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Abstract. The random walk model was introduced and investigated by D. Heyer [1]. It is a loss development model, where the geometric Brownian motion, which is frequently used in Mathematical Finance (for example, recall the famous Black-Sholes option pricing formula), is applied to cumulative losses. In this paper, as an application of the random walk model, the conditional distribution and the conditional confidence interval of the total loss to be paid in the specific future year, being given the cumulative losses of the present, will be investigated.

Keywords. random walk model, geometric Brownian motion, distribution, confidence interval, cumulative loss

1. Introduction

In the random walk model introduced by D. Heyer [1], it is assumed that the cumulative loss development P_t of age t obeys the stochastic differential equation (SDE in short)

$$dP_t = \mu(t)P_t dt + \sigma(t)P_t dB_t,$$

where $\mu, \sigma : [0,\infty) \to \mathbb{R}$ are continuous, and dB_t stands for the Itô integral with respect to the 1-dimensional Brownian motion $\{B_t\}_{t\geqq 0}$. For Itô integrals, see Section A. If $\mu(t)$ and $\sigma(t)$ are both constant functions, then $\{P_t\}_{t\geqq 0}$ is the geometric Brownian motion, which plays a key role in the famous Black-Sholes model in Mathematical Finance.

Let $n \in \mathbb{N}$. For $1 \leqq i \leqq n$, $t \geqq 0$, denote by S_t^i the cumulative loss from accident year k as of age t. See Figure 1 below. In this paper, we assume that each cumulative loss

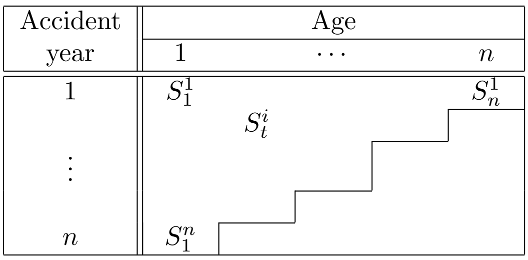

Figure 1: Run-Off Triangle

S_t^i obeys the same SDE that P_t does;

$$dS_t^i = \mu(t)S_t^i dt + \sigma(t)S_t^i dB_t^i, \quad i = 1,2,\dots,n, \tag{1}$$

where $\{B_t^1\}_{t\geqq 0}, \dots, \{B_t^n\}_{t\geqq 0}$ are independent 1-dimensional Brownian motions with $B_0^i = 0$, $i = 1,2,\dots,n$. Due to Itô's formula, it holds (Proposition 3, Sect.B) that

$$S_t^i = S_s^i \exp\Big(\nu(s;t) + \int_s^t \sigma(u)dB_u^i\Big), \tag{2}$$

where

$$\nu(s;t) = \int_s^t \{\mu(u) - \tfrac{1}{2}\sigma^2(u)\}du.$$

For k with $n+1 \leqq k \leqq 2n$, define T_k by

$$T_k = \sum_{i=0}^{2n-k} \{S_{k-n+i}^{n-i} - S_{k-n+i-1}^{n-i}\}. \tag{3}$$

See Figure 2 below. By definition, T_k stands for the total

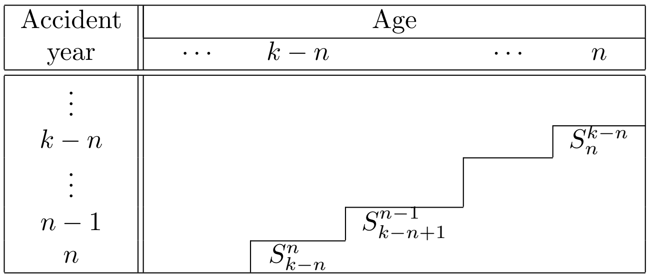

Figure 2: T_k

loss to be paid in the year k. The aim of this paper is to investigate the conditional distribution and the conditional confidence intervals of T_k given the present cumulative loss. More precisely, for $r = (r_1, \dots, r_n)$ with $r_j > 0$, $1 \leqq j \leqq n$, denote by $P^{(r)}$ the conditional probability given $S_\ell^j = r_j$, $1 \leqq j, \ell \leqq n$ with $j + \ell = n+1$;

$$P^{(r)}(A) = P(A \,|\, S_{n-j+1}^j = r_j, j = 1,\dots,n), \quad A \in \mathcal{F},$$

where $(\Omega, \mathcal{F}, P)$ is the underlying probability space, on which the Brownian motions $\{B_t^i\}_{t\geqq 0}$, $1 \leqq i \leqq n$, are defined. The conditions are given on the diagonal as below (Figure 3); In this paper, we investigate the distribution and the confidence intervals of T_k under $P^{(r)}$.

The organization of the paper is as follows. In the first three sections, main observations will be made by assuming

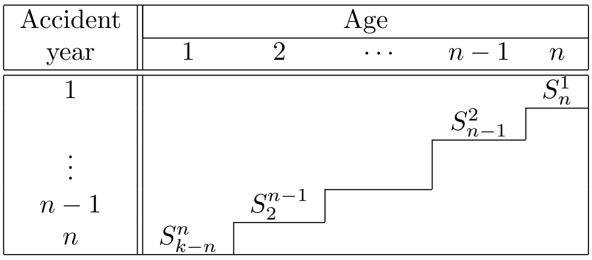

Figure 3: Conditioning

several mathematical facts. The proofs of the facts are given in Sections A and B at the end of the paper.

In the sequel, the "age" parameter t in S_t^i's are assumed to run over natural numbers $\mathbb{N}$; For example, $\{S_t\}_{t\geqq 0}$ is equal to $\{S_t\}_{t=1,2,\ldots}$.

2. Distribution

Let S_t^i, T_k, and $P^{(r)}$ be as in the previous section. In this section, we investigate the distribution of T_k under $P^{(r)}$.

We start this section with introducing some notations. For a random variable X, we write $X \sim LN(m,v)$ if X obeys the log-normal distribution, i.e., the probability density function $f_{m,v}$ of X is of the form

$$f_{m,v}(x) = \frac{1}{\sqrt{2\pi v}\,x} e^{-(\log x - m)^2/2v} \mathcal{X}_{(0,\infty)}(x), \quad x \in \mathbb{R}$$

where $\mathcal{X}_A$ stands for the indicator function of A;

$$\mathcal{X}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

Define $f_{m,v;\ell,u} : \mathbb{R} \to [0,\infty)$ by

$$f_{m,v;\ell,u}(z) = \int_{-\infty}^{\infty} f_{m,v}\Big(\frac{z}{w} + 1\Big) f_{\ell,u}(w) \frac{1}{w} dw, \quad z \in \mathbb{R}.$$

Since $S_{k-n+i}^{n-i} - S_{k-n+i-1}^{n-i}$, $i = 0, \ldots, 2n-k$, are independent under $P^{(r)}$ (Proposition 4, Sect.B), the distribution of T_k is obtained as the convolution of those of $S_{k-n+i}^{n-i} - S_{k-n+i-1}^{n-i}$'s. Thus it suffices to specify the distribution of $S_{k-n+i}^{n-i} - S_{k-n+i-1}^{n-i}$ under $P^{(r)}$.

By the very definition of conditional probability, under $P^{(r)}$, it holds that

$$\begin{aligned} &S_{k-n+i}^{n-i} - S_{k-n+i-1}^{n-i} \\ &= \left[\frac{S_{k-n+i}^{n-i}}{S_{k-n+i-1}^{n-i}} - 1\right] \times \frac{S_{k-n+i-1}^{n-i}}{S_{i+1}^{n-i}} \times r_{n-i}. \end{aligned} \tag{4}$$

We first assume that $k > n+2$. By Proposition 4 in Sect.B, $S_{k-n+i}^{n-i}/S_{k-n+i-1}^{n-i}$ and $S_{k-n+i-1}^{n-i}/S_{i+1}^{n-i}$ are independent, and obey the distributions

$$LN(\nu(k-n+i-1; k-n+i), \sigma^2(k-n+i-1; k-n+i)),$$

and

$$LN(\nu(i+1; k-n+i-1), \sigma^2(i+1; k-n+i-1)),$$

respectively, where

$$\sigma^2(s;t) = \int_s^t \sigma(u)^2 du.$$

If we set

$$\begin{aligned} m(i,k) &= \nu(k-n+i-1; k-n+i), \\ v(i,k) &= \sigma^2(k-n+i-1; k-n+i), \\ \ell(i,k;r) &= \nu(i+1; k-n+i-1) + \log r_{n-i}, \\ u(i,k) &= \sigma^2(i+1; k-n+i-1), \end{aligned}$$

and

$$f_{k,r}^i = f_{m(i,k),v(i,k);\ell(i,k;r),u(i,k)},$$

then, by virtue of Proposition 5 in Sect.B, $f_{k,r}^i$ is the probability density function of $S_{k-n+i}^{n-i} - S_{k-n+i-1}^{n-i}$ under $P^{(r)}$;

$$P^{(r)}(S_{k-n+i}^{n-i} - S_{k-n+i-1}^{n-i} \leqq a) = \int_{-\infty}^{a} f_{k,r}^i(x)dx \quad \text{for every } a \in \mathbb{R}.$$

We next assume that $k = n+2$. Then, the quantity $S_{k-n+i-1}^{n-i}/S_{i+1}^{n-i}$ in the right hand side of (4) equals to 1. Hence we have that

$$\begin{aligned} &P^{(r)}(S_{k-n+i}^{n-i} - S_{k-n+i-1}^{n-i} \leqq a) \\ &\quad = P^{(r)}\bigg(r_{n-i}\frac{S_{n-k+i}^{n-i}}{S_{n-k+i-1}^{n-i}} \leqq a + r_{n-i}\bigg). \end{aligned}$$

If we put

$$\begin{aligned} m(i;r) &= \nu(i+1; i+2) + \log r_{n-i}, \\ v(i) &= \sigma^2(i+1; i+2), \end{aligned}$$

and

$$f_{n+2,r}^i(x) = f_{m(i;r),v(i)}(x - r_{n-i}), \quad x \in \mathbb{R},$$

then, by Proposition 5 in Sect.B, $f_{n+2,r}^i$ is the probability density function of $S_{k-n+i}^{n-i} - S_{k-n+i-1}^{n-i}$ under $P^{(r)}$;

$$P(S_{k-n+i}^{n-i} - S_{k-n+i-1}^{n-i} \leqq a) = \int_{-\infty}^{a} f_{n+2,r}^i(x)dx, \quad \text{for any } a \in \mathbb{R}.$$

Summing up the above observations, we arrive at

Theorem 1. *Let $f_{k,r}^i$'s be as above. Define*

$$f_{k,r} = f_{k,r}^0 * f_{k,r}^1 * \cdots * f_{k,r}^{2n-k},$$

where $*$ indicates the convolution. Then, $f_{k,r}$ is the probability density function of T_k under $P^{(r)}$;

$$P^{(r)}(T_k \leqq a) = \int_{-\infty}^{a} f_{k,r}(x)dx, \quad \textit{for every } a \in \mathbb{R}.$$

3. Confidence interval

We first compute the expectation $t_k(r)$ of T_k under $P^{(r)}$. It holds that

$$S_t^{n-i} = r_{n-i}\frac{S_t^{n-i}}{S_{i+1}^{n-i}} \quad \text{under } P^{(r)}.$$

By Proposition 4 in Sect.B, under $P^{(r)}$,

$$\frac{S_t^{n-i}}{S_{i+1}^{n-i}} \sim LN(\nu(i+1;t), \sigma^2(i+1;t)).$$

We then have that

$$E_{P^{(r)}}\left[\frac{S_t^{n-i}}{S_{i+1}^{n-i}}\right] = e^{\mu(i+1;k-n+i)},$$

where $E_{P^{(r)}}$ stands for the expectation with respect to $P^{(r)}$, and

$$\mu(s;t) = \int_s^t \mu(u)du = \nu(s;t) + \frac{1}{2}\sigma^2(s;t).$$

Hence

$$\begin{aligned} t_k(r) &= E_{P^{(r)}}[T_k] \\ &= \sum_{i=0}^{2n-k} r_{n-i}\big\{e^{\mu(i+1;k-n+i)} - e^{\mu(i+1;k-n+i-1)}\big\}. \end{aligned}$$

For $\alpha \in (0, 100)$, choose B_α^r and R_α^r so that

$$\int_{t_k(r)-B_\alpha^r}^{t_k(r)+B_\alpha^r} f_{k,r}(x)dx = 1 - \frac{\alpha}{100},$$
$$\int_{R_\alpha^r}^{\infty} f_{k,r}(x)dx = \frac{\alpha}{100}.$$

Then it holds that

$$P^{(r)}\big(T_k \notin (t_k(r) - B_\alpha^r, t_k(r) + B_\alpha^r)\big) = \frac{\alpha}{100},$$

and

$$P^{(r)}\big(T_k \geqq R_\alpha^r\big) = \frac{\alpha}{100}.$$

Hence we have that

Proposition 1. *(i) The two-sided* $(100-\alpha)\%$ *confidence interval of* T_k *is* $(t_k(r) - B_\alpha^r, t_k(r) + B_\alpha^r)$.
(ii) The one-sided $(100-\alpha)\%$ *confidence interval of* T_k *is* $[0, R_\alpha^r)$.

From the second assertion, we see that the total loss of the year k is at most R_α^r with $(100-\alpha)\%$ reliability.

Using the Chebychev inequality, we shall compute more concrete confidence interval. In what follows, we denote by $V_{P^{(r)}}(X)$ the variance of X under $P^{(r)}$.

Due to the Chebychev inequality, it holds that

$$(5) \qquad P^{(r)}(|T_k - t_k(r)| \geqq C) \leqq \frac{1}{C^2}V_{P^{(r)}}(T_k), \quad C > 0.$$

By Proposition 4 in Sect.B, $S_{k-n+i}^{n-i} - S_{k-n+i-1}^{n-i}$, $0 \leqq i \leqq 2n-k$, are independent, and hence

$$(6) \qquad V_{P^{(r)}}(T_k) = \sum_{i=0}^{2n-k} V_{P^{(r)}}(S_{k-n+i}^{n-i} - S_{k-n+i-1}^{n-i}).$$

By virtue of (4), we see that

$$\begin{aligned} &V_{P^{(r)}}(S_{k-n+i}^{n-i} - S_{k-n+i-1}^{n-i}) \\ &= r_{n-i}^2 V_{P^{(r)}}\left(\left[\frac{S_{k-n+i}^{n-i}}{S_{k-n+i-1}^{n-i}} - 1\right] \times \frac{S_{k-n+i-1}^{n-i}}{S_{i+1}^{n-i}}\right). \end{aligned}$$

Plugging this, Propositions 4 and 5, and (6) into (5), we obtain that

$$P^{(r)}\big(|T_k - t_k(r)| \geqq C\big) \leqq \frac{1}{C^2}\sum_{i=0}^{2n-k} r_{n-i}^2 V_i^k,$$

where

$$\begin{aligned} V_i^k = {} & e^{2\mu(i+1;k-n+i-1)+\sigma^2(i+1;k-n+i-1)} \\ & \times e^{2\mu(k-n+i-1;k-n+i)} \\ & \times \big\{e^{\sigma^2(k-n+i-1;k-n+i)} - 1\big\} \\ & + e^{2\mu(i+1;k-n+i-1)}\big\{e^{\sigma^2(i+1;k-n+i-1)} - 1\big\} \\ & \times \big\{e^{\mu(k-n+i-1;k-n+i)} - 1\big\}^2. \end{aligned}$$

For $\alpha \in (0, 100)$, choose C_α^r so that

$$\frac{1}{(C_\alpha^r)^2}\sum_{i=0}^{2n-k} r_{n-i}^2 V_i^k \leqq \frac{\alpha}{100}.$$

Then it holds that

$$P^{(r)}\big(T_k \notin (t_k(r) - C_\alpha^r, t_k(r) + C_\alpha^r)\big) \leqq \frac{\alpha}{100}.$$

Thus we have that

Proposition 2. *The interval*

$$(t_k(r) - C_\alpha^r, t_k(r) + C_\alpha^r)$$

is the two-sided more than $(100-\alpha)\%$ *confidence interval of* T_k.

Remark 1. For a square integrable random variable X, $E[(X-a)^2]$, $a \in \mathbb{R}$, is minimized when $a = E[X]$, i.e., the variance of X is the minimum of $E[(X-a)^2]$, $a \in \mathbb{R}$. In this sense, the above two-sided confidence interval is the best one as far as one controls the probability $P(|X-a| \geqq C)$ by $E[(X-a)^2]$ via the Chebychev inequality.

A. Itô integrals of deterministic continuous function

In this section, we shall give a brief review on Itô integrals of deterministic continuous function. For general Itô integrals, see [2].

Let $\{\beta_t\}_{t\geqq 0}$ be a 1-dimensional Brownian motion on a probability space $(\Omega,\mathcal{F},P)$, and $\sigma:[0,\infty)\to\mathbb{R}$ a continuous function. For $s<t$, set $T^{s,t}_{k,i}=s+i(t-s)2^{-k}$, and define

$$I_k(s;t)=\sum_{i=0}^{2^k-1}\sigma(T^{s,t}_{k,i})\Big\{\beta_{T^{s,t}_{k,i+1}}-\beta_{T^{s,t}_{k,i}}\Big\}.$$

If $m>k$, then it holds that

$$\begin{aligned} &I_k(s;t)-I_m(s;t)\\ &\quad=\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^{m-k}-1}\big\{\sigma(T^{s,t}_{k,i})-\sigma(T^{s,t}_{m,2^{m-k}i+j})\big\}\\ &\qquad\qquad\times\Big\{\beta_{T^{s,t}_{m,2^{m-k}i+j+1}}-\beta_{T^{s,t}_{m,2^{m-k}i+j}}\Big\}. \end{aligned}$$

By the independent increments property of the Brownian motion and the fact that $\beta_u-\beta_v$ obeys the normal distribution of mean 0 and covariance $u-v$, it holds that

$$\begin{aligned} &E\Big[\Big\{\beta_{T^{s,t}_{m,2^{m-k}i+j+1}}-\beta_{T^{s,t}_{m,2^{m-k}i+j}}\Big\}\\ &\qquad\times\Big\{\beta_{T^{s,t}_{m,2^{m-k}p+q+1}}-\beta_{T^{s,t}_{m,2^{m-k}p+q}}\Big\}\Big]\\ &=\delta_{ip}\delta_{jq}2^{-m}(t-s). \end{aligned}$$

Plugging this into the above identity, we have that

$$\begin{aligned} &E[(I_k(s;t)-I_m(s;t))^2]\\ &=\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^{m-k}-1}\big\{\sigma(T^{s,t}_{k,i})-\sigma(T^{s,t}_{m,2^{m-k}i+j})\big\}^2 2^{-m}(t-s)\\ &\leqq\Big[\max_{0\leqq i\leqq k,0\leqq j\leqq 2^{m-k}}\{\sigma(T^{s,t}_{k,i})-\sigma(T^{s,t}_{m,2^{m-k}i+j})\}^2\Big](t-s)\\ &\longrightarrow 0\quad(k,m\to\infty). \end{aligned}$$

Thus, $I_k(s;t)$ admits L^2-limit, say $\int_s^t\sigma(u)d\beta_u$, which is the Itô integral of $\sigma(u)$ over $[s,t]$;

$$\lim_{k\to\infty}E\bigg[\bigg(I_k(s;t)-\int_s^t\sigma(u)d\beta_u\bigg)^2\bigg]=0. \tag{7}$$

Since stochastic integrals of deterministic function was first considered by N. Wiener, an Itô integral of deterministic function is often called the Wiener integral.

B. Mathematical observations

We first give an expression of S^i_t. As remarked at the end of the first section, we assume that the age parameter t runs over $\mathbb{N}$.

Proposition 3. *S^i_t satisfies that*

$$S^i_t=S^i_s\exp\bigg(\nu(s;t)+\int_s^t\sigma(u)dB^i_u\bigg), \tag{8}$$

$i=1,2,\dots,n$.

Proof. Let $0<s$ and

$$X_t=\int_s^t\nu(u)du+\int_s^t\sigma(u)dB^i_u,$$

where

$$\nu(u)=\mu(u)-\frac{1}{2}\sigma(u)^2.$$

Applying Itô's formula ([2]) to $f(X_t)$, $t\geqq s$, where $f(x)=e^x$, we see that $Y_t=f(X_t)$ obeys the SDE

$$\begin{aligned} dY_t&=f'(X_t)dX_t+\frac{1}{2}f''(X_t)dX_t\cdot dX_t\\ &=Y_t\{\nu(t)dt+\sigma(t)dB^i_t\}+\frac{1}{2}Y_t\sigma(t)^2dt\\ &=\mu(t)Y_tdt+\sigma(t)Y_tdB^i_t. \end{aligned}$$

Due to the uniqueness of solution ([2]), we have that

$$S^i_t=S^i_sY_t=S^i_s\exp\bigg(\int_s^t\nu(u)du+\int_s^t\sigma(u)dB^i_u\bigg).$$

This implies the desired expression of S^i_t. □

We next investigate on S^i_t under the conditional probability $P^{(r)}$.

Proposition 4. *Under the probability measure $P^{(r)}$, the following assertions hold.*
(i) For $i\leqq n$ and $s\geqq i+1$, the processes $\{S^{n-i}_t/S^{n-i}_s\}_{t>s}$ and $\{S^{n-i}_v\}_{i+1\leqq v\leqq s}$ are independent. Moreover,

$$S^{n-i}_t/S^{n-i}_s\sim LN(\nu(s;t),\sigma^2(s;t)).$$

(ii) $\{S^2_t\}_{t\geqq n},\{S^3_t\}_{t\geqq n-1},\dots,\{S^n_t\}_{t\geqq 2}$ are independent.

The proof of the proposition is broken into several steps, each step being a lemma.

Lemma 1. *Let $s<t$. Under the probability measure P,*

$$S^i_t/S^i_s\sim LN(\nu(s;t),\sigma^2(s;t)).$$

Proof. We continue to use the same notation as used in Sect.A. Due to the observation in the previous section, if we set

$$I^i_k(s;t)=\sum_{j=0}^{2^k-1}\sigma(T^{s,t}_{k,j})\Big\{B^i_{T^{s,t}_{k,j+1}}-B^i_{T^{s,t}_{k,j}}\Big\},$$

then we obtain that

$$\begin{aligned} &\lim_{k\to\infty}E\bigg[\bigg(I^i_k(s;t)-\int_s^t\sigma(u)dB^i_u\bigg)^2\bigg]=0,\\ &E[I^i_k(s;t)]=0,\\ &E[(I^i_k(s;t))^2]=\sum_{j=0}^{2^k-1}\sigma(T_{k,j})^2 2^{-k}(t-s)\to\sigma^2(s;t). \end{aligned}$$

Since each $I^i_k(s;t)$ obeys the normal distribution, these yield that

$$\begin{aligned} &E\left[\exp\left(\sqrt{-1}\lambda\int_s^t \sigma(u)dB^i_u\right)\right] \\ &= \lim_{k\to\infty} E[\exp(\sqrt{-1}\lambda I^i_k(s;t)] \\ &= \lim_{k\to\infty}\exp\left(-\frac{E[(I^i_k(s;t))^2]\lambda^2}{2}\right) \\ &= \exp\left(-\frac{\sigma^2(s;t)\lambda^2}{2}\right) \quad \text{for any } \lambda\in\mathbb{R}. \end{aligned}$$

Thus $\int_s^t \sigma(u)dB^i_u$ obeys the normal distribution of mean 0 and variance $\sigma^2(s;t)$. In conjunction with (2), this implies the desired assertion. □

Lemma 2. *Let $s > 0$. Under P, $\{S^i_t/S^i_s\}_{t>s}$ and $\{S^i_v\}_{v\leqq s}$ are independent.*

Proof. We continue to use the same notation as used in the proof of Lemma 1. Due to the independent increment property of the Brownian motion, we see that $\{I^i_k(0;v)\}_{v\leqq s}$ and $\{I^i_k(s;t)\}_{t>s}$ are independent. By (7), this implies that so are $\{\int_0^v \sigma(u)dB^i_u\}_{v\leqq s}$ and $\{\int_s^t \sigma(u)dB^i_u\}_{t>s}$. Plugging this independence into (8), we see that $\{S^i_t/S^i_s\}_{t>s}$ and $\{S^i_v\}_{v\leqq s}$ are independent. □

Lemma 3. *Under P, $\{S^1_t\}_{t\geqq 0}, \{S^2_t\}_{t\geqq 0}, \dots, \{S^n_t\}_{t\geqq 0}$ are independent.*

Proof. We use the same notation as used in the proof of Lemma 1. Due to the independence of components of n-dimensional Brownian motion,

$$\{I^1_k(0;t)\}_{t\geqq 0}, \{I^2_k(0;t)\}_{t\geqq 0}, \dots, \{I^n_k(0;t)\}_{t\geqq 0}$$

are independent. Letting $k\to\infty$, we obtain the desired assertion. □

Proof of Proposition 4. (i) Let $i+1\leqq v_1\leqq\cdots\leqq v_m\leqq s < t_1<\cdots<t_k$. For $a_1,\dots,a_m,b_1,\dots,b_k\in\mathbb{R}$, put

$$\begin{aligned} A &= \{S^{n-i}_{t_p}/S^{n-i}_s \leqq b_p, 1\leqq p\leqq k\}, \\ B &= \{S^{n-i}_{v_q}\leqq a_q, 1\leqq q\leqq m\}. \end{aligned}$$

It follows from Lemmas 2 and 3 that

$$\begin{aligned} P^{(r)}(A\cap B) &= P(A\cap B|S^{n-i}_{i+1} = r_{n-i}) \\ &= P(A)P(B|S^{n-i}_{i+1} = r_{n-i}) \\ &= P(A|S^{n-i}_{i+1} = r_{n-i})P(B|S^{n-i}_{i+1} = r_{n-i}) \\ &= P^{(r)}(A)P^{(r)}(B). \end{aligned}$$

This implies first the desired independence, and then in conjunction with Lemma 1, does the desired conclusion on the distribution.
(ii) The assertion follows from Lemmas 2 and 3 by the similar argument as employed to prove the assertion (i). The details are omitted. □

We finally make some observations on log-normal distributions.

Proposition 5. *Let X, Y be independent random variables such that $X\sim LN(m,v)$ and $Y\sim LN(\ell,u)$. Then,*
(i) $f_{m,v;\ell,u}$ is the probability density function of the random variable $Y(X-1)$.
(ii) The variance $V(Y(X-1))$ of $Y(X-1)$ is given by

$$\begin{aligned} V(Y(X-1)) = e^{2\ell+2u}e^{2m+v}(e^v-1) \\ + e^{2\ell+u}(e^u-1)(e^{m+(v/2)}-1)^2. \end{aligned}$$

Proof. (i) Let $g:\mathbb{R}\to\mathbb{R}$ be a bounded continuous function. Due to the independence of X and Y, we have that

$$E[g(Y(X-1))] = \int_{-\infty}^\infty\int_{-\infty}^\infty g(y(x-1))f_{m,v}(x)f_{\ell,u}(y)dxdy.$$

The Jacobi matrix for the change of variables $z = y(x-1)$ and $w = y$ is

$$\begin{pmatrix} 1/w & -z/w^2 \\ 0 & 1 \end{pmatrix}.$$

Hence we have that

$$E[g(Y(X-1))] = \int_{-\infty}^\infty g(z)f_{m,v;\ell,u}(z)dz.$$

This completes the proof.
(ii) Since $E[X] = e^{m+(v/2)}$ and $E[X^2] = e^{2m+2v}$, due to the independence of X and Y, we have that

$$\begin{aligned} &V(Y(X-1)) \\ &= E[Y^2]E[(X-1)^2] - \{E[Y]E[X-1]\}^2 \\ &= e^{2\ell+2u}(e^{2m+2v}-2e^{m+(v/2)}+1) \\ &\quad - e^{2\ell+u}(e^{m+(v/2)}-1)^2 \\ &= e^{2\ell+2u}e^{2m+v}(e^v-1) + e^{2\ell+u}(e^u-1)(e^{m+(v/2)}-1)^2. \end{aligned}$$

□

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