by Ohta, Ramani and Grammaticos [15]. Also, we gave a geometric description of the elliptic difference Painlevé equation in terms of plane curves.

Similarly to the Painlevé differential equations, discrete Painlevé equations admit particular solutions expressible in terms of various hypergeometric functions. In fact, we showed that the elliptic difference Painlevé equation has special Riccati type solutions expressed by the elliptic hypergeometric function

${}_{10}E_9$ [7]. Regarding the q-difference Painlevé equations, we constructed hypergeometric solutions to those equations by means of a geometric approach and direct linearization of q-difference Riccati equations [8, 9]. In particular, the Riccati solution to the q-difference Painlevé equation whose symmetry is affine Weyl group $W(E_8^{(1)})$ is expressed in terms of the q-hypergeometric series

$$\begin{aligned}
&{}_{10}W_9(a_0; a_1, \ldots, a_7; q, z) \\
&= {}_{10}\phi_9\left(\begin{matrix} a_0,\ qa_0^{1/2},\ -qa_0^{1/2},\ a_1,\ \ldots,\ a_7 \\ a_0^{1/2}, -a_0^{1/2}, qa_0/a_1, \ldots, qa_0/a_7 \end{matrix} ; q, z \right) \\
&= \sum_{k=0}^{\infty} \frac{1-q^{2k}a_0}{1-a_0} \frac{(a_0;q)_k}{(q;q)_k} \prod_{i=1}^{7} \frac{(a_i;q)_k}{(qa_0/a_i;q)_k} z^k,
\end{aligned} \tag{1.1}$$

where $(a;q)_k = \prod_{i=0}^{k-1}(1-aq^i)$. However, from the viewpoint of τ-functions, what we have done in [8, 9] is to consider the evolution equation with respect to a certain translation and to determine the ratio of τ-functions for the hypergeometric solutions up to multiplication by some normalization factors.

The aim of this paper is to completely determine the τ-functions for the hypergeometric solutions (hypergeometric τ-functions for short) to the q-Painlevé system of type $E_8^{(1)}$.

This paper is organized as follows. In Section 2, we give a brief review on the formulation of the discrete Painlvé equation of type $E_8^{(1)}$ in terms of the lattice τ-functions. Section 3 is devoted to a preparation for constructing the hypergeometric τ-functions. We decompose the lattice on which the τ-functions are defined into a family of seven-dimensional lattices, and explain a basic idea for constructing hypergeometric τ-functions.

In Section 4 – 6, we construct the hypergeometric τ-functions. We find that a discrete analogue of the double gamma function appears as the normalization factor of the hypergeometric τ-functions in Section 4. In Section 5, we find that a class of bilinear equations for the lattice τ-functions yields the contiguity relations for Rahman's q-hypergeometric integral that is expressed as a sum of two balanced ${}_{10}W_9$ series. As is well-known, Rahman's q-hypergeometric integral possesses $W(E_6)$-symmetry [13]. From that, we can construct a set of fifty-six solutions corresponding to the coset $W(E_7)/W(E_6)$, and describe the action of $W(E_7^{(1)})$ on the solutions. It is known that many of hypergeometric solutions to the continuous and discrete Painlevé equations admit a determinant expression [3, 4, 11, 6, 16]. In Section 6, we construct a determinant formula for the hypergeometric τ-functions and show that they are expressed by a "two-directional Casorati determinant".

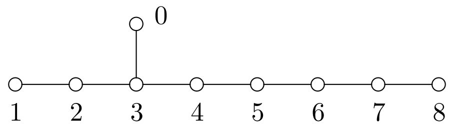
2 The discrete Painlevé system of type $E_8^{(1)}$

In this section, we give a brief review on the formulation of the discrete Painlevé equation of type $E_8^{(1)}$ in terms of the lattice τ-functions [7, 10].

Let $L = \bigoplus_{i=0}^{9} \mathbb{Z}\,\mathrm{e}_i$ be a lattice with basis $\{\mathrm{e}_0, \mathrm{e}_1, \ldots, \mathrm{e}_9\}$, and define a symmetric bilinear form $\langle\ ,\ \rangle\ :\ L \times L \to \mathbb{Z}$ by

$$\begin{aligned} &\langle \mathrm{e}_0, \mathrm{e}_0 \rangle = -1, \quad \langle \mathrm{e}_i, \mathrm{e}_i \rangle = 1 \quad (i = 1, 2, \ldots, 9), \\ &\langle \mathrm{e}_i, \mathrm{e}_j \rangle = 0 \qquad\quad (i, j = 0, 1, \ldots, 9;\ i \neq j). \end{aligned} \tag{2.1}$$

Consider the affine Weyl group $W(E_8^{(1)}) = \langle s_0, s_1, \ldots, s_8 \rangle$ associated with the Dynkin diagram:



. (2.2)

The lattice L admits a natural linear action of $W(E_8^{(1)})$ defined by

$$s_i.\Lambda = \Lambda - \langle h_i, \Lambda \rangle\, h_i \qquad (\Lambda \in L) \tag{2.3}$$

for each $i = 0, 1, \ldots, 8$, where the $h_i\,(i = 0, 1, \ldots, 8)$ are the simple coroots defined by

$$h_0 = \mathrm{e}_0 - \mathrm{e}_1 - \mathrm{e}_2 - \mathrm{e}_3, \quad h_i = \mathrm{e}_i - \mathrm{e}_{i+1} \quad (i = 1, \ldots, 8). \tag{2.4}$$

The canonical central element $c = 3\mathrm{e}_0 - \mathrm{e}_1 - \cdots - \mathrm{e}_9$ is orthogonal to all the $h_i\,(i = 0, 1, \ldots, 8)$, and hence $W(E_8^{(1)})$-invariant.

The parameter space for the discrete Painlevé system of type $E_8^{(1)}$ is the ten-dimensional vector space $\oplus_{i=0}^{9} \mathbb{C}\,\mathrm{e}_i$, whose coordinates are denoted by $\varepsilon_i = \langle \mathrm{e}_i, \cdot \rangle\,(i = 0, 1, \ldots, 9)$. The root lattice $Q(E_8^{(1)}) = \oplus_{i=0}^{8} \mathbb{Z}\,\alpha_i$ is generated by the simple roots

$$\alpha_0 = \varepsilon_0 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3, \quad \alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad (i = 1, \ldots, 8). \tag{2.5}$$

The affine Weyl group $W(E_8^{(1)})$ acts on the coordinate functions ε_i in similar way as on the basis e_i. The $W(E_8^{(1)})$-invariant element corresponding to c is given by $\delta = \langle c, \cdot \rangle = 3\varepsilon_0 - \varepsilon_1 - \cdots - \varepsilon_9$, which is called the null root and plays the role of the scaling constant for difference equations in the context of discrete Painlevé equations. For simplicity, we denote the reflection s_α with respect to $\alpha = \varepsilon_{ij} = \varepsilon_i - \varepsilon_j$ or $\alpha = \varepsilon_{ijk} = \varepsilon_0 - \varepsilon_i - \varepsilon_j - \varepsilon_k$ for $i, j, k \in \{1, 2, \ldots, 9\}$ by s_{ij} or s_{ijk}, respectively.

Let us introduce a family of dependent variables $\tau_\Lambda = \tau_\Lambda(\varepsilon)$, $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_9)$, indexed by $\Lambda \in M$, where M is the $W(E_8^{(1)})$-orbit defined by

$$M = W(E_8^{(1)}).\,\mathrm{e}_1 = \{\Lambda \in L \,|\, \langle c, \Lambda \rangle = -1, \langle \Lambda, \Lambda \rangle = 1\} \subset L. \tag{2.6}$$

The action of $W(E_8^{(1)})$ on τ_Λ is defined by

$$w\ :\ \tau_\Lambda \mapsto \tau_{w.\Lambda} \quad (w \in W(E_8^{(1)}),\ \Lambda \in M). \tag{2.7}$$

The discrete Painlevé system of type $E_8^{(1)}$ is equivalent to the overdetermined system defined by the bilinear equations

$$[\varepsilon_{jk}][\varepsilon_{jkl}]\tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_l} + [\varepsilon_{ki}][\varepsilon_{kil}]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_l} + [\varepsilon_{ij}][\varepsilon_{ijl}]\tau_{\mathrm{e}_k}\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_l} = 0 \quad (2.8)$$

for any mutually distinct indices $i, j, k, l \in \{1, 2, \ldots, 9\}$, as well as their $W(E_8^{(1)})$-transforms

$$\begin{aligned} &[w(\varepsilon_{jk})][w(\varepsilon_{jkl})]\tau_{w.\mathrm{e}_i}\tau_{w.(\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_l)} \\ &+[w(\varepsilon_{ki})][w(\varepsilon_{kil})]\tau_{w.\mathrm{e}_j}\tau_{w.(\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_l)} \\ &+[w(\varepsilon_{ij})][w(\varepsilon_{ijl})]\tau_{w.\mathrm{e}_k}\tau_{w.(\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_l)} = 0 \end{aligned} \quad (2.9)$$

for any $w \in W(E_8^{(1)})$. Here, $[x]$ is a nonzero odd holomorphic function on $\mathbb{C}$ satisfying the Riemann relation

$$\begin{aligned} &[x+y][x-y][u+v][u-v] \\ &= [x+u][x-u][y+v][y-v] - [x+v][x-v][y+u][y-u] \end{aligned} \quad (2.10)$$

for any $x, y, u, v \in \mathbb{C}$. There are three classes of such functions; elliptic, trigonometric and rational. These three cases correspond to the three types of difference equations, namely, elliptic difference, q-difference and ordinal difference, respectively. The lattice part of $W(E_8^{(1)})$ gives rise to the difference Painlevé equation.

In the trigonometric case, it is possible to fix the function $[x]$ as $[x] = e^{\pi\sqrt{-1}x} - e^{-\pi\sqrt{-1}x}$ without loss of generality. Introducing the dependent variables f and g by

$$\begin{aligned} f &= \frac{[\varepsilon_{112}]_+\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_3} - [\varepsilon_{233}]_+\tau_{\mathrm{e}_1}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}}{\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_3} - \tau_{\mathrm{e}_1}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}}, \\ g &= \frac{[\varepsilon_{122}]_+\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_3} - [\varepsilon_{133}]_+\tau_{\mathrm{e}_2}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}}{\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_3} - \tau_{\mathrm{e}_2}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}}, \end{aligned} \quad (2.11)$$

where $[x]_+ = e^{\pi\sqrt{-1}\,x} + e^{-\pi\sqrt{-1}\,x}$, one get an explicit expression of the q-difference Painlevé equation of type $E_8^{(1)}$ [15, 14]

$$\begin{aligned} \frac{(\overline{g}\overline{s}t - f)(gst - f) - (\overline{s}^2t^2 - 1)(s^2t^2 - 1)}{\left(\dfrac{\overline{g}}{\overline{s}t} - f\right)\left(\dfrac{g}{st} - f\right) - \left(1 - \dfrac{1}{\overline{s}^2t^2}\right)\left(1 - \dfrac{1}{s^2t^2}\right)} &= \frac{P(f, t, m_1, \ldots, m_7)}{P(f, t^{-1}, m_7, \ldots, m_1)}, \\ \frac{(\underline{f}s\underline{t} - g)(fst - g) - (s^2\underline{t}^2 - 1)(s^2t^2 - 1)}{\left(\dfrac{\underline{f}}{s\underline{t}} - g\right)\left(\dfrac{f}{st} - g\right) - \left(1 - \dfrac{1}{s^2\underline{t}^2}\right)\left(1 - \dfrac{1}{s^2t^2}\right)} &= \frac{P(g, s, m_7, \ldots, m_1)}{P(g, s^{-1}, m_1, \ldots, m_7)}. \end{aligned} \quad (2.12)$$

Here, s and t are independent variables with $t = \tilde{q}^{1/2}s$ and the time evolution of the dependent variables is given by $\overline{g} = g(\tilde{q}t)$ and $\underline{f} = f(t/\tilde{q})$. The polynomial

$P(f,t,m_1,\ldots,m_7)$ is given by

$$\begin{aligned}P(f,t,m_1,\ldots,m_7) = {} & f^4 - m_1 t f^3 + (m_2 t^2 - 3 - t^8) f^2 \\ & + (m_7 t^7 - m_3 t^3 + 2 m_1 t) f \\ & + (t^8 - m_6 t^6 + m_4 t^4 - m_2 t^2 + 1),\end{aligned} \tag{2.13}$$

where m_k $(k = 1, 2, \ldots 7)$ are the elementary symmetric functions of k-th degree in the parameters b_i $(i = 1, 2, \ldots, 8)$ with $b_1 b_2 \cdots b_8 = 1$. The derivation of the equation (2.12) is discussed in Appendix A.

3 Preliminaries

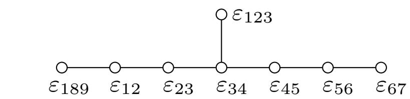
As a preparation for constructing the hypergeometric τ-functions, we decompose the lattice M defined by (2.6) into a family of seven-dimensional lattices. One can see that the affine Weyl group $W(E_7^{(1)})$ acts on each lattice. According to location of the lattice τ-functions, we classify the bilinear equations into four types and discuss the relationships among them. Note that the discussion in this section is independent of the class of function $[x]$.

3.1 A family of seven-dimensional lattices and bilinear equations

We decompose the lattice M defined by (2.6) into a family of seven-dimensional lattices according to the value of the symmetric bilinear form with the coroot vector $\mathrm{e}_{89} = \mathrm{e}_8 - \mathrm{e}_9$;

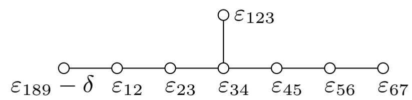
$$M = \coprod_{n\in\mathbb{Z}} M_n, \quad M_n = \{\Lambda \in M \mid \langle \Lambda, \mathrm{e}_{89} \rangle = n\}. \tag{3.1}$$

Parallel to this decomposition, let us consider the orthogonal complement of ε_{89} in the root lattice $Q(E_8^{(1)})$. Then we get the root lattice $Q(E_7^{(1)})$ corresponding to the Dynkin diagram:



. (3.2)

Since we have $\varepsilon_{189} + 2\varepsilon_{12} + 3\varepsilon_{23} + 4\varepsilon_{34} + 3\varepsilon_{45} + 2\varepsilon_{56} + \varepsilon_{67} + 2\varepsilon_{123} = \delta$, the same δ denotes the null root of $Q(E_7^{(1)})$. The corresponding simple reflections generate the affine Weyl group $W(E_7^{(1)}) = \langle s_{189}, s_{12}, \ldots, s_{67}, s_{123} \rangle$, which acts transitively on each M_n. Also, we have the finite Weyl group $W(E_7) = \langle s_{12}, \ldots, s_{67}, s_{123} \rangle$, whose extended Dynkin diagram is given by



. (3.3)

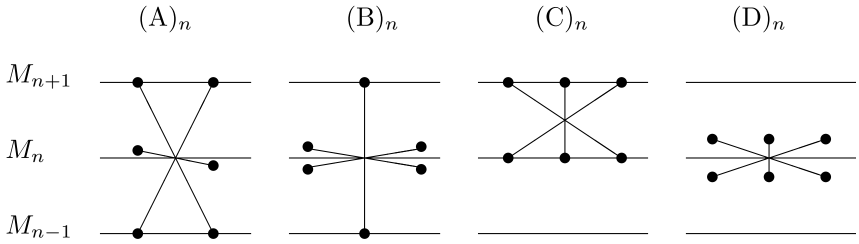


Figure 1: Types of bilinear equations

Note that the Weyl group $W(E_7)$ includes the symmetric group $\mathfrak{S}_8 = \langle s_{01}, s_{12}, \cdots, s_{67} \rangle$ as a subgroup, where s_{01} denotes a reflection with respect to $\varepsilon_{189} - \delta$. The action of the central element $w_c \in W(E_7)$, which can be defined by

$$w_c = s_{23} s_{45} s_{123} s_{145} s_{01} s_{67} s_{167}, \tag{3.4}$$

on the variables $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_9)$ is given by

$$\begin{aligned} w_c \quad : \quad & \varepsilon_0 \mapsto -\varepsilon_0 + 3(\delta + \varepsilon_8 + \varepsilon_9), \\ & \varepsilon_i \mapsto \delta - \varepsilon_i + \varepsilon_8 + \varepsilon_9 \quad (i = 1, 2, \ldots, 7), \\ & \varepsilon_i \mapsto \varepsilon_i \quad (i = 8, 9). \end{aligned} \tag{3.5}$$

Typical bilinear equations for the discrete Painlevé system of type $E_8^{(1)}$ are given by

$$[\varepsilon_{jk}][\varepsilon_{jkl}]\tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_l} + [\varepsilon_{ki}][\varepsilon_{kil}]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_l} + [\varepsilon_{ij}][\varepsilon_{ijl}]\tau_{\mathrm{e}_k}\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_l} = 0 \tag{3.6}$$

for mutually distinct indices $i, j, k, l \in \{1, 2, \ldots, 9\}$. According to location of the lattice τ-functions, one can classify the bilinear equations into the following four types:

$$\begin{array}{ll} (\mathrm{A})_n: & \text{Two on each of } M_{n-1}, M_n \text{ and } M_{n+1}\text{, respectively} \\ (\mathrm{B})_n: & \text{Four on } M_n\text{, and one on } M_{n+1} \text{ and } M_{n-1}\text{, respectively} \\ (\mathrm{C})_n: & \text{Three on } M_{n+1} \text{ and } M_n\text{, respectively} \\ (\mathrm{D})_n: & \text{Six on } M_n \end{array} . \tag{3.7}$$

See Figure 1. Typical bilinear equations of each type are given by

$$\begin{aligned} (\mathrm{A})_0: \quad & [\varepsilon_{89}][\varepsilon_{j89}]\tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j} \\ & = [\varepsilon_{i9}][\varepsilon_{ij9}]\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_8} - [\varepsilon_{i8}][\varepsilon_{ij8}]\tau_{\mathrm{e}_9}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_9}, \\ (\mathrm{B})_0: \quad & [\varepsilon_{ij}][\varepsilon_{ijk}]\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_8} \\ & = [\varepsilon_{i8}][\varepsilon_{ik8}]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_k} - [\varepsilon_{j8}][\varepsilon_{jk8}]\tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_k}, \\ (\mathrm{C})_0: \quad & [\varepsilon_{jk}][\varepsilon_{jk9}]\tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_9} + (i,j,k)\text{-cyclic} = 0, \\ (\mathrm{D})_0: \quad & [\varepsilon_{jk}][\varepsilon_{jkl}]\tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_l} + (i,j,k)\text{-cyclic} = 0 \end{aligned} \tag{3.8}$$

for mutually distinct indices $i, j, k, l \in \{1, 2, \ldots, 7\}$.

Lemma 3.1. All the bilinear equations of type $(\mathrm{A})_0$ can be obtained by $W(E_7^{(1)})$ action on the first equation of (3.8). Also, we have a similar situation regarding each case of type $(\mathrm{B})_0$, $(\mathrm{C})_0$ and $(\mathrm{D})_0$, respectively.

Proof. Any of the lattice τ-functions on M_1 can be transformed to τ_{e_8} by an action of $W(E_7^{(1)})$. Searching for $\Lambda \in M_{-1}$ such that $\langle \Lambda + \mathrm{e}_8, \Lambda + \mathrm{e}_8 \rangle = 0$, we find that the lattice τ-functions on M_{-1} which can pair with τ_{e_8} are following:

$$\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_8},\ \tau_{2\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j-\mathrm{e}_k-\mathrm{e}_l-\mathrm{e}_8},\ \tau_{c+\mathrm{e}_9+\mathrm{e}_i-\mathrm{e}_j},\ \tau_{c+\mathrm{e}_{ijk}+\mathrm{e}_9},\ \tau_{2c-\mathrm{e}_{i89}+\mathrm{e}_9} \tag{3.9}$$

for mutually distinct indices $i, j, k, l \in \{1, 2, \ldots, 7\}$. Any of them can be transformed to $\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_8}$ by an action of $W(E_7)$. Since τ_{e_8} is invariant under the action of $W(E_7)$, we find that one of the pairs of lattice τ-functions in the bilinear equations of type $(\mathrm{A})_0$ can be transformed to $\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_8}$ by an action of $W(E_7^{(1)})$. Note that three pairs of lattice τ-functions in a bilinear equation have a common barycenter. Therefore, the bilinear equations of type $(\mathrm{A})_0$ including the term $\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_8}$ are reduced to the first equation of (3.8). Proofs for the other types of bilinear equations are given in a similar way. ∎

From this lemma, we immediately get the following Proposition.

Proposition 3.2. For fixed $n \in \mathbb{Z}$, all the bilinear equations of type $(\mathrm{A})_n$ can be transformed by actions of $W(E_7^{(1)})$ to one another. Also, we have a similar situation regarding each case of type $(\mathrm{B})_n$, $(\mathrm{C})_n$ and $(\mathrm{D})_n$, respectively.

Let us discuss the relationships among the four types of bilinear equations.

Proposition 3.3. If the lattice τ-functions satisfy all the bilinear equations of type $(\mathrm{B})_n$, then they also satisfy those of type $(\mathrm{A})_n$; that is,

$$1. \qquad (\mathrm{B})_n \Rightarrow (\mathrm{A})_n. \tag{3.10}$$

Similarly, if $\tau_\Lambda \neq 0$ for $\Lambda \in M_{n-1}$, we have the following:

$$\begin{aligned} &2. \quad (\mathrm{A})_n, (\mathrm{C})_{n-1} \Rightarrow (\mathrm{C})_n. \\ &3. \quad (\mathrm{C})_{n-1} \Rightarrow (\mathrm{D})_n. \end{aligned} \tag{3.11}$$

Proof. It is sufficient to verify the statement in the case of $n = 0$.

1. $(\mathrm{B})_0 \Rightarrow (\mathrm{A})_0$: Let us consider the following bilinear equations of type $(\mathrm{B})_0$

$$\begin{aligned} [\varepsilon_{ij}][\varepsilon_{ijk}]\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_8} &= [\varepsilon_{i8}][\varepsilon_{ik8}]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_k} - [\varepsilon_{j8}][\varepsilon_{jk8}]\tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_k}, \\ [\varepsilon_{ij}][\varepsilon_{ijk}]\tau_{\mathrm{e}_9}\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_9} &= [\varepsilon_{i9}][\varepsilon_{ik9}]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_k} - [\varepsilon_{j9}][\varepsilon_{jk9}]\tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_k}. \end{aligned} \tag{3.12}$$

Eliminating the term $\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_k}$, we get the first equation of (3.8).

2. $(\mathrm{A})_0, (\mathrm{C})_{-1} \Rightarrow (\mathrm{C})_0$: Let us consider the following bilinear equation of type $(\mathrm{C})_{-1}$

$$[\varepsilon_{jk}][\varepsilon_{jk8}]\tau_{\mathrm{e}_9}\tau_{\mathrm{e}_0-\mathrm{e}_8-\mathrm{e}_9} = [\varepsilon_{j9}][\varepsilon_{j89}]\tau_{\mathrm{e}_k}\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_8} - [\varepsilon_{k9}][\varepsilon_{k89}]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_8}. \tag{3.13}$$

Multiplying both right and left-hand sides by $[\varepsilon_{jk9}]\tau_{\mathrm{e}_8}$ and using the first equation of (3.8), we get

$$
\begin{aligned}
&[\varepsilon_{jk8}]\tau_{\mathrm{e}_9} \times [\varepsilon_{jk}][\varepsilon_{jk9}]\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_0-\mathrm{e}_8-\mathrm{e}_9} \\
&= [\varepsilon_{j89}]\tau_{\mathrm{e}_k} \times [\varepsilon_{j9}][\varepsilon_{jk9}]\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_8} - \{j \leftrightarrow k\} \\
&= [\varepsilon_{j89}]\tau_{\mathrm{e}_k} \times \Big([\varepsilon_{89}][\varepsilon_{k89}]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_k} + [\varepsilon_{j8}][\varepsilon_{jk8}]\tau_{\mathrm{e}_9}\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_9} \Big) \\
&\quad - \{j \leftrightarrow k\} \\
&= [\varepsilon_{jk8}]\tau_{\mathrm{e}_9} \times \Big([\varepsilon_{j8}][\varepsilon_{j89}]\tau_{\mathrm{e}_k}\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_9} - [\varepsilon_{k8}][\varepsilon_{k89}]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_9} \Big),
\end{aligned} \tag{3.14}
$$

which is equivalent to the bilinear equation of type $(\mathrm{C})_0$

$$
[\varepsilon_{jk}][\varepsilon_{jk9}]\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_0-\mathrm{e}_8-\mathrm{e}_9} = [\varepsilon_{j8}][\varepsilon_{j89}]\tau_{\mathrm{e}_k}\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_9} - [\varepsilon_{k8}][\varepsilon_{k89}]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_9}. \tag{3.15}
$$

3. $(\mathrm{C})_{-1} \Rightarrow (\mathrm{D})_0$: Let us consider the following bilinear equations

$$
[\varepsilon_{jk}][\varepsilon_{jk8}]\tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_8} + [\varepsilon_{ki}][\varepsilon_{ki8}]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_8} + [\varepsilon_{ij}][\varepsilon_{ij8}]\tau_{\mathrm{e}_k}\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_8} = 0 \tag{3.16}
$$

and

$$
\begin{aligned}
&[\varepsilon_{ij}][\varepsilon_{l8}]\tau_{\mathrm{e}_k}\tau_{2\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j-\mathrm{e}_k-\mathrm{e}_l-\mathrm{e}_8} \\
&= [\varepsilon_{jkl}][\varepsilon_{ki8}]\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_l}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_8} - [\varepsilon_{ikl}][\varepsilon_{jk8}]\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_l}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_8}
\end{aligned} \tag{3.17}
$$

of type $(\mathrm{C})_{-1}$. Multiplying the equation (3.16) by $[\varepsilon_{jkl}]\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_l}$, we have

$$
\begin{aligned}
&[\varepsilon_{jk8}]\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_8} \times [\varepsilon_{jk}][\varepsilon_{jkl}]\tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_l} \\
&+[\varepsilon_{ki}]\tau_{\mathrm{e}_j} \times [\varepsilon_{jkl}][\varepsilon_{ki8}]\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_l}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_8} \\
&+[\varepsilon_{ij}]\tau_{\mathrm{e}_k} \times [\varepsilon_{jkl}][\varepsilon_{ij8}]\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_l}\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_8} = 0.
\end{aligned} \tag{3.18}
$$

Applying the equation (3.17) to the second and third terms of (3.18), we get

$$
\begin{aligned}
&[\varepsilon_{jk8}]\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_8} \times [\varepsilon_{jk}][\varepsilon_{jkl}]\tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_l} \\
&+[\varepsilon_{ki}]\tau_{\mathrm{e}_j} \times \Big([\varepsilon_{ikl}][\varepsilon_{jk8}]\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_l}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_8} \\
&\qquad\qquad +[\varepsilon_{ij}][\varepsilon_{l8}]\tau_{\mathrm{e}_k}\tau_{2\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j-\mathrm{e}_k-\mathrm{e}_l-\mathrm{e}_8} \Big) \\
&+[\varepsilon_{ij}]\tau_{\mathrm{e}_k} \times \Big([\varepsilon_{ijl}][\varepsilon_{jk8}]\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_l}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_8} \\
&\qquad\qquad +[\varepsilon_{ik}][\varepsilon_{l8}]\tau_{\mathrm{e}_j}\tau_{2\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j-\mathrm{e}_k-\mathrm{e}_l-\mathrm{e}_8} \Big) = 0,
\end{aligned} \tag{3.19}
$$

which is reduced to the fourth equation of (3.8). ∎

3.2 The idea for constructing hypergeometric τ-functions

Let us explain the basic idea for constructing hypergeometric τ-functions. On the basis of our previous experiences, we impose the boundary condition $\tau_\Lambda = 0$

for any $\Lambda \in M_{-1}$. Then, the bilinear equations of type $(\mathrm{B})_0$ yield the functional equations for the lattice τ-functions on M_0. By solving them, we find that the τ-functions on M_0 are expressed in terms of a discrete analogue of the double gamma function. Next, we consider the bilinear equations of type $(\mathrm{C})_0$. These yield the linear equations for the τ-functions on M_1, since the τ-functions on M_0 are already known. We find that these linear equations are reduced to the contiguity relations for the corresponding hypergeometric function, such as Rahman's q-hypergeometric integral in the case of the q-Painlevé system of type $E_8^{(1)}$. The hypergeometric τ-functions on M_n $(n \geq 2)$ can be constructed recursively by using the bilinear equations of type $(\mathrm{B})_n$. Proposition 3.3 guarantees that such hypergeometric τ-functions satisfy all the bilinear equations.

4 Construction of τ-functions on M_0

Hereafter, we construct the hypergeometric τ-functions for the q-Painlevé system of type $E_8^{(1)}$ by imposing the following boundary condition

$$\tau_{\Lambda_{-1}} = 0 \qquad \text{for any} \quad \Lambda_{-1} \in M_{-1} \tag{4.1}$$

and $\tau_{\Lambda_0} \neq 0$ for any $\Lambda_0 \in M_0$. We fix the coefficients of the bilinear equations as

$$[x] = e(\tfrac{1}{2}x) - e(-\tfrac{1}{2}x), \quad e(x) = e^{2\pi\sqrt{-1}\,x}. \tag{4.2}$$

In this section, we discuss construction of the τ-functions on the lattice M_0.

First, let us consider the following bilinear equations of type $(\mathrm{A})_0$

$$[\varepsilon_{89}][\varepsilon_{j89}]\tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j} = [\varepsilon_{i9}][\varepsilon_{ij9}]\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_8} - [\varepsilon_{i8}][\varepsilon_{ij8}]\tau_{\mathrm{e}_9}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_9} \tag{4.3}$$

for $i,j \in \{1,2,\ldots,7\}$. The conditions $\tau_{\mathrm{e}_9} = \tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_8} = 0$ and $\tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j} \neq 0$ lead us to $[\varepsilon_{89}][\varepsilon_{j89}] = 0\,(j = 1,2,\ldots,7)$. We consider the case

$$[\varepsilon_{89}] = 0 \quad \Leftrightarrow \quad \varepsilon_{89} = \omega \in \mathbb{Z}. \tag{4.4}$$

All the bilinear equations of type $(\mathrm{A})_0$ hold under the boundary conditions (4.1) and (4.4), since they can be obtained by the action of $W(E_7^{(1)}) = \langle s_{189}, s_{12}, \ldots, s_{67}, s_{123}\rangle$ on (4.3) and the coefficient $[\varepsilon_{89}]$ is $W(E_7^{(1)})$-invariant.

Let us introduce the variables x_i $(i = 0,1,\ldots,7)$ by

$$x_0 = \delta - \frac{1}{2}\,\varepsilon_{889}, \quad x_i = \frac{1}{2}\,\varepsilon_{ii9}\ (i = 1,2,\ldots,7), \tag{4.5}$$

where we have $x_0 + x_1 + \cdots + x_7 = 2\delta + 2\omega$. The symmetric group $\mathfrak{S}_8 = \langle s_{01}, s_{12}, \ldots, s_{67}\rangle$ acts on x_i as permutations of the indices. Under the condition (4.1) and (4.4), we consider the functions τ_Λ depending on x_i (and ω).

Under the boundary condition (4.1), the bilinear equations of type $(\mathrm{B})_0$ and $(\mathrm{D})_0$ are expressed in terms of the lattice τ-functions on M_0. We have the bilinear equations of type $(\mathrm{B})_0$

$$\begin{aligned}
&[\varepsilon_{ij}][\varepsilon_{ijk}]\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_8} = [\varepsilon_{i8}][\varepsilon_{ik8}]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_k} - [\varepsilon_{j8}][\varepsilon_{jk8}]\tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_k},\\
&[\varepsilon_{ij}][\varepsilon_{ijk}]\tau_{\mathrm{e}_9}\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_9} = [\varepsilon_{i9}][\varepsilon_{ik9}]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_k} - [\varepsilon_{j9}][\varepsilon_{jk9}]\tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_k}
\end{aligned} \tag{4.6}$$

for mutually distinct indices $i, j, k \in \{1, 2, \ldots, 7\}$, both of which are reduced to

$$[x_0 + x_j - \delta][x_j + x_k]\tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_k} = [x_0 + x_i - \delta][x_i + x_k]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_k} \tag{4.7}$$

due to $\tau_{\mathrm{e}_9} = \tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_8} = 0$ and $\varepsilon_{89} = \omega \in \mathbb{Z}$. The bilinear equations of type $(\mathrm{D})_0$

$$\begin{aligned} &[x_j - x_k][x_0 + x_j + x_k + x_l - \delta]\tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_l} \\ +&[x_k - x_i][x_0 + x_k + x_i + x_l - \delta]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_l} \\ +&[x_i - x_j][x_0 + x_i + x_j + x_l - \delta]\tau_{\mathrm{e}_k}\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_l} = 0 \end{aligned} \tag{4.8}$$

for mutually distinct indices $i, j, k, l \in \{1, 2, \ldots, 7\}$ and their $W(E_7^{(1)})$-transforms can be derived from the equations (4.7) and their $W(E_7^{(1)})$-transforms by using the Riemann relations for their coefficients. Then, it is sufficient to consider the equations (4.7) and their $W(E_7^{(1)})$-transforms for constructing the τ-functions on M_0.

Let us consider a pair of non-zero meromorphic functions $(G(x), F(x))$ satisfying the difference equations

$$G(x + \delta) = \epsilon\,[x]\,G(x), \quad F(x + \delta) = G(x)F(x) \tag{4.9}$$

with a constant $\epsilon \in \mathbb{C}^*$. When $\operatorname{Im}\delta > 0$, a typical choice of such functions is given by

$$G(x) = \frac{e(-\frac{\delta}{2}\binom{x/\delta}{2})}{(u; q)_\infty}, \quad F(x) = e(-\tfrac{\delta}{2}\tbinom{x/\delta}{3})\,(u; q, q)_\infty, \tag{4.10}$$

where $u = e(x)$, $q = e(\delta)$, $(u; q, q) = \prod_{i,j=0}^{\infty}(1 - uq^{i+j})$ and $\epsilon = -1$. For other choice of $(G(x), F(x), \epsilon)$, see Appendix B. In what follows, we fix two triplets $(G_+(x), F_+(x), \epsilon_+)$ and $\left(\widehat{G}_+(x), \widehat{F}_+(x), \epsilon_+\right)$ with the common constant factor ϵ_+, namely we have

$$\begin{aligned} G_+(x + \delta) &= \epsilon_+[x]\,G_+(x), \quad F_+(x + \delta) = G_+(x)F_+(x), \\ \widehat{G}_+(x + \delta) &= \epsilon_+[x]\,\widehat{G}_+(x), \quad \widehat{F}_+(x + \delta) = \widehat{G}_+(x)\widehat{F}_+(x). \end{aligned} \tag{4.11}$$

Also, we introduce two pairs of functions $(G_-(x), F_-(x))$ and $\left(\widehat{G}_-(x), \widehat{F}_-(x)\right)$ by the relations

$$\begin{aligned} F_-(x) &= F_+(2\delta + \omega - x), \quad G_-(x)G_+(\delta + \omega - x) = 1, \\ \widehat{F}_-(x) &= \widehat{F}_+(2\delta + \omega - x), \quad \widehat{G}_-(x)\widehat{G}_+(\delta + \omega - x) = 1. \end{aligned} \tag{4.12}$$

Note that these functions satisfy the difference equations

$$\begin{aligned} G_-(x + \delta) &= \epsilon_-[x]\,G_-(x), \quad F_-(x + \delta) = G_-(x)F_-(x), \\ \widehat{G}_-(x + \delta) &= \epsilon_-[x]\,\widehat{G}_-(x), \quad \widehat{F}_-(x + \delta) = \widehat{G}_-(x)\widehat{F}_-(x), \end{aligned} \tag{4.13}$$

where $\epsilon_- = (-1)^{\omega+1}\epsilon_+$. Although the function $F_+(x)$ (resp.$F_-(x)$) satisfies the same difference equation as that for $\widehat{F}_+(x)$ (resp. $\widehat{F}_-(x)$), they need not be the same functions.

Definition 4.1. For each $\Lambda_0 \in M_0$, we define the fifty-six functions $\tau^{(ab;\pm)}_{\Lambda_0}(x)$ by

$$\begin{aligned}\tau^{(ab;\pm)}_{\Lambda_0}(x) &= \widehat{F}_{\pm}(x_a + x_b + (\langle v_a + v_b, \Lambda_0\rangle + 1)\delta) \\ &\quad \times \prod_{\substack{0\le r<s\le 7\\ \{r,s\}\ne\{a,b\}}} F_{\pm\kappa^{(ab)}_{rs}}(x_r + x_s + (\langle v_r + v_s, \Lambda_0\rangle + 1)\delta),\end{aligned} \tag{4.14}$$

where $a, b \in \{0, 1, \ldots, 7\}$ are mutually distinct indices, v_i is the vector corresponding to the variables x_i given by

$$v_0 = c - \frac{1}{2}\,\mathrm{e}_{889}, \quad v_i = \frac{1}{2}\,\mathrm{e}_{ii9} \ (i = 1, 2, \ldots, 7), \tag{4.15}$$

and $\kappa^{(ab)}_{ij}$ is the sign factor defined by $\kappa^{(ab)}_{ij} = (-1)^{\sharp(\{i,j\}\cap\{a,b\})}$.

Proposition 4.2. The action of $W(E^{(1)}_7)$ on the functions $\tau^{(ab;\pm)}_{\Lambda_0}(x)$ is described as follows:

1. For any translation operator $T \in W(E^{(1)}_7)$, we have
$$\tau^{(ab;\pm)}_{T.\Lambda_0}(x) = \tau^{(ab;\pm)}_{\Lambda_0}(T(x)). \tag{4.16}$$
2. For any permutation $\sigma \in \mathfrak{S}_8$, we have
$$\tau^{(\sigma(a)\sigma(b);\pm)}_{\sigma.\Lambda_0}(x) = \tau^{(ab;\pm)}_{\Lambda_0}(\sigma(x)). \tag{4.17}$$
3. Take three mutually distinct indices $i, j, k \in \{1, 2, \ldots, 7\}$.
   (a) If $a \in \{0, i, j, k\}$ and $b \notin \{0, i, j, k\}$, then
$$\tau^{(ab;\pm)}_{s_{ijk}.\Lambda_0}(x) = \tau^{(ab;\pm)}_{\Lambda_0}(s_{ijk}(x)). \tag{4.18}$$
   (b) If either $a, b \in \{0, i, j, k\}$ or $a, b \notin \{0, i, j, k\}$, then
$$\tau^{(ab;\pm)}_{s_{ijk}.\Lambda_0}(x) = \tau^{(cd;\mp)}_{\Lambda_0}(s_{ijk}(x)), \tag{4.19}$$
   where c and d are indices such that $\{a, b, c, d\} = \{0, i, j, k\}$ or $\{a, b, c, d\} = \{1, 2, \ldots, 7\}\backslash\{i, j, k\}$, respectively.
4. The action of the central element $w_c \in W(E_7)$ defined by (3.4) is given by
$$\tau^{(ab;\mp)}_{w_c.\Lambda_0}(x) = \tau^{(ab;\pm)}_{\Lambda_0}(w_c(x)). \tag{4.20}$$

Proof. The first and second statements are obvious from the definition of $\tau^{(ab;\pm)}_{\Lambda_0}(x)$. The third statement is guaranteed by the relations (4.12). Since the action of w_c on the variables x_i is given by

$$w_c(x_i) = \frac{1}{2}\,\delta - x_i - \frac{1}{2}\,\omega, \tag{4.21}$$

one can verify the fourth statement by using the relations (4.12). ∎

Let S be the label set defined by

$$S = \{(a, b\,;\,\epsilon) \,|\, a, b \in \{0, 1, \ldots, 7\},\ a \neq b,\ \epsilon = \pm 1\} \tag{4.22}$$

under the convention $(a, b\,;\,\epsilon) = (b, a\,;\,\epsilon)$. By using the difference equations (4.11) and (4.13), one can verify that the family of functions $\{\tau^{(\eta)}_{\Lambda_0}(x)\}_{\Lambda_0 \in M_0}$ for each label $\eta \in S$ satisfies the bilinear equations (4.7). Also, the set of all functions $\{\tau^{(\eta)}_{\Lambda_0}(x) \,|\, \eta \in S,\ \Lambda_0 \in M_0\}$ is consistent with respect to the action of $W(E^{(1)}_7)$ in the sense of Proposition 4.2. Since any bilinear equation of type $(\mathrm{B})_0$ (regardless of τ-functions on M_1) and of type $(\mathrm{D})_0$ can be derived by the $W(E^{(1)}_7)$-action on (4.7) and (4.8), respectively, we have the following Theorem.

Theorem 4.3. For each label $\eta \in S$, the family of functions $\{\tau^{(\eta)}_{\Lambda_0}(x)\}_{\Lambda_0 \in M_0}$ defined by (4.14) satisfies all the bilinear equations of type $(\mathrm{B})_0$ and $(\mathrm{D})_0$.

Before discussing construction of the hypergeometric τ-functions on M_n for $n \in \mathbb{Z}_{\geq 1}$, we mention those τ-functions on M_n for $n \in \mathbb{Z}_{<0}$.

Lemma 4.4. For any fixed $n \in \mathbb{Z}_{<0}$, we have $\tau_{\Lambda_n}(x) = 0$ for any $\Lambda_n \in M_n$ under the conditions (4.1) and (4.4).

This lemma is proved by using the bilinear equations of type $(\mathrm{B})_n$.

5 Rahman's q-hypergeometric integral and τ-functions on M_1

In this section, we construct the hypergeometric τ-functions on M_1. We find that a class of bilinear equations for the lattice τ-functions yields the contiguity relations for Rahman's q-hypergeometric integral [1]. As is well-known, Rahman's q-hypergeometric integral possesses $W(E_6)$-symmetry [13]. From that, we can construct a set of fifty-six solutions corresponding to the coset $W(E_7)/W(E_6)$, and describe the action of $W(E^{(1)}_7)$ on the solutions.

5.1 Rahman's q-hypergeometric integral and Bailey's four-term transformation formula

Fix a complex number q with $0 < |q| < 1$. Let us consider the following integral

$$I(u_0; u_1, \ldots u_6; u_7; q) = \frac{(q;q)_\infty}{4\pi\sqrt{-1}} \int_C \frac{(z^{\pm 2}, qu_0^{-1} z^{\pm 1}, qu_7^{-1} z^{\pm 1}; q)_\infty}{(u_1 z^{\pm 1}, \ldots, u_6 z^{\pm 1}; q)_\infty} \frac{dz}{z} \tag{5.1}$$

under the balancing condition $u_0 u_1 \cdots u_7 = q^2$. Here, C is a contour such that the poles of integrand $q^k u_i$ $(i = 1, \ldots, 6\,;\, k = 0, 1, 2, \ldots)$ are located inside of it and $q^{-k} u_i^{-1}$ $(i = 1, \ldots, 6\,;\, k = 0, 1, 2, \ldots)$ are located outside of it. The double sign in $(uz^{\pm 1}; q)_\infty$ denotes the product $(uz^{\pm 1}; q)_\infty = (uz, uz^{-1}; q)_\infty$.

Theorem 5.1 (Rahman)**.** Under the balancing condition $u_0u_1\cdots u_7=q^2$, the above integral is expressed by the sum of two q-hypergeometric series ${}_{10}W_9$ as follows [1, Exercise 6.7, p.169]:

$$\begin{aligned}
&\prod_{1\le k<l\le 6}(u_ku_l;q)_\infty\, I(u_0;u_1,\ldots,u_6;u_7;q)\\
&=\frac{\prod_{k=1}^{6}(qu_k/u_0,q/u_ku_7;q)_\infty}{(u_0/u_7,q^2/u_0^2;q)_\infty}\\
&\quad\times{}_{10}W_9\big(q/u_0^2;q/u_0u_1,\ldots,q/u_0u_6,q/u_0u_7;q,q\big)\\
&+\frac{\prod_{k=1}^{6}(q/u_0u_k,qu_k/u_7;q)_\infty}{(u_7/u_0,q^2/u_7^2;q)_\infty}\\
&\quad\times{}_{10}W_9\big(q/u_7^2;q/u_0u_7,q/u_1u_7,\ldots,q/u_6u_7;q,q\big).
\end{aligned}\tag{5.2}$$

In terms of Rahman's q-hypergeometric integral $I(u_0;u_1,\ldots,u_6;u_7;q)$, Bailey's four-term transformation formula for ${}_{10}W_9$ [1, (2.12.9), p.57] is expressed by

$$\begin{aligned}
&I(u_0;u_1,\ldots,u_6;u_7;q)\\
&=I(\widetilde{u}_0;\widetilde{u}_1,\ldots,\widetilde{u}_6;\widetilde{u}_7;q)\,\frac{\prod_{k=1}^{3}(q/u_0u_k;q)_\infty\prod_{k=4}^{6}(q/u_ku_7;q)_\infty}{\prod_{1\le k<l\le 3}(u_ku_l;q)_\infty\prod_{4\le k<l\le 6}(u_ku_l;q)_\infty},
\end{aligned}\tag{5.3}$$

where

$$\widetilde{u}_i=\begin{cases} u_i(q/u_0u_1u_2u_3)^{1/2} & (i=0,1,2,3)\\ u_i(u_0u_1u_2u_3/q)^{1/2}=u_i(q/u_4u_5u_6u_7)^{1/2} & (i=4,5,6,7)\end{cases}.\tag{5.4}$$

Assume that $\operatorname{Im}\delta>0$. We relate the variables u_i to x_i, defined by (4.5), by

$$u_i=e(x_i),\quad q=e(\delta)\qquad (i=0,1,\ldots,7).\tag{5.5}$$

Then the relation $x_0+x_1+\cdots+x_7=2\delta+2\omega\,(\omega\in\mathbb{Z})$ corresponds to the balancing condition $u_0u_1\cdots u_7=q^2$. Since the action of $s_{123}\in W(E_7)=\langle s_{12},\ldots,s_{67},s_{123}\rangle$ on the variables x_i is given by

$$s_{123}(x_i)=\begin{cases} x_i-\frac{1}{2}(x_0+x_1+x_2+x_3-\delta-\omega) & (i=0,1,2,3)\\ x_i+\frac{1}{2}(x_0+x_1+x_2+x_3-\delta-\omega) & (i=4,5,6,7)\end{cases},\tag{5.6}$$

we have

$$s_{123}(u_i)=\begin{cases} (-1)^\omega u_i(q/u_0u_1u_2u_3)^{1/2} & (i=0,1,2,3)\\ (-1)^\omega u_i(u_0u_1u_2u_3/q)^{1/2} & (i=4,5,6,7)\end{cases}.\tag{5.7}$$

Noticing that $I_{07}(-u)=I_{07}(u)$, we see that the action of s_{123} leads us to the change of variables in Bailey's transformation formula (5.3), namely

$$s_{123}\left(I(u_0;u_1,\ldots,u_6;u_7)\right)=I(\widetilde{u}_0;\widetilde{u}_1,\ldots,\widetilde{u}_6;\widetilde{u}_7),\tag{5.8}$$

regardless of the parity of ω.

5.2 Contiguity relations for q-hypergeometric integrals

Define the q-hypergeometric integral $I_{ab}(u)$ by

$$\begin{aligned} I_{ab}(u) &= I_{ab}(u_0, u_1, \dots, u_6, u_7; q) \\ &= \frac{(q;q)_\infty}{4\pi\sqrt{-1}} \int_C \frac{(z^{\pm 2}, q u_a^{-1} z^{\pm 1}, q u_b^{-1} z^{\pm 1}; q)_\infty}{\prod_{0 \le l \le 7;\, l \ne a,b} (u_l z^{\pm 1}; q)_\infty} \frac{dz}{z} \end{aligned} \tag{5.9}$$

for mutually distinct indices $a, b \in \{0, 1, \dots, 7\}$. Note that

$$I_{07}(u) = I(u_0; u_1, \dots, u_6; u_7; q), \tag{5.10}$$

where $I(u_0; u_1, \dots, u_6; u_7; q)$ is Rahman's q-hypergeometric integral defined by (5.1). In this subsection, we discuss two types of contiguity relations for the q-hypergeometric integrals.

Denote the integrand of $I_{ab}(u)$ by

$$H_{ab}(u; z) = \frac{(z^{\pm 2}, q u_a^{-1} z^{\pm 1}, q u_b^{-1} z^{\pm 1}; q)_\infty}{\prod_{0 \le l \le 7;\, l \ne a,b} (u_l z^{\pm 1}; q)_\infty}. \tag{5.11}$$

Then, for a translation operator T_{q,u_i} whose action is given by $T_{q,u_i} u_i = q u_i$ and $T_{q,u_i} u_j = u_j$ $(i \ne j)$, we have

$$T_{q,u_i} H_{ab}(u; z) = \begin{cases} u_i^{-1} \langle u_i z \rangle \langle u_i / z \rangle H_{ab}(u; z) & (i = a, b) \\ u_i \, \langle u_i z \rangle \langle u_i / z \rangle H_{ab}(u; z) & (i \ne a, b) \end{cases}, \tag{5.12}$$

where $\langle u \rangle = u^{-1/2} - u^{1/2}$. Let us introduce the "corrected q-hypergeometric integrals" $\Phi_{ab}(x)$ by $\Phi_{ab}(x) = \nu_{ab}(x)\, I_{ab}(u)$ under the correspondence of the variables (5.5) so that the function $\Phi_{ab}(x)$ behaves equally for the action of T_{q,u_i} for any index $i \in \{0, 1, \dots, 7\}$. If the correction factor $\nu_{ab}(x)$ satisfies the difference equations

$$\nu_{ab}(x)|_{x_i \mapsto x_i + \delta} = \begin{cases} c(x) e(x_i)\, \nu_{ab}(x) & (i = a, b) \\ c(x) e(-x_i)\, \nu_{ab}(x) & (i \ne a, b) \end{cases}, \tag{5.13}$$

where $c(x)$ is a non-zero function, then we have for $\widetilde{H}_{ab}(u; z) = \nu_{ab}(x)\, H_{ab}(u; z)$ the formula

$$T_{q,u_i} \widetilde{H}_{ab}(u; z) = c(x)\, \langle u_i z \rangle \langle u_i / z \rangle\, \widetilde{H}_{ab}(u; z) \quad (i = 0, 1, \dots, 7). \tag{5.14}$$

A typical example of $\nu_{ab}(x)$ is given by

$$\begin{aligned} &\nu_{ab}(x) = e\big(Q_{ab}(x)\big), \\ &Q_{ab}(x) = \delta \Big[\tbinom{x_a/\delta}{2} + \tbinom{x_b/\delta}{2} - \sum_{0 \le l \le 7;\, l \ne a,b} \tbinom{x_l/\delta}{2} \Big], \end{aligned} \tag{5.15}$$

where $c(x) = 1$. From the Riemann relation

$$\langle u_j u_k^{-1} \rangle \langle u_j u_k \rangle \langle u_i z^{-1} \rangle \langle u_i z \rangle + (i,j,k)\text{-cyclic} = 0, \tag{5.16}$$

we get

$$\langle u_j u_k^{-1} \rangle \langle u_j u_k \rangle T_{q,u_i} \widetilde{H}_{ab}(u;z) + (i,j,k)\text{-cyclic} = 0. \tag{5.17}$$

This yields the contiguity relation for $\Phi_{ab}(x)$

$$[x_j - x_k][x_j + x_k]\, \Phi_{ab}(x)|_{x_i \mapsto x_i+\delta, x_l \mapsto x_l - \delta} + (i,j,k)\text{-cyclic} = 0 \tag{5.18}$$

for mutually distinct indices $i, j, k, l \in \{0, 1, \dots, 7\}$.

Next, we derive another contiguity relation for the corrected hypergeometric integrals $\Phi_{ab}(x)$ by using Bailey's transformation formula. Suppose that the functions $\nu_{ab}(x)$ satisfies

$$\nu_{ab}(\sigma(x)) = \nu_{\sigma(a)\sigma(b)}(x), \quad \sigma \in \mathfrak{S}_8 = \langle s_{01}, \dots, s_{67} \rangle \tag{5.19}$$

and

$$\frac{\nu_{07}(s_{123}(x))}{\nu_{07}(x)} = \frac{\prod_{i=1}^{3} g_+(\delta + \omega - x_0 - x_i) \prod_{i=4}^{6} g_+(\delta + \omega - x_i - x_7)}{\prod_{1 \le i < j \le 3} g_+(x_i + x_j) \prod_{4 \le i < j \le 6} g_+(x_i + x_j)}, \tag{5.20}$$

where $g_+(x)$ is given by

$$G_+(x) = \frac{g_+(x)}{(u;q)_\infty}, \qquad u = e(x), \quad q = e(\delta). \tag{5.21}$$

For instance, these conditions are in fact satisfied by the functions defined by (5.15), when $\omega = 0$ and $g_+(x) = e(-\frac{\delta}{2}\binom{x/\delta}{2})$. Then the corrected q-hypergeometric integral $\Phi_{07}(x) = \nu_{07}(x) I_{07}(u)$ with (5.5) is invariant under the action of $\mathfrak{S}_6 = \langle s_{12}, s_{23}, \dots, s_{56} \rangle$ and satisfies

$$\Phi_{07}(s_{123}(x)) = \frac{\prod_{i=1}^{3} G_+(\delta + \omega - x_0 - x_i) \prod_{i=4}^{6} G_+(\delta + \omega - x_i - x_7)}{\prod_{1 \le i < j \le 3} G_+(x_i + x_j) \prod_{4 \le i < j \le 6} G_+(x_i + x_j)} \Phi_{07}(x). \tag{5.22}$$

It is easy to see that we have

$$\Phi_{07}(s_{145} s_{123}(x)) = \prod_{i=2}^{5} \frac{G_+(\delta + \omega - x_0 - x_i) G_+(\delta + \omega - x_i - x_7)}{G_+(x_1 + x_i) G_+(x_i + x_6)} \Phi_{07}(x). \tag{5.23}$$

On the other hand, introducing the variable y_i by

$$y_i = \begin{cases} \frac{1}{2}(x_0 + x_1 + x_6 + x_7) - x_i & (i = 0, 1, 6, 7) \\ \frac{1}{2}(x_2 + x_3 + x_4 + x_5) - x_i & (i = 2, 3, 4, 5) \end{cases}, \tag{5.24}$$

we see that $\Phi_{07}(s_{145} s_{123}(x)) = \Phi_{16}(y)$. Considering the more general case

$$y_i = \begin{cases} \frac{1}{2}(x_0 + x_a + x_b + x_7) - x_i & (i = 0, a, b, 7) \\ \frac{1}{2}(x_c + x_d + x_e + x_f) - x_i & (i = c, d, e, f) \end{cases} \tag{5.25}$$

associated with the decomposition $\{0,1,\ldots,7\} = \{0,a,b,7\} \cup \{c,d,e,f\}$, we have

$$\Phi_{07}(x) = \Phi_{ab}(y) \prod_{r\in\{c,d,e,f\}} \frac{G_+(x_a+x_r)G_+(x_r+x_b)}{G_+(\delta+\omega-x_0-x_r)G_+(\delta+\omega-x_r-x_7)}. \quad (5.26)$$

For mutually distinct indices $i,j,k \in \{0,1,\ldots,7\}$, one can choose the decomposition $\{0,1,\ldots,7\} = \{0,a,b,7\} \cup \{c,d,e,f\}$ such that all the indices i,j and k belong to either $\{0,a,b,7\}$ or $\{c,d,e,f\}$. Assume that we have either $\{i,j,k,l\} = \{0,a,b,7\}$ or $\{i,j,k,l\} = \{c,d,e,f\}$ for given $\{i,j,k\}$ and $l \neq i,j,k$. Then, we have

$$\begin{aligned}
&\Phi_{07}(x)|_{x_i\mapsto x_i-\delta,\, x_l\mapsto x_l+\delta} \\
&= \Phi_{ab}(y)|_{y_i\mapsto y_i+\delta,\, y_l\mapsto y_l-\delta} \prod_{r\notin\{i,j,k,l\}} \frac{[x_l+x_r]}{[x_i+x_r-\delta]} \\
&\qquad \times \prod_{r\in\{c,d,e,f\}} \frac{G_+(x_a+x_r)G_+(x_r+x_b)}{G_+(\delta+\omega-x_0-x_r)G_+(\delta+\omega-x_r-x_7)}.
\end{aligned} \quad (5.27)$$

Since we already have

$$[y_j-y_k][y_j+y_k]\Phi_{ab}(y)|_{y_i\mapsto y_i+\delta,\, y_l\mapsto y_l-\delta} + (i,j,k)\text{-cyclic} = 0, \quad (5.28)$$

we obtain the contiguity relations

$$\begin{aligned}
&[x_j-x_k][x_i+x_l] \prod_{0\le r\le 7;\, r\neq i,j,k,l} [x_i+x_r-\delta]\, \Phi_{07}(x)|_{x_i\mapsto x_i-\delta, x_l\mapsto x_l+\delta} \\
&\quad +(i,j,k)\text{-cyclic} = 0.
\end{aligned} \quad (5.29)$$

It is obvious that each of functions $\Phi_{ab}(x)$ satisfies the same contiguity relations.

Proposition 5.2. Each of the functions $\Phi_{ab}(x)$ for mutually distinct indices $a,b \in \{0,1,\ldots,7\}$ satisfies the contiguity relations

$$\begin{aligned}
&[x_j-x_k][x_j+x_k]\, \Phi_{ab}(x)|_{x_i\mapsto x_i+\delta, x_l\mapsto x_l-\delta} + (i,j,k)\text{-cyclic} = 0, \\
&[x_j-x_k][x_i+x_l] \prod_{0\le r\le 7;\, r\neq i,j,k,l} [x_i+x_r-\delta]\, \Phi_{ab}(x)|_{x_i\mapsto x_i-\delta, x_l\mapsto x_l+\delta} \\
&\quad +(i,j,k)\text{-cyclic} = 0
\end{aligned} \quad (5.30)$$

for mutually distinct indices $i,j,k,l \in \{0,1,\ldots,7\}$.

Here, we give a remark on choice of the correction factors $\nu_{ab}(x)$. The function $\nu_{07}(x)$ in the form

$$\nu_{07}(x) = \frac{\widehat{h}(x_0+x_7) \prod_{\substack{1\le r\le 6\\ s=0,7}} h(\delta+\omega-x_r-x_s) \prod_{1\le r<s\le 6} h(x_r+x_s)}{\prod_{1\le r\le 6} g_+(\delta+\omega-x_0-x_r)g_+(\delta+\omega-x_r-x_7)}, \quad (5.31)$$

where $h(x)$ and $\widehat{h}(x)$ are arbitrary functions, is manifestly $\mathfrak{S}_6$-invariant and satisfies the relation (5.20). Note that the function $g_+(x)$ satisfies the difference equation $g_+(x+\delta) = -\epsilon_+ e(-\frac{1}{2}x)\, g_+(x)$. If the functions $h(x)$ and $\widehat{h}(x)$ satisfy the difference equations

$$h(x+\delta) = e(-\tfrac{1}{3}x)\, h(x), \quad \widehat{h}(x+\delta) = \epsilon_+^{-4} e(\tfrac{2}{3}x - \tfrac{1}{2}\,\delta + \tfrac{2}{3}\,\omega)\, \widehat{h}(x), \tag{5.32}$$

the function $\nu_{07}(x)$ satisfies the difference equations

$$\begin{aligned} &\nu_{07}(x)|_{x_0 \mapsto x_0-\delta, x_i \mapsto x_i+\delta} = e(\delta - x_0 - x_i)\, \nu_{07}(x) \qquad (i = 1, 2, \ldots, 6), \\ &\nu_{07}(x)|_{x_0 \mapsto x_0-\delta, x_7 \mapsto x_7+\delta} = e(\delta - u_0 + u_7)\, \nu_{07}(x), \end{aligned} \tag{5.33}$$

which are consistent with (5.13). Thus, we find that the functions $\nu_{ab}(x)$ can be given by

$$\nu_{ab}(x) = \frac{\widehat{h}(x_a + x_b) \displaystyle\prod_{\substack{0 \le r < s \le 7 \\ \{r,s\} \ne \{a,b\}}} h_{\kappa_{rs}^{(ab)}}(x_r + x_s)}{\displaystyle\prod_{\substack{0 \le r \le 7;\, r \ne a,b \\ s = a,b}} g_+(\delta + \omega - x_r - x_s)}, \tag{5.34}$$

where $h_+(x) = h(x)$ and $h_-(x) = h(\delta + \omega - x)$. It is possible to determine $\nu_{ab}(x)$ according to the choice of the functions $h_+(x), \widehat{h}(x)$ and $G_+(x)$.

5.3 The fifty-six solutions

Hereafter, we denote $\Phi_{ab}(x)$ by $\Phi^{(ab;+)}(x)$. Due to (4.21), the action of the central element $w_c \in W(E_7)$ on the contiguity relations (5.30) leads us to

$$\begin{aligned} &[x_j - x_k][x_j + x_k - \delta]\, \check{\Phi}^{(ab;+)}(x)|_{x_i \mapsto x_i - \delta,\, x_l \mapsto x_l + \delta} + (i,j,k)\text{-cyclic} = 0, \\ &[x_j - x_k][x_i + x_l - \delta] \prod_{\substack{0 \le r \le 7 \\ r \ne i,j,k,l}} [x_i + x_r]\, \check{\Phi}^{(ab;+)}(x)|_{x_i \mapsto x_i + \delta, x_l \mapsto x_l - \delta} \\ &\quad + (i,j,k)\text{-cyclic} = 0, \end{aligned} \tag{5.35}$$

where $\check{\Phi}^{(ab;+)}(x) = \Phi^{(ab;+)}(w_c(x))$. Let us introduce the function $\Phi^{(ab;-)}(x)$ by

$$\begin{aligned} &\Phi^{(ab;-)}(x) = \mathcal{G}_{ab}^-(x)\, \check{\Phi}^{(ab;+)}(x), \\ &\mathcal{G}_{ab}^-(x) = \widehat{G}_-(x_a + x_b) \prod_{\substack{0 \le r < s \le 7 \\ \{r,s\} \ne \{a,b\}}} G_{-\kappa_{rs}^{(ab)}}(x_r + x_s). \end{aligned} \tag{5.36}$$

Noticing that the factor $\mathcal{G}_{ab}^-(x)$ satisfies the difference equations

$$\mathcal{G}_{ab}^-(x)|_{x_i \mapsto x_i + \delta,\, x_j \mapsto x_j - \delta} = \prod_{\substack{0 \le r \le 7 \\ r \ne i,j}} \frac{[x_i + x_r]}{[x_j + x_r - \delta]}\, \mathcal{G}_{ab}^-(x) \tag{5.37}$$

for mutually distinct indices $i, j \in \{0, 1, \ldots, 7\}$, we see that each of the functions $\Phi^{(ab;-)}(x)$ satisfies the contiguity relations

$$\begin{aligned}
&[x_j - x_k][x_i + x_l] \prod_{\substack{0\le r\le 7\\ r\ne i,j,k,l}} [x_i + x_r - \delta]\, \Phi^{(ab;-)}(x)|_{x_i\mapsto x_i-\delta, x_l\mapsto x_l+\delta} \\
&\quad +(i,j,k)\text{-cyclic} = 0, \\
&[x_j - x_k][x_j + x_k]\Phi^{(ab;-)}(x)|_{x_i\mapsto x_i+\delta,\, x_l\mapsto x_l-\delta} + (i,j,k)\text{-cyclic} = 0,
\end{aligned} \tag{5.38}$$

which are the same as those for $\Phi^{(ab;+)}(x)$.

Theorem 5.3. The fifty-six functions $\Phi(x) = \Phi^{(ab;\pm)}(x)$ give rise to the solutions of the set of contiguity relations

$$\begin{aligned}
&[x_j - x_k][x_j + x_k]\Phi(x)|_{x_i\mapsto x_i+\delta,\, x_l\mapsto x_l-\delta} + (i,j,k)\text{-cyclic} = 0, \\
&[x_j - x_k][x_i + x_l] \prod_{0\le r\le 7;\, r\ne i,j,k,l} [x_i + x_r - \delta]\, \Phi(x)|_{x_i\mapsto x_i-\delta, x_l\mapsto x_l+\delta} \\
&\quad +(i,j,k)\text{-cyclic} = 0
\end{aligned} \tag{5.39}$$

for mutually distinct indices $i, j, k, l \in \{0, 1, \ldots, 7\}$.

From these two types of contiguity relations, one can get the q-hypergeometric equations of the second order. The functions $\Phi^{(ab;\pm)}(x)$ coincide with the fifty-six pairwise linearly independent solutions to the q-hypergeometric equations constructed by Gupta and Masson [2].

By construction, we have the following Proposition.

Proposition 5.4. Under the balancing condition $u_0 u_1 \cdots u_7 = q^2$, the action of $W(E_7)$ on the functions $\Phi^{(ab;\pm)}(x)$ is described as follows:

1. For any permutation $\sigma \in \mathfrak{S}_8$, we have $\Phi^{(ab;\pm)}(\sigma(x)) = \Phi^{(\sigma(a)\sigma(b);\pm)}(x)$.
2. Take three mutually distinct indices $i, j, k \in \{1, 2, \ldots, 7\}$.
   (a) If $a \in \{0, i, j, k\}$ and $b \notin \{0, i, j, k\}$, then

$$\begin{aligned}
&\Phi^{(ab;\pm)}(s_{ijk}(x)) \\
&= \frac{\prod_{r\in I\setminus\{a\}} G_\pm(\delta + \omega - x_a - x_r) \prod_{r\in I'\setminus\{b\}} G_\pm(\delta + \omega - x_r - x_b)}{\prod_{\substack{r,s\in I\setminus\{a\}\\ r<s}} G_\pm(x_r + x_s) \prod_{\substack{r,s\in I'\setminus\{b\}\\ r<s}} G_\pm(x_r + x_s)} \\
&\times \Phi^{(ab;\pm)}(x),
\end{aligned} \tag{5.40}$$

   where $I = \{0, i, j, k\}$ and $I' = \{0, 1, \ldots, 7\}\setminus I$.

(b) If $a,b \in \{0,i,j,k\}$, then

$$\begin{aligned}&\Phi^{(ab;\pm)}(s_{ijk}(x))\\&= \frac{\Phi^{(cd;\mp)}(x)}{\widehat{G}_\pm(x_c+x_d)G_\mp(x_a+x_b)\displaystyle\prod_{\substack{r=a,b\\s=c,d}} G_\pm(x_r+x_s)\prod_{\substack{r,s\in I'\\r<s}} G_\mp(x_r+x_s)},\end{aligned} \tag{5.41}$$

where c and d are indices such that $\{a,b,c,d\} = \{0,i,j,k\}$, and $I' = \{1,2,\ldots,7\}\backslash\{i,j,k\}$.

(c) If $a,b \notin \{0,i,j,k\}$, then

$$\begin{aligned}&\Phi^{(ab;\pm)}(s_{ijk}(x))\\&= \frac{\Phi^{(cd;\mp)}(x)}{\widehat{G}_\pm(x_c+x_d)G_\mp(x_a+x_b)\displaystyle\prod_{\substack{r=a,b\\s=c,d}} G_\pm(x_r+x_s)\prod_{\substack{r,s\in I\\r<s}} G_\mp(x_r+x_s)},\end{aligned} \tag{5.42}$$

where c and d are indices such that $\{a,b,c,d\} = \{1,2,\ldots,7\}\backslash\{i,j,k\}$, and $I = \{0,i,j,k\}$.

3. The action of the central element $w_c \in W(E_7)$ is given by

$$\Phi^{(ab;\mp)}(x) = \widehat{G}_\mp(x_a+x_b) \prod_{\substack{0\le r<s\le 7\\ \{r,s\}\neq\{a,b\}}} G_{\mp\kappa_{rs}^{(ab)}}(x_r+x_s)\,\Phi^{(ab;\pm)}(w_c(x)). \tag{5.43}$$

The set of fifty-six functions $\Phi^{(ab;\pm)}(x)$ corresponds to the coset $W(E_7)/W(E_6)$, as we will see below. Note that $|W(E_7)/W(E_6)| = 56$.

5.4 The τ-functions on M_1

In this subsection, we construct the functions $\tau_{\Lambda_1}(x)\,(\Lambda_1 \in M_1)$ on the basis of above discussions. The bilinear equations to be considered are of type $(\mathrm{C})_0$ and $(\mathrm{D})_1$, since the functions $\tau_{\Lambda_0}(x)\,(\Lambda_0 \in M_0)$ are already known.

Definition 5.5. For each $\Lambda_1 \in M_1$, we define the fifty-six functions $\tau^{(ab;\pm)}_{\Lambda_1}(x)$ by

$$\tau^{(ab;\pm)}_{\Lambda_1}(x) = \mathcal{N}^{(ab;\pm)}_{\Lambda_1}(x)\,\Phi^{(ab;\pm)}_{\Lambda_1}(x), \tag{5.44}$$

with

$$\begin{aligned}\mathcal{N}^{(ab;\pm)}_{\Lambda_1}(x) &= \widehat{F}_\pm(x_a+x_b+\langle v_a+v_b,\Lambda_1\rangle\,\delta)\\ &\quad\times \prod_{\substack{0\le r<s\le 7\\ \{r,s\}\neq\{a,b\}}} F_{\pm\kappa_{rs}^{(ab)}}(x_r+x_s+\langle v_r+v_s,\Lambda_1\rangle\delta),\end{aligned} \tag{5.45}$$

$$\Phi^{(ab;\pm)}_{\Lambda_1}(x) = \Phi^{(ab;\pm)}(x+\langle v,\Lambda_1\rangle\delta),$$

$$x+\langle v,\Lambda_1\rangle\delta = (x_0+\langle v_0,\Lambda_1\rangle\delta,\ldots,x_7+\langle v_7,\Lambda_1\rangle\delta).$$

Theorem 5.6. The action of $W(E_7^{(1)})$ on the functions $\tau_{\Lambda_1}^{(ab;\pm)}(x)$ is described as follows:

1. For any translation operator $T \in W(E_7^{(1)})$, we have
$$\tau_{T.\Lambda_1}^{(ab;\pm)}(x) = \tau_{\Lambda_1}^{(ab;\pm)}(T(x)). \tag{5.46}$$

2. For any permutation $\sigma \in \mathfrak{S}_8$, we have
$$\tau_{\sigma.\Lambda_1}^{(\sigma(a)\sigma(b);\pm)}(x) = \tau_{\Lambda_1}^{(ab;\pm)}(\sigma(x)). \tag{5.47}$$

3. Take three mutually distinct indices $i, j, k \in \{1, 2, \ldots, 7\}$.

   (a) If $a \in \{0, i, j, k\}$ and $b \notin \{0, i, j, k\}$, then
   $$\tau_{s_{ijk}.\Lambda_1}^{(ab;\pm)}(x) = \tau_{\Lambda_1}^{(ab;\pm)}(s_{ijk}(x)). \tag{5.48}$$

   (b) If either $a, b \in \{0, i, j, k\}$ or $a, b \notin \{0, i, j, k\}$, then
   $$\tau_{s_{ijk}.\Lambda_1}^{(ab;\pm)}(x) = \tau_{\Lambda_1}^{(cd;\mp)}(s_{ijk}(x)), \tag{5.49}$$
   where c and d are indices such that $\{a, b, c, d\} = \{0, i, j, k\}$ or $\{a, b, c, d\} = \{1, 2, \ldots, 7\}\backslash\{i, j, k\}$, respectively.

4. The action of the central element $w_c \in W(E_7)$ is given by
$$\tau_{w_c.\Lambda_1}^{(ab;\mp)}(x) = \tau_{\Lambda_1}^{(ab;\pm)}(w_c(x)). \tag{5.50}$$

Proof. The first and second statements are obvious from the definition of $\tau_{\Lambda_1}^{(ab;\pm)}(x)$. The third and fourth statements are guaranteed by Proposition 5.4 and (5.44). ∎

Corollary 5.7. For the particular element $\mathrm{e}_8 \in M_1$, the set of fifty-six functions
$$\tau_{\mathrm{e}_8}^{(ab;\pm)}(x) = \widehat{F}_{\pm}(x_a + x_b) \prod_{\substack{0 \le r < s \le 7 \\ \{r,s\} \ne \{a,b\}}} F_{\pm\kappa_{rs}^{(ab)}}(x_r + x_s)\, \Phi^{(ab;\pm)}(x) \tag{5.51}$$
is stabilised by $W(E_7)$[1]. For each label $(ab;\pm)$, the isotropy subgroup of $\tau_{\mathrm{e}_8}^{(ab;\pm)}(x)$ is isomorphic to $W(E_6)$;
$$\tau_{\mathrm{e}_8}^{(07;\pm)}(w(x)) = \tau_{\mathrm{e}_8}^{(07;\pm)}(x), \quad w \in W(E_6) = \langle s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{123}\rangle, \tag{5.52}$$
for instance.

[1] Note that $\mathrm{e}_8 \in M_1$ is $W(E_7)$-invariant.

Let us consider the bilinear equations of type $(\mathrm{C})_0$

$$[x_j - x_k][x_j + x_k]\tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_9} + (i,j,k)\text{-cyclic} = 0 \tag{5.53}$$

for mutually distinct indices $i, j, k \in \{1, 2, \ldots, 7\}$. Substituting (4.14) and (5.44) into (5.53) and their $\mathfrak{S}_8$-transforms, we get for $\Phi^{(ab;\pm)}(x)$ the linear relations

$$[x_j - x_k][x_j + x_k]\,\Phi^{(ab;\pm)}(x)|_{x_i \mapsto x_i+\delta,\, x_l \mapsto x_l-\delta} + (i,j,k)\text{-cyclic} = 0 \tag{5.54}$$

for mutually distinct indices $i, j, k, l \in \{0, 1, \ldots, 7\}$, which are precisely the contiguity relations satisfied by the corrected hypergeometric integrals. Similarly, the action of the central element $w_c \in W(E_7)$ on the bilinear equations (5.53) (and their $\mathfrak{S}_8$-transforms) lead us to

$$\begin{aligned}&[x_j - x_k][x_i + x_l] \prod_{\substack{0\le r\le 7\\ r\neq i,j,k,l}} [x_i + x_r - \delta]\,\Phi^{(ab;\pm)}(x)|_{x_i \mapsto x_i-\delta, x_l \mapsto x_l+\delta}\\ &\quad +(i,j,k)\text{-cyclic} = 0\end{aligned} \tag{5.55}$$

for mutually distinct indices $i, j, k, l \in \{0, 1, \ldots, 7\}$.

Also, the set of functions $\{\tau^{(\eta)}_{\Lambda_1}(x)\,|\,\eta \in S,\ \Lambda_1 \in M_1\}$ is consistent with respect to the action of $W(E^{(1)}_7)$ in the sense of Proposition 5.6. Since any bilinear equation of type $(\mathrm{C})_0$ can be derived from (5.53) by applying $W(E^{(1)}_7)$, we see that all the bilinear equations of type $(\mathrm{C})_0$ and $(\mathrm{D})_1$ hold due to Proposition 3.3.

Theorem 5.8. For each label $\eta \in S$, the families of functions $\{\tau^{(\eta)}_{\Lambda_0}(x)\}_{\Lambda_0\in M_0}$ and $\{\tau^{(\eta)}_{\Lambda_1}(x)\}_{\Lambda_1\in M_1}$ defined by (4.14) and (5.44), respectively, satisfy all the bilinear equations of type $(\mathrm{C})_0$ and $(\mathrm{D})_1$.

Remark 5.9. From the above Theorem, we see that all the bilinear equations of type $(\mathrm{C})_0$ imply the contiguity relations for Rahman's q-hypergeometric integral (or sum of two q-hypergeometric series ${}_{10}W_9$). For example, let us consider the bilinear difference equation

$$[\varepsilon_{12}][\varepsilon_{129}]\tau_{\mathrm{e}_0-\mathrm{e}_8-\mathrm{e}_9}\tau_{\mathrm{e}_8} = [\varepsilon_{18}][\varepsilon_{189}]\tau_{\mathrm{e}_2}\tau_{\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_9} - [\varepsilon_{28}][\varepsilon_{289}]\tau_{\mathrm{e}_1}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_9}, \tag{5.56}$$

and its w_c-transform. Rewrite these equations into the relations for the function

$$\begin{aligned}\psi(a_0; a_1, \ldots, a_7) &= {}_{10}W_9(a_0; a_1, \ldots, a_6, a_7; q, q)\\ &\quad + \frac{(qa_0, a_7/a_0; q)_\infty}{(a_0/a_7, qa_7^2/a_0; q)_\infty} \prod_{k=1}^{6} \frac{(a_k, qa_7/a_k; q)_\infty}{(qa_0/a_k, a_k a_7/a_0; q)_\infty}\\ &\quad \times {}_{10}W_9(a_7^2/a_0; a_1a_7/a_0, \ldots, a_6a_7/a_0, a_7; q, q),\end{aligned} \tag{5.57}$$

where the variable a_i are given by $a_i = q/u_0u_i$ $(i = 0, 1, \ldots, 7)$. Then, we obtain the well-known contiguity relations

$$\begin{aligned}&\psi(a_0; a_1/q, qa_2, a_3, \ldots, a_7) - \psi(a_0; a_1, a_2, a_3, \ldots, a_7)\\ &= V_1\,\psi(q^2a_0; a_1, qa_2, \ldots, qa_7),\\ &V_2\,\psi(q^2a_0; a_1, qa_2, qa_3, \ldots, qa_7) - V_3\,\psi(q^2a_0; qa_1, a_2, qa_3, \ldots, qa_7)\\ &= V_4\,\psi(a_0; a_1, \ldots, a_7),\end{aligned} \tag{5.58}$$

where $V_i\,(i=1,2,3,4)$ are given by

$$V_1=\frac{qa_0/a_2(1-qa_2/a_1)(1-a_1a_2/qa_0)(1-qa_0)(1-q^2a_0)\prod_{j=3}^{7}(1-a_j)}{(1-qa_0/a_1)(1-q^2a_0/a_1)(1-a_0/a_2)(1-qa_0/a_2)\prod_{j=3}^{7}(1-qa_0/a_j)},$$
$$V_2=\frac{a_1^2(1-a_2)\prod_{j=3}^{7}(1-qa_0/a_1a_j)}{(1-qa_0/a_1)(1-q^2a_0/a_1)},\quad V_3=V_2|_{\,a_1\leftrightarrow a_2},$$
$$V_4=\frac{a_1(1-a_2/a_1)\prod_{j=3}^{7}(1-qa_0/a_j)}{(1-qa_0)(1-q^2a_0)}. \tag{5.59}$$

The bilinear equations of type $(\mathrm{C})_0$ include some other types of contiguity relations, which may not be found in literature.

Theorem 5.10. Fix two triplets $(G_+(x),F_+(x),\epsilon_+)$ and $\big(\widehat{G}_+(x),\widehat{F}_+(x),\epsilon_+\big)$, and the correction factor $\nu_{ab}(x)$. For each label $\eta\in S$, there is a unique family of functions $\{\tau^{(\eta)}_{\Lambda}(x)\}_{\Lambda\in M}$ such that $\tau^{(\eta)}_{\Lambda_{-1}}(x)=0\,(\Lambda_{-1}\in M_{-1})$, and $\tau^{(\eta)}_{\Lambda_0}(x)\,(\Lambda_0\in M_0)$ and $\tau^{(\eta)}_{\Lambda_1}(x)\,(\Lambda_1\in M_1)$ are given by (4.14) and (5.44), respectively. By using Proposition 3.3, the functions $\tau^{(\eta)}_{\Lambda_n}(x)\,(\Lambda_n\in M_n)$ for $n\in\mathbb{Z}_{\geq 2}$ are uniquely determined.

The next section is devoted to construction of the functions $\tau^{(\eta)}_{\Lambda_n}(x)\,(\Lambda_n\in M_n)$ for $n\in\mathbb{Z}_{\geq 2}$.

6 A determinant formula for the hypergeometric τ-functions

As is well-known, many of hypergeometric solutions to the continuous and discrete Painlevé equations admit a determinant expression [3, 4, 11, 6, 16]. In this section, we show that the hypergeometric τ-functions on $M_n\,(n\in\mathbb{Z}_{\geq 2})$ are expressed by a "two-directional Casorati determinant" of order n. In what follows, we denote a function $f^{(\eta)}(x)\,(\eta\in S)$ by $f(\eta;x)$ for convenience.

For each $n\in\mathbb{Z}_{\geq 0}$, we define the fifty-six functions $K_n(\eta;x)=K_n^{(ab;\pm)}(x)$ by the following "two-directional Casorati determinant"

$$\begin{aligned}
&K_{2m}(\eta;x+\tfrac{2m-1}{4}\delta)|_{x_i\mapsto x_i-(m-1)\delta\,(i=1,2,3,4)}\\
&=\det\Big(\Phi(b-m,m+1-b,a-m,m+1-a)\Big)_{a,b=1}^{2m},\\
&K_{2m+1}(\eta;x+\tfrac{m}{2}\delta)|_{x_i\mapsto x_i-m\delta\,(i=1,2,3,4)}\\
&=\det\Big(\Phi(b-m-1,m+1-b,a-m-1,m+1-a)\Big)_{a,b=1}^{2m+1},
\end{aligned} \tag{6.1}$$

where $\Phi(m_1,m_2,m_3,m_4)=\Phi(\eta;x)|_{x_i\mapsto x_i+m_i\delta\,(i=1,2,3,4)}$, and $\Phi(\eta;x)$ is the "corrected hypergeometric integral" introduced in the previous section. Some first members of $K_n(\eta;x)$ are given as follows:

$$K_0(x)=1,\quad K_1(x)=\Phi(x),\quad K_2(x+\tfrac{\delta}{4})=\begin{vmatrix}\Phi^{24}(x) & \Phi^{13}(x)\\ \Phi^{23}(x) & \Phi^{14}(x)\end{vmatrix},$$
$$K_3(x+\tfrac{\delta}{2})|_{x_i\mapsto x_i-\delta\,(i=1,2,3,4)}=\begin{vmatrix}\Phi^{24}_{13}(x) & \Phi^{4}_{3}(x) & \Phi^{14}_{23}(x)\\ \Phi^{2}_{1}(x) & \Phi(x) & \Phi^{1}_{2}(x)\\ \Phi^{23}_{14}(x) & \Phi^{3}_{4}(x) & \Phi^{13}_{24}(x)\end{vmatrix}, \tag{6.2}$$

where we omit the label η for simplicity, and

$$\Phi^{i_1,\ldots,i_r}_{j_1,\ldots,j_r}(x)=\Phi(x)\,|_{\substack{x_i\mapsto x_i+\delta\,(i=i_1,\ldots,i_r)\\ x_j\mapsto x_j-\delta\,(j=j_1,\ldots,j_r)}}. \tag{6.3}$$

By using Jacobi's identity, one can easily see that the functions $K_n(\eta;x)$ satisfy the relation

$$\begin{aligned}&K_{n+1}(\eta;x)K^{(1234)}_{n-1}(\eta;x-\tfrac{\delta}{2})\\ &=K^{(24)}_n(\eta;x-\tfrac{\delta}{4})K^{(13)}_n(\eta;x-\tfrac{\delta}{4})-K^{(14)}_n(\eta;x-\tfrac{\delta}{4})K^{(23)}_n(\eta;x-\tfrac{\delta}{4}),\end{aligned} \tag{6.4}$$

where the superscripts denote the shift of variables, namely $K^{(i_1,\ldots,i_r)}_n(\eta;x)=K_n(\eta;x)\,|_{x_i\mapsto x_i+\delta\,(i=i_1,\ldots,i_r)}$.

Definition 6.1. For each $n\in\mathbb{Z}_{\geq 0}$, we define the fifty-six functions $\tau_n(\eta;x)$ by

$$\tau_n(\eta;x)=\Upsilon_n(\eta;x)\,K_n(\eta;x). \tag{6.5}$$

Here, the normalization factor $\Upsilon_n(\eta;x)=\Upsilon^{(ab;\pm)}_n(x)$ is given by

$$\begin{aligned}\Upsilon^{(ab;\pm)}_n(x)=&\frac{1}{c_n(x)}\,\widehat{F}_{\pm}(x_a+x_b-\tfrac{n-1}{2}\delta)\\ &\times\prod_{\substack{0\leq r<s\leq 7\\ \{r,s\}\neq\{a,b\}}}F_{\pm\kappa^{(ab)}_{rs}}(x_r+x_s-\tfrac{n-1}{2}\delta)\end{aligned} \tag{6.6}$$

with

$$\begin{aligned}c_n(x)=\epsilon_{+}^{2n(n-1)}&\prod_{r=1}^{n-1}[x_1-x_2+I_r\delta][x_3-x_4+I_r\delta]\\ &\times\prod_{r=1}^{n-1}[x_1+x_2+(r-\tfrac{n+1}{2})\delta]^r[x_3+x_4+(r-\tfrac{n-1}{2})\delta]^r,\end{aligned} \tag{6.7}$$

where I_r $(r=1,2,\ldots)$ is the subset of $\mathbb{Z}$ defined by $I_r=\{-r+1,-r+3,\ldots,r-3,r-1\}$ and $[x+I_r\delta]=\prod_{k\in I_r}[x+k\delta]$.

Proposition 6.2. We have the following bilinear equations

$$\begin{aligned}&[x_1-x_2][x_3-x_4]\tau_{n+1}(\eta;x)\tau^{(1234)}_{n-1}(\eta;x-\tfrac{\delta}{2})\\ &=[x_1+x_4-\tfrac{n}{2}\delta][x_2+x_3-\tfrac{n}{2}\delta]\tau^{(24)}_n(\eta;x-\tfrac{\delta}{4})\tau^{(13)}_n(\eta;x-\tfrac{\delta}{4})\\ &\ -[x_2+x_4-\tfrac{n}{2}\delta][x_1+x_3-\tfrac{n}{2}\delta]\tau^{(14)}_n(\eta;x-\tfrac{\delta}{4})\tau^{(23)}_n(\eta;x-\tfrac{\delta}{4}).\end{aligned} \tag{6.8}$$

This Proposition is easily verified by noticing that the normalization factors satisfy the relations

$$\begin{aligned}
&[x_1-x_2][x_3-x_4]\Upsilon_{n+1}(\eta;x)\Upsilon^{(1234)}_{n-1}(\eta;x-\tfrac{\delta}{2})\\
&=[x_1+x_4-\tfrac{n}{2}\delta][x_2+x_3-\tfrac{n}{2}\delta]\Upsilon^{(24)}_{n}(\eta;x-\tfrac{\delta}{4})\Upsilon^{(13)}_{n}(\eta;x-\tfrac{\delta}{4})\\
&=[x_2+x_4-\tfrac{n}{2}\delta][x_1+x_3-\tfrac{n}{2}\delta]\Upsilon^{(14)}_{n}(\eta;x-\tfrac{\delta}{4})\Upsilon^{(23)}_{n}(\eta;x-\tfrac{\delta}{4}).
\end{aligned}\tag{6.9}$$

Definition 6.3. For each $\Lambda_n \in M_n$ ($n \in \mathbb{Z}$), we define the fifty-six functions $\tau_{\Lambda_n}(\eta;x)$ by

$$\tau_{\Lambda_n}(\eta;x)=\tau_n(\eta;x+l^{(n)}\delta),\quad l^{(n)}_i=\langle v_i,\Lambda_n\rangle+\frac{1-n}{4}\tag{6.10}$$

under the condition (4.1) and (4.4).

We show below that the functions $\tau_{\Lambda_n}(\eta;x)$ are precisely the hypergeometric τ-functions on M_n. As a preparation, let us define the action of $W(E^{(1)}_7)=\langle s_{189},s_{12},\dots,s_{67},s_{123}\rangle$ on the label set S.

Definition 6.4. We define the action of $W(E^{(1)}_7)$ on the label $\eta\in S$ as follows:

1. The label is invariant under the action of any translation.
2. The action of permutations $\sigma\in\mathfrak{S}_8$ is defined by
$$\sigma\ :\ (a,b;\pm)\mapsto(\sigma(a),\sigma(b);\pm).\tag{6.11}$$
3. Take three mutually distinct indices $i,j,k\in\{1,2,\dots,7\}$.
   (a) If $a\in\{0,i,j,k\}$ and $b\notin\{0,i,j,k\}$, then
   $$s_{ijk}\ :\ (a,b;\pm)\mapsto(a,b;\pm).\tag{6.12}$$
   (b) If either $a,b\in\{0,i,j,k\}$ or $a,b\notin\{0,i,j,k\}$, then
   $$s_{ijk}\ :\ (a,b;\pm)\mapsto(c,d;\mp),\tag{6.13}$$
   where c and d are indices such that $\{a,b,c,d\}=\{0,i,j,k\}$ or $\{a,b,c,d\}=\{1,2,\dots,7\}\backslash\{i,j,k\}$, respectively.
4. The action of central element w_c is defined by
$$w_c\ :\ (a,b;\pm)\mapsto(a,b;\mp).\tag{6.14}$$

Theorem 6.5.

1. For each $\eta\in S$, the family of functions $\{\tau_\Lambda(\eta;x)\}_{\Lambda\in M}$ satisfies all the bilinear equations for the q-Painlevé system of type $E^{(1)}_8$ under the conditions (4.1) and (4.4).

2. For each $n \in \mathbb{Z}$, the action of $W(E_7^{(1)})$ on the set of functions $\{\tau_{\Lambda_n}(\eta; x) \,|\, \eta \in S,\ \Lambda_n \in M_n\}$ is described by

$$\tau_{w.\Lambda_n}(w(\eta); x) = \tau_{\Lambda_n}(\eta; w(x)), \quad w \in W(E_7^{(1)}). \tag{6.15}$$

First, let us consider the bilinear equations

$$\begin{aligned}
&[\varepsilon_{12}][\varepsilon_{34}]\tau_{L_{m,1}+e_8}\tau_{L_{m,-1}+2e_0-e_1-e_2-e_3-e_4-e_8} \\
&= [\varepsilon_{148} - m\delta][\varepsilon_{238} - m\delta]\tau_{L_{m,0}+e_0-e_2-e_4}\tau_{L_{m,0}+e_0-e_1-e_3} \\
&\ - [\varepsilon_{248} - m\delta][\varepsilon_{138} - m\delta]\tau_{L_{m,0}+e_0-e_1-e_4}\tau_{L_{m,0}+e_0-e_2-e_3}
\end{aligned} \tag{6.16}$$

and

$$\begin{aligned}
&[\varepsilon_{12}][\varepsilon_{34}]\tau_{L_{m,2}+c+e_0-2e_9}\tau_{L_{m,0}+c+3e_0-e_1-e_2-e_3-e_4-2e_8-2e_9} \\
&= [\varepsilon_{148} - m\delta][\varepsilon_{238} - m\delta]\tau_{L_{m,1}+c+2e_0-e_2-e_4-e_8-2e_9}\tau_{L_{m,1}+c+2e_0-e_1-e_3-e_8-2e_9} \\
&\ - [\varepsilon_{248} - m\delta][\varepsilon_{138} - m\delta]\tau_{L_{m,1}+c+2e_0-e_1-e_4-e_8-2e_9}\tau_{L_{m,1}+c+2e_0-e_2-e_3-e_8-2e_9},
\end{aligned} \tag{6.17}$$

where $L_{m,n} = m(m+n)c + me_{89}$ ($m \in \mathbb{Z}$), which are of type $(\mathrm{B})_{2m}$ and $(\mathrm{B})_{2m+1}$, respectively. Substituting (6.10), we see that these bilinear equations are satisfied thanks to (6.8).

Lemma 6.6. Suppose that the functions $\tau_{\Lambda_{n-1}}(\eta; x)$ and $\tau_{\Lambda_n}(\eta; x)$ obey all the bilinear equations of type $(\mathrm{D})_n$ and $(\mathrm{C})_{n-1}$, and satisfy the relations

$$\tau_{w.\Lambda_{n-1}}(w(\eta); x) = \tau_{\Lambda_{n-1}}(\eta; w(x)), \quad \tau_{w.\Lambda_n}(w(\eta); x) = \tau_{\Lambda_n}(\eta; w(x)) \tag{6.18}$$

for any $w \in W(E_7)$. Then the function $\tau_{\Lambda_{n+1}}(\eta; x)$ determined by the bilinear equation of type $(\mathrm{B})_n$ also satisfies

$$\tau_{w.\Lambda_{n+1}}(w(\eta); x) = \tau_{\Lambda_{n+1}}(\eta; w(x)) \tag{6.19}$$

for any $w \in W(E_7^{(1)})$.

Proof. From the assumption, we have

$$[x_1 - x_2][x_3 - x_4]\tau_n^{(12)}(\eta; x)\tau_n^{(34)}(\eta; x) + (1,2,3)\text{-cyclic} = 0, \tag{6.20}$$

$$\begin{aligned}
&[x_3 - x_4][x_1 + x_5 - \tfrac{n-1}{2}\delta]\tau_{n-1}^{(1234)}(\eta; x - \tfrac{\delta}{4})\tau_n^{(25)}(\eta; x) + (3,4,5)\text{-cyclic} = 0, \\
&[x_3 - x_4][x_2 + x_5 - \tfrac{n-1}{2}\delta]\tau_{n-1}^{(1234)}(\eta; x - \tfrac{\delta}{4})\tau_n^{(15)}(\eta; x) + (3,4,5)\text{-cyclic} = 0,
\end{aligned} \tag{6.21}$$

and

$$\tau_{n-1}(w(\eta); x) = \tau_{n-1}(\eta; w(x)), \quad \tau_n(w(\eta); x) = \tau_n(\eta; w(x)) \tag{6.22}$$

for any $w \in W(E_7) = \langle s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{67}, s_{123} \rangle$. What we have to do is to show that the function $\tau_{n+1}(\eta; x)$ determined by the recurrence relation (6.8) also satisfies

$$\tau_{n+1}(w(\eta); x) = \tau_{n+1}(\eta; w(x)) \tag{6.23}$$

for any $w \in W(E_7)$. It is obvious that we have (6.23) for $w = s_{12}, s_{34}, s_{56}, s_{67}$ and s_{125} under the assumption. Then, it is sufficient to verify (6.23) for $w = s_{23}$ and s_{45}. Replacing x by $\widetilde{x} = s_{23}(x)$ in the recurrence relation (6.8), we get

$$\begin{aligned}
&[x_1 - x_3][x_2 - x_4]\tau_{n+1}(\eta;\widetilde{x})\tau_{n-1}^{(1234)}(\widetilde{\eta}; x - \tfrac{\delta}{2}) \\
&= [x_1 + x_4 - \tfrac{n}{2}\delta][x_2 + x_3 - \tfrac{n}{2}\delta]\tau_n^{(34)}(\widetilde{\eta}; x - \tfrac{\delta}{4})\tau_n^{(12)}(\widetilde{\eta}; x - \tfrac{\delta}{4}) \\
&-[x_3 + x_4 - \tfrac{n}{2}\delta][x_1 + x_2 - \tfrac{n}{2}\delta]\tau_n^{(14)}(\widetilde{\eta}; x - \tfrac{\delta}{4})\tau_n^{(23)}(\widetilde{\eta}; x - \tfrac{\delta}{4}),
\end{aligned} \tag{6.24}$$

where $\widetilde{\eta} = s_{23}(\eta)$. Then, the bilinear equation (6.20) yields $\tau_{n+1}(\widetilde{\eta}; x) = \tau_{n+1}(\eta; \widetilde{x})$. Similarly, replacing x by $\widetilde{x} = s_{45}(x)$ in the recurrence relation (6.8), we get

$$\begin{aligned}
&[x_1 - x_2][x_3 - x_5]\tau_{n+1}(\eta;\widetilde{x})\tau_{n-1}^{(1235)}(\widetilde{\eta}; x - \tfrac{\delta}{2}) \\
&= [x_1 + x_5 - \tfrac{n}{2}\delta][x_2 + x_3 - \tfrac{n}{2}\delta]\tau_n^{(25)}(\widetilde{\eta}; x - \tfrac{\delta}{4})\tau_n^{(13)}(\widetilde{\eta}; x - \tfrac{\delta}{4}) \\
&-[x_2 + x_5 - \tfrac{n}{2}\delta][x_1 + x_3 - \tfrac{n}{2}\delta]\tau_n^{(15)}(\widetilde{\eta}; x - \tfrac{\delta}{4})\tau_n^{(23)}(\widetilde{\eta}; x - \tfrac{\delta}{4}),
\end{aligned} \tag{6.25}$$

where $\widetilde{\eta} = s_{45}(\eta)$. From the bilinear equations (6.21), we get $\tau_{n+1}(\widetilde{\eta}; x) = \tau_{n+1}(\eta; \widetilde{x})$. ∎

We already have

$$\tau_{w.\Lambda_0}(w(\eta); x) = \tau_{\Lambda_0}(\eta; w(x)), \quad \tau_{w.\Lambda_1}(w(\eta); x) = \tau_{\Lambda_1}(\eta; w(x)) \tag{6.26}$$

for any $w \in W(E_7^{(1)})$ from Proposition 4.2 and 5.6. Also, these functions satisfy all the bilinear equations of type $(\mathrm{C})_0$ and $(\mathrm{D})_1$. Then we have $\tau_{w.\Lambda_2}(w(\eta); x) = \tau_{\Lambda_2}(\eta; w(x))$ for any $w \in W(E_7^{(1)})$ from Lemma 6.6. Applying Proposition 3.3 and Lemma 6.6 repeatedly, we can verify Theorem 6.5.

With respect to the q-difference equation (2.12), we have the following.

Corollary 6.7. Define the functions $f_n(x)$ and $g_n(x)$ by

$$f_n(x) = (-1)^{\omega}\frac{N_{f,n}(x)}{D_{f,n}(x)}, \quad g_n(x) = (-1)^{\omega}\frac{N_{g,n}(x)}{D_{g,n}(x)}, \tag{6.27}$$

where

$$\begin{aligned}
N_{f,n}(x) &= \langle u_0 u_1^2 u_2/q\rangle_+ \tau_n^{[03]}(x + \tfrac{\delta}{4})\tau_n^{(23)}(x - \tfrac{\delta}{4}) \\
&\quad -\langle u_0 u_2 u_3^2/q\rangle_+ \tau_n^{[01]}(x + \tfrac{\delta}{4})\tau_n^{(12)}(x - \tfrac{\delta}{4}), \\
D_{f,n}(x) &= \tau_n^{[03]}(x + \tfrac{\delta}{4})\tau_n^{(23)}(x - \tfrac{\delta}{4}) \\
&\quad -\tau_n^{[01]}(x + \tfrac{\delta}{4})\tau_n^{(12)}(x - \tfrac{\delta}{4}), \\
N_{g,n}(x) &= \langle u_0 u_1 u_2^2/q\rangle_+ \tau_n^{[03]}(x + \tfrac{\delta}{4})\tau_n^{(13)}(x - \tfrac{\delta}{4}) \\
&\quad -\langle u_0 u_1 u_3^2/q\rangle_+ \tau_n^{[02]}(x + \tfrac{\delta}{4})\tau_n^{(12)}(x - \tfrac{\delta}{4}), \\
D_{g,n}(x) &= \tau_n^{[03]}(x + \tfrac{\delta}{4})\tau_n^{(13)}(x - \tfrac{\delta}{4}) \\
&\quad -\tau_n^{[02]}(x + \tfrac{\delta}{4})\tau_n^{(12)}(x - \tfrac{\delta}{4}),
\end{aligned} \tag{6.28}$$

where $\tau_n^{[i_1 i_2]}(x) = \tau_n(x)|_{x_{i_r}\mapsto x_{i_r}-\delta\,(r=1,2)}$ and $\langle u\rangle_+ = u^{1/2}+u^{-1/2}$. Let $b_{i,n}$ $(i = 1,2,\ldots,8)$ be the parameters defined by

$$\begin{aligned}
b_{1,n} &= (-1)^\omega q^{3/8} u_0^{-1/2} u_1^{-3/4} u_2^{-3/4},\\
b_{i,n} &= (-1)^\omega q^{-5/8} u_0^{1/2} u_1^{1/4} u_2^{1/4} u_{i+1} \quad (i = 2,\ldots,6),\\
b_{7,n} &= (-1)^\omega q^{-n/2+3/8} u_0^{-1/2} u_1^{1/4} u_2^{1/4},\\
b_{8,n} &= (-1)^\omega q^{n/2+3/8} u_0^{-1/2} u_1^{1/4} u_2^{1/4}.
\end{aligned} \tag{6.29}$$

Then, $f = f_n(x)$ and $g = g_n(x)$ with $b_i = b_{i,n}$ give rise to a solution of the q-difference Painlevé equation of type $E_8^{(1)}$ given by (2.12).

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A Derivation of the difference equation (2.12)

The discrete Painlevé system can be also formulated as a discrete dynamical system on a family of rational surfaces obtained by blowing-up at eight points on $\mathbb{P}^1\times\mathbb{P}^1$. The dependent variables f and g of the q-difference equation (2.12) are the inhomogeneous coordinates of $\mathbb{P}^1\times\mathbb{P}^1$. In this formalism, the lattice L is denoted by $L = \mathbb{Z}H_1 \oplus \mathbb{Z}H_2 \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_8$. The basis is related to that of $\mathbb{P}^2$-formalism by

$$\begin{aligned}
&H_1 = \mathrm{e}_0 - \mathrm{e}_2, \quad H_2 = \mathrm{e}_0 - \mathrm{e}_1,\\
&E_1 = \mathrm{e}_0 - \mathrm{e}_1 - \mathrm{e}_2, \quad E_i = \mathrm{e}_{i+1}\,(i = 2,\ldots,8)
\end{aligned} \tag{A.1}$$

or

$$\begin{aligned}
&\mathrm{e}_0 = H_1 + H_2 - E_1, \quad \mathrm{e}_1 = H_1 - E_1, \quad \mathrm{e}_2 = H_2 - E_1,\\
&\mathrm{e}_i = E_{i-1}\,(i = 3,\ldots,9).
\end{aligned} \tag{A.2}$$

We denote the coordinate functions corresponding to the basis H_1, H_2 and E_i by h_1, h_2 and e_i, respectively.

The q-Painlevé system of type $E_8^{(1)}$ is associated with the following eight points [14]

$$f_i = b_i t + \frac{1}{b_i t}, \quad g_i = \frac{s}{b_i} + \frac{b_i}{s} \qquad (i = 1,2,\ldots,8) \tag{A.3}$$

on $\mathbb{P}^1\times\mathbb{P}^1$, where s, t and b_i are expressed by $s = e(\frac{1}{4}(h_1 - h_2) - \frac{1}{8}\delta), t = e(\frac{1}{4}(h_1 - h_2) + \frac{1}{8}\delta)$ and $b_i = e(\frac{1}{4}(h_1 + h_2) - e_i - \frac{1}{8}\delta)$, respectively. Consider the translation operator with respect to $h_2 - h_1 = \varepsilon_2 - \varepsilon_1$ denoted by T_{21}. Introducing the transformation $\mu = s_{234}s_{256}s_{278}s_{37}s_{48}s_{19}s_{21} \in W(E_8^{(1)})$, we see that $\mu^2 = T_{21}$ and that the action on the parameters is given by

$$\begin{aligned}
&\mu(b_i) = b_{9-i}^{-1}, \quad \overline{b}_i = b_i,\\
&\mu(s) = t, \quad \overline{t} = \tilde{q}t, \quad \overline{s} = \tilde{q}s, \qquad \tilde{q} = e(\tfrac{1}{2}\delta),
\end{aligned} \tag{A.4}$$

where $T_{21}(x) = \overline{x}$.

In this setting, the dependent variables f and g are expressed by

$$
\begin{aligned}
f &= \frac{f_1\,\tau_{E_2}\tau_{H_1-E_2} - f_2\,\tau_{E_1}\tau_{H_1-E_1}}{\tau_{E_2}\tau_{H_1-E_2} - \tau_{E_1}\tau_{H_1-E_1}} \\
&= \frac{[\varepsilon_{112}]_+\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_3} - [\varepsilon_{233}]_+\tau_{\mathrm{e}_1}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}}{\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_3} - \tau_{\mathrm{e}_1}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}}, \\
g &= \frac{g_1\,\tau_{E_2}\tau_{H_2-E_2} - g_2\,\tau_{E_1}\tau_{H_2-E_1}}{\tau_{E_2}\tau_{H_2-E_2} - \tau_{E_1}\tau_{H_2-E_1}} \\
&= \frac{[\varepsilon_{122}]_+\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_3} - [\varepsilon_{133}]_+\tau_{\mathrm{e}_2}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}}{\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_3} - \tau_{\mathrm{e}_2}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}},
\end{aligned}
\tag{A.5}
$$

in terms of the lattice τ-functions. The action of $W(E_8^{(1)})$ on these variables is given by

$$
s_{12} : f \leftrightarrow g, \quad s_{23} : f \to \widetilde{f}, \tag{A.6}
$$

where $\widetilde{f}$ is defined by

$$
\frac{\widetilde{f} - \widetilde{f}_2}{\widetilde{f} - \widetilde{f}_1} = \frac{f - f_2}{f - f_1}\,\frac{g - g_1}{g - g_2}. \tag{A.7}
$$

The invariance of f (resp. g) with respect to the action of s_{34} (resp. s_{23} and s_{34}) is a consequence of the bilinear equations for the lattice τ-functions. Therefore, by writing down the action of the translation operator T_{21} on the variables f and g, we get the q-difference equation (2.12) [14].

B Examples of $G_+(x)$ and $F_+(x)$

(I) Let $G_+(x)$ and $F_+(x)$ be

$$
G_+(x) = \frac{e(-\frac{\delta}{2}\binom{x/\delta}{2})}{(u;q)_\infty}, \quad F_+(x) = e(-\tfrac{\delta}{2}\tbinom{x/\delta}{3})\,(u;q,q)_\infty, \tag{B.1}
$$

with $u = e(x)$ and $q = e(\delta)$. These satisfy the difference equations

$$
G_+(x+\delta) = -[x]\,G_+(x), \quad F_+(x+\delta) = G_+(x)F_+(x). \tag{B.2}
$$

(II) Let $G_+(x)$ and $F_+(x)$ be

$$
G_+(x) = \frac{\theta(u^{1/2};q^{1/2})}{(u;q)_\infty}, \quad F_+(x) = \Gamma(u^{1/2};q^{1/2},q^{1/2})(u;q,q)_\infty, \tag{B.3}
$$

where

$$
\theta(u;q) = (u, qu^{-1};q)_\infty, \quad \Gamma(u;p,q) = \frac{(pqu^{-1};p,q)_\infty}{(u;p,q)_\infty}. \tag{B.4}
$$

Due to

$$
\theta(qu;q) = -u^{-1}\theta(u;q), \quad \Gamma(qu;q,q) = \theta(u;q)\Gamma(u;q,q), \tag{B.5}
$$

we have the difference equations

$$G_+(x+\delta) = [x]\, G_+(x), \quad F_+(x+\delta) = G_+(x)F_+(x). \tag{B.6}$$

If ω is odd, we have $G_-(x) = G_+(x), F_-(x) = F_+(x)$ and the relations

$$G_+(x)G_+(\delta+\omega-x) = 1, \quad F_+(x) = F_+(2\delta+\omega-x). \tag{B.7}$$

(III) It is possible to construct the functions $G_+(x)$ and $F_+(x)$ in terms of the multiple sine functions $S_r(x;\omega_1,\dots,\omega_r)$ [12, 18]. These satisfy the difference equations

$$S_r(x+\omega_r;\omega_1,\dots,\omega_r) = S_{r-1}(x;\omega_1,\dots,\omega_{r-1})^{-1} S_r(x;\omega_1,\dots,\omega_r) \tag{B.8}$$

and the following relations

$$S_r(\omega_1+\cdots+\omega_r-x;\omega_1,\dots,\omega_r) = S_r(x;\omega_1,\dots,\omega_r)^{(-1)^{r-1}}. \tag{B.9}$$

The first two members are given by

$$\begin{aligned} S_1(x;\omega_1) &= 2\sin\frac{\pi x}{\omega_1}, \\ S_2(x;\omega_1,\omega_2) &= e(\tfrac{1}{2}\, B_2(x;\omega_1,\omega_2)) \frac{(e(x/\omega_1); e(\omega_2/\omega_1))_\infty}{(e((x-\omega_1)/\omega_2); e(-\omega_1/\omega_2))_\infty}, \end{aligned} \tag{B.10}$$

where $B_2(x;\omega_1,\omega_2)$ is the Bernoulli polynomial of degree 2 defined by

$$B_2(x;\omega_1,\omega_2) = \frac{1}{2\omega_1\omega_2}\left(x^2 - (\omega_1+\omega_2)x + \frac{\omega_1^2+3\omega_1\omega_2+\omega_2^2}{6}\right). \tag{B.11}$$

Let $G_+(x)$ and $F_+(x)$ be

$$G_+(x) = \frac{1}{S_2(x;1,\delta)}, \quad F_+(x) = S_3(x;1,\delta,\delta). \tag{B.12}$$

Then, we have the difference equations

$$G(x+\delta) = -\sqrt{-1}\,[x]\, G(x), \quad F(x+\delta) = G(x)F(x). \tag{B.13}$$

It is possible to define the functions $G_-(x)$ and $F_-(x)$ by (4.12) for any $\omega \in \mathbb{Z}$. If $\omega = 1$, we see that $G_-(x) = G_+(x)$ and $F_-(x) = F_+(x)$ due to the relations

$$G_+(x)G_+(1+\delta-x) = 1, \quad F_+(x) = F(1+2\delta-x). \tag{B.14}$$

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