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Y. Kagei & Y. Maekawa

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Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

ON ASYMPTOTIC BEHAVIORS OF SOLUTIONS TO PARABOLIC SYSTEMS MODELLING CHEMOTAXIS

YOSHIYUKI KAGEI[1] AND YASUNORI MAEKAWA[2]

Abstract. This paper deals with large time behavior of solutions to a Keller-Segel system which possesses self-similar solutions. By taking into account the invariant properties of the equation with respect to a scaling and translations, we show that suitably shifted self-similar solutions give more precise asymptotic profiles of general solutions at large time. The convergence rate is also computed in details.

1. Introduction

In this paper we consider the two dimensional parabolic systems modelling chemotaxis:

$$\begin{cases} \partial_t\Omega^{(1)} - \Delta\Omega^{(1)} + \nabla\cdot(\Omega^{(1)}\nabla\Omega^{(2)}) = 0, & t>0, \quad x\in\mathbb{R}^2, \\ \partial_t\Omega^{(2)} - \Delta\Omega^{(2)} - \Omega^{(1)} = 0, & t>0, \quad x\in\mathbb{R}^2, \end{cases} \tag{1.1}$$

with the initial condition

$$\Omega^{(1)}\,|_{t=0} = \Omega_0^{(1)},\ \Omega^{(2)}\,|_{t=0} = \Omega_0^{(2)}, \qquad x\in\mathbb{R}^2. \tag{1.2}$$

The system (1.1)-(1.2) is called a Keller-Segel system. Unknown functions $\Omega^{(1)}$ and $\Omega^{(2)}$ represent the population of the organism and the concentration of the chemical at $x\in\mathbb{R}^2$ and $t>0$, respectively. This system possesses an invariant property under the scaling

$$\Theta_\lambda \begin{pmatrix} \Omega^{(1)}(x,t) \\ \Omega^{(2)}(x,t) \end{pmatrix} := \begin{pmatrix} \lambda\Omega^{(1)}(\lambda^{\frac12}x,\lambda t) \\ \Omega^{(2)}(\lambda^{\frac12}x,\lambda t) \end{pmatrix}, \qquad \lambda>0.$$

Indeed, it is easy to check that if $\Omega = (\Omega^{(1)},\Omega^{(2)})^\top$ solves (1.1) then $\Theta_\lambda\Omega$ also satisfies (1.1). If Ω satisfies (1.1) and $\Theta_\lambda\Omega=\Omega$ for all $\lambda>0$, then Ω is called a self-similar solution to (1.1). The existence of self-similar solutions to (1.1) is proved by Biler [1], and Naito [12] showed that general solutions to (1.1)-(1.2) behave like self-similar solutions at time infinity. Let us be

[1]Faculty of Mathematics, Kyushu University
6-10-1, Hakozaki, Higashi-ku, Fukuoka, 812-8581, Japan
kagei@math.kyushu-u.ac.jp
[2]Department of Mathematics, Graduate School of Science, Kobe University
1-1, Rokkodai, Nada-ku, Kobe, 657-8501, Japan
yasunori@math.kobe-u.ac.jp

more precise. We mean by $\nabla f \in L^p(\mathbb{R}^2)$ that $\partial_j f$ belongs to $L^p(\mathbb{R}^2)$ for each $j = 1, 2$. Then the following theorem is proved in [12].

Theorem 1 ([12]). *If $\|\Omega_0^{(1)}\|_{L^1}$ and $\|\nabla\Omega_0^{(2)}\|_{L^2}$ are sufficiently small, then there is a unique solution Ω to (1.1)-(1.2) such that*

$$\sup_{t>0} t^{1-\frac{1}{p}}\|\Omega^{(1)}(t)\|_{L^p} + \sup_{t>0} t^{\frac{1}{2}-\frac{1}{q}}\|\nabla\Omega^{(2)}(t)\|_{L^q} < \infty, \qquad 1 \le p \le \infty,\ 2 \le q \le \infty. \tag{1.3}$$

Moreover, if $(1+|x|^2)\Omega_0^{(1)} \in L^1(\mathbb{R}^2)$ and $\nabla\Omega_0^{(2)} \in L^1(\mathbb{R}^2)$ in addition, then it follows that

$$t^{1-\frac{1}{p}}\|\Omega^{(1)}(t) - \frac{1}{t}U_\delta^{(1)}(\frac{\cdot}{\sqrt{t}})\|_{L^p} = O(t^{-\sigma}), \qquad t \to \infty, \tag{1.4}$$

for some $\sigma \in (0, \frac{1}{2})$. Here $(t^{-1}U_\delta^{(1)}(\frac{x}{\sqrt{t}}), U_\delta^{(2)}(\frac{x}{\sqrt{t}}))^\top$ is the self-similar solution to (1.1) with $\int_{\mathbb{R}^2} U_\delta^{(1)}(x)dx = \delta$, where $\delta = \int_{\mathbb{R}^2}\Omega_0^{(1)}(x)dx$.

In this paper we study the asymptotic behavior of solutions estimated as (1.4) in more details. Especially, we will give a more accurate asymptotic profile in terms of the profile function U_δ. For $m, m' > 0$ set

$$L^2_m = L^2((1+|x|^2)^{\frac{m}{2}}dx), \qquad H^1_{m'} = H^1((1+|x|^2)^{\frac{m'}{2}}dx).$$

Our main result is

Theorem 2. *Let $m > 2$. Assume that $\|\Omega_0\|_{L^2_m \times H^1_{m-2}} \ll 1$ and that $\int_{\mathbb{R}^2}\Omega_0^{(1)}(x)dx = \delta \ne 0$. Let Ω be the solution to (1.1)-(1.2) in Theorem 1. Then Ω belongs to $C([0,\infty); L^2_m \times H^1_{m-2})$ and there exist $\tilde\eta(\delta) \in \mathbb{R}$ and $\tilde y^* \in \mathbb{R}^2$ such that*

$$\|\Omega^{(1)}(t) - (1+t)^{-1}U_\delta^{(1)}(\frac{\cdot + \tilde y^*}{\sqrt{1+t}})\|_{L^p} \le C_\epsilon(1+t)^{-1+\frac{1}{p}-\min\{1,\frac{m-1}{2}\}+\tilde\eta(\delta)+\epsilon}, \tag{1.5}$$

holds for all $\epsilon > 0$ and $1 \le p \le 2$. Here the number $\tilde\eta(\delta)$ depends only on δ and satisfies $\lim_{\delta\to 0}\tilde\eta(\delta) = 0$. Furtheremore, if $m > 3$ then there are $\eta(\delta) \in \mathbb{R}$ and $y^* = (\tilde y^*, y_3^*) \in \mathbb{R}^2 \times \mathbb{R}$ such that

$$\|\Omega^{(1)}(t) - (1+t+y_3^*)^{-1}U_\delta^{(1)}(\frac{\cdot + \tilde y^*}{\sqrt{1+t+y_3^*}})\|_{L^p} \le C(1+t)^{-2+\frac{1}{p}+\eta(\delta)}, \tag{1.6}$$

holds for $1 \le p \le 2$. Here the number $\eta(\delta)$ depends only on δ and satisfies $\lim_{\delta\to 0}\eta(\delta) = 0$. Moreover, $\eta(\delta)$ is positive if δ is positive, and negative if δ is negative.

Remark 1. Let $m > 3$. Then, since

$$\|(1+t)^{-1}U_\delta^{(1)}(\frac{\cdot + \tilde y^*}{\sqrt{1+t}}) - (1+t+y_3^*)^{-1}U_\delta^{(1)}(\frac{\cdot + \tilde y^*}{\sqrt{1+t+y_3^*}})\|_{L^p} \le C(1+t)^{-2+\frac{1}{p}},$$

(1.6) implies that $\tilde\eta(\delta) + \epsilon = 0$ if $\delta < 0$ and $\tilde\eta(\delta) + \epsilon = \eta(\delta)$ if $\delta > 0$ in (1.5). Therefore, (1.6) gives more precise asymptotic profile than (1.5) if $m > 3$ and $\delta < 0$.

Theorem 2 states that if the self-similar solution is suitably shifted, then it describes the large time behavior of solutions more precisely. This phenomenon was observed also for the one dimensional viscous Burgers equation; see Miller-Bernoff [10] and Yanagisawa [14]. The proof of [10] and [14] heavily depends on the use of the Hopf-Cole transformations which reduces the problem to a linear heat equation. Recently an observation similar to (1.5) was made for a one dimensional KS:

$$\left\{ \begin{array}{ll} \partial_t \Omega^{(1)} - \Delta \Omega^{(1)} + \nabla \cdot (\Omega^{(1)} \nabla \Omega^{(2)}) = 0, & t > 0, \quad x \in \mathbb{R}, \\ \partial_t \Omega^{(2)} - \Delta \Omega^{(2)} + \Omega^{(2)} - \Omega^{(1)} = 0, & t > 0, \quad x \in \mathbb{R}. \end{array} \right. \tag{1.7}$$

As for (1.7) it is proved in Nagai-Yamada [11] and Kato [7] that general solutions behave like constant multiples of the heat kernel as time goes to infinity. Nishihara [13] then proved that a suitably shifted heat kernel gives more precise asymptotics as in (1.5) by introducing a time dependent shift parameter in the spatial variable in a heuristic way.

We will prove Theorem 2 by applying the abstract results in [6] which are based on the spectral properties of the linearized operator around the profile function U_δ in connection with the symmetries of the equation: translation and scaling invariances. Our approach delineates the nature of the equation to yield the phenomenan as those in Theorem 2. However, as is seen in Sections 3 and 4, in order to verify this application one needs detailed analysis of (1.1) around the self-similar solution, which highly depends on the nonlinearity.

In general, when evolution equations possess a scaling invariant property there is an associated "similarity transform"; see (2.13). For (1.1) this transform is written as

$$\begin{pmatrix} u^{(1)}(x,t) \\ u^{(2)}(x,t) \end{pmatrix} = \begin{pmatrix} e^t \Omega^{(1)}(e^{\frac{t}{2}} x, e^t - 1) \\ \Omega^{(2)}(e^{\frac{t}{2}} x, e^t - 1) \end{pmatrix},$$

and the problem is converted to the stability problem of the profile function U_δ in this new variables. The linearized operator around U_δ, denoted by L_δ, is then given by

$$L_\delta = \mathcal{A} + B - \mathcal{N}'(U_\delta),$$

where

$$\mathcal{A}f = \begin{pmatrix} \Delta f^{(1)} \\ \Delta f^{(2)} + f^{(1)} \end{pmatrix}, \quad Bf = \begin{pmatrix} (\frac{x}{2} \cdot \nabla + I) f^{(1)} \\ \frac{x}{2} \cdot \nabla f^{(2)} \end{pmatrix},$$

and

$$\mathcal{N}'(U_\delta) f = \begin{pmatrix} \nabla \cdot (U_\delta^{(1)} \nabla f^{(2)}) + \nabla \cdot (f^{(1)} \nabla U_\delta^{(2)}) \\ 0 \end{pmatrix}.$$

See Section 3 for details. The next result on the spectrum of L_δ is important to obtain the decay rate in (1.6). For $m \geq 0$ let L^2_m and H^2_m be the Hilbert spaces defined by

$$\begin{aligned} L^2_m &= \{\phi \in L^2(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} (1+|x|^2)^m |\phi(x)|^2 dx < \infty\}, \\ H^2_m &= \{\phi \in L^2_m \mid \partial_x^l \phi \in L^2_m,\ |l| \le 2\}, \end{aligned}$$

where the inner products of L^2_m and H^2_m are defined in a natural way. Set $X_m = L^2_m \times H^2_{m-2}$ and let $\sigma(L_\delta)$ be the spectrum of L_δ in X_m.

Theorem 3. *Assume that $m > 2$. If $|\delta|$ is sufficiently small then there exist $\eta'(\delta) \in \mathbb{R}$ and $\lambda_i(\delta) \in \mathbb{C}$, $i = 1, 2$, such that*

(1.8)

$$\sigma(L_\delta) \subset \{0, -\frac{1}{2}, -1, \lambda_1(\delta), \lambda_2(\delta)\} \cup \{\mathrm{Re}(\lambda) \le -\min\{\frac{3}{2}, \frac{m-1}{2}\} + \eta'(\delta)\}.$$

Here $\eta'(\delta)$ satisfies $\lim_{\delta\to 0} \eta'(\delta) = 0$, and each $\lambda_i(\delta)$ has the asymptotics at $\delta \to 0$ as follows.

$$\lambda_1(\delta) = -1 + \frac{1}{16\pi}\delta + o(\delta), \qquad \lambda_2(\delta) = -1 - \frac{1}{2^8 3\pi^2}\delta^2 + o(\delta^2). \tag{1.9}$$

Especially, 0 is a simple eigenvalue whose eigenfunction is $\partial_\delta U_\delta$, $-\frac{1}{2}$ is a semisimple eigenvalue with multiplicity 2 whose eigenfunctions are $\partial_j U_\delta$, $j = 1, 2$. Moreover, if $m > 3$ then: -1 is a simple eigenvalue whose eigenfunction is BU_δ; $\lambda_1(\delta)$ is a simple eigenvalue; $\lambda_2(\delta)$ is a semisimple eigenvalue with multiplicity 2.

The eigenvalues $\lambda_1(\delta)$ and $\lambda_2(\delta)$ given in Theorem 3 are closely related to the decay rate in (1.6). In fact, we can see from [6, Lemma 6.2] that the value of $\eta(\delta)$ in (1.6) is given by

$$\eta(\delta) = \frac{1}{16\pi}\delta + o(\delta) \quad \text{if } \delta > 0, \qquad \eta(\delta) = -\frac{1}{2^8 3\pi^2}\delta^2 + o(\delta^2) \quad \text{if } \delta < 0.$$

The asymptotics of $\lambda_i(\delta)$ in Theorem 3 are established through the power series representation of U_δ and the reduction process in the perturbation theory of linear operators. However, some delicate calculations are required in order to obtain the exact asymptotics (1.9) due to the nonlinear interaction of U_δ.

This paper is organized as follows. In Section 2 we summarize the results of [6]. In Section 3 we check several conditions assumed in the abstract results of [6], which gives (1.5) and a part of Theorem 3. In Section 4 we complete the proof of Theorem 3 and (1.6) by investigating the spectrum of L_δ.

2. Preliminaries

In this section we recall the results in [6], where the nonlinear evolution equation in a Banach space X is discussed:

$$\frac{d}{dt}\Omega - \mathcal{A}\Omega + \mathcal{N}(\Omega) = 0, \qquad t > 0. \tag{E}$$

Here $\mathcal{A}$ is a closed linear operator in X and $\mathcal{N}$ is a nonlinear operator. In [6] a scaling and translations in abstract settings are introduced as follows.

We denote by $\mathbb{R}^\times$ the multiplicative group $\{\lambda \in \mathbb{R} \mid \lambda > 0\}$ and by $\mathbb{R}^+$ the additive group $\mathbb{R}$. Both groups are endowed with the usual Eucledian topology. Let $\mathcal{B}(X)$ be the Banach space of all bounded linear operators in X. Then

Definition 1 (Definition 2.1 [6]). (1) *We say* $\mathcal{R} = \{R_\lambda\}_{\lambda\in\mathbb{R}^\times} \subset \mathcal{B}(X)$ *a scaling if* $\mathcal{R}$ *is a strongly continuous action of* $\mathbb{R}^\times$ *on* X*, i.e.,*

$$R_{\lambda_1\lambda_2} = R_{\lambda_1}R_{\lambda_2}, \qquad \lambda_1,\ \lambda_2 \in \mathbb{R}^\times \tag{2.1}$$

$$R_1 = I, \tag{2.2}$$

$$R_{\lambda'}u \to R_\lambda u \ \text{ in } X \text{ when } \lambda' \to \lambda \quad \text{ for each } u \in X. \tag{2.3}$$

(2) *We say* $\mathcal{T} = \{\tau_a\}_{a\in\mathbb{R}^+} \subset \mathcal{B}(X)$ *a translation if* $\mathcal{T}$ *is a strongly continuous group acting on* X.

For one-parameter family of translations $\{\mathcal{T}_\theta\}_{\theta\in\mathbb{R}}$ *with* $\mathcal{T}_\theta = \{\tau_{a,\theta}\}_{a\in\mathbb{R}^+}$*, we say that it is strongly continuous if* $\tau_{a,\theta'}(f) \to \tau_{a,\theta}(f)$ *in* X *as* $\theta' \to \theta$ *for each* $a \in \mathbb{R}^+$ *and* $f \in X$*. When there are* n *one-parameter families of translations* $\{\mathcal{T}^{(j)}_\theta\}_{\theta\in\mathbb{R}}$, $j = 1, \cdots, n$*, we say that they are independent if for all* $a, a', \theta \in \mathbb{R}^+$ *it follows that*

$$\tau^{(i)}_{a,\theta}\tau^{(j)}_{a',\theta} = \tau^{(j)}_{a',\theta}\tau^{(i)}_{a,\theta}, \qquad 1 \le i, j \le n. \tag{2.4}$$

The generator of $\{R_\lambda\}_{\lambda\in\mathbb{R}^\times}$ and $\{\mathcal{T}^{(j)}_\theta\}_{\theta\in\mathbb{R}}$ are denoted by B and $D^{(j)}_\theta$, respectively. We also consider a linear operator $\Gamma_{a,\theta}$ which is a derivative of $\tau_{a,\theta}$ with respect to θ:

$$\begin{aligned} \mathrm{Dom}(\Gamma_{a,\theta}) &= \{f \in X \mid \lim_{h\to 0} \frac{\tau_{a,\theta+h}(f) - \tau_{a,\theta}(f)}{h} \text{ exists }\}, \\ \Gamma_{a,\theta}(f) &= \lim_{h\to 0} \frac{\tau_{a,\theta+h}(f) - \tau_{a,\theta}(f)}{h}, \qquad f \in \mathrm{Dom}(\Gamma_{a,\theta}). \end{aligned} \tag{2.5}$$

A scaling $\mathcal{R} = \{R_\lambda\}_{\lambda\in\mathbb{R}^\times}$ naturally induces an action on $C((0,\infty); X)$ as follows. For $f \in C((0,\infty); X)$ we set

$$\Theta_\lambda(f)(t) = R_\lambda(f(\lambda t)), \qquad \lambda \in \mathbb{R}^\times. \tag{2.6}$$

We call $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$ the scaling induced by $\mathcal{R}$.

Definition 2. *Let* $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$ *be the scaling induced by* $\mathcal{R} = \{R_\lambda\}_{\lambda\in\mathbb{R}^\times}$*. We say that* $f \in C((0,\infty); X)$ *is self-similar with respect to* $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$ *if*

$$\Theta_\lambda(f) = f, \qquad \lambda \in \mathbb{R}^\times. \tag{2.7}$$

Then we easily see that

Proposition 1. *The function* $f \in C((0,\infty); X)$ *is self-similar with respect to* $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$ *if and only if* f *can be expressed as*

$$f(t) = R_{\frac{1}{t}}(h) \tag{2.8}$$

with a function $h \in X$.

In [6] it is assumed that $\mathcal{A}$ generates a strongly continuous (C_0) semigroup $e^{t\mathcal{A}}$ in X, which gives a mild solution to the linear equation

$$\frac{d}{dt}\Omega - \mathcal{A}\Omega = 0, \qquad t > 0. \tag{E$_0$}$$

Definition 3. *We say that* $\Omega(t) \in C((0,\infty); X)$ *is a mild solution to (E) if* $\int_0^t \|e^{(t-s)\mathcal{A}}\mathcal{N}(\Omega(s))\|_X ds < \infty$ *for any* $t > 0$ *and* $\Omega(t)$ *satisfies the equality*

$$\Omega(t) = e^{(t-s)\mathcal{A}}\Omega(s) - \int_s^t e^{(t-\tau)\mathcal{A}}\mathcal{N}(\Omega(\tau))d\tau, \qquad \text{for all } t > s > 0. \tag{2.9}$$

Moreover, if $\Omega(t)$ *satisfies in addition*

$$\lim_{t\to 0}\Omega(t) = \Omega_0 \in X, \qquad \lim_{t\to 0}\int_0^t \|e^{(t-s)\mathcal{A}}\mathcal{N}(\Omega(s))\|_X ds = 0, \tag{2.10}$$

then we say that $\Omega(t)$ *is a mild solution to (E) with initial data* Ω_0*.*

Definition 4. *We call* $\Omega(t) \in C((0,\infty); X)$ *a self-similar solution to (E) with respect to* $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$ *if* $\Omega(t)$ *is a mild solution to (E) and is self-similar with respect to* $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$*.*

With the definition of mild solutions we define the invariance of (E) with respect to a scaling or translations as follows.

Definition 5. *Let* $\mathcal{R} = \{R_\lambda\}_{\lambda\in\mathbb{R}^\times}$ *be a scaling and let* $\{\mathcal{T}_\theta\}_{\theta\in\mathbb{R}}$ *be a strongly continuous one-parameter family of translations.*

(i) We say that (E) is invariant with respect to $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$ *if* $\Theta_\lambda(\Omega)(t)$ *is a mild solution to (E) with initial data* $R_\lambda(\Omega_0)$ *for each* $\lambda \in \mathbb{R}^\times$ *whenever* $\Omega(t)$ *is a mild solution to (E) with initial data* Ω_0*.*

(ii) We say that (E) is invariant with respect to $\{\mathcal{T}_\theta\}_{\theta\in\mathbb{R}}$ *if* $\tau_{a,t+\theta}(\Omega(t))$ *is a mild solution to (E) with initial data* $\tau_{a,\theta}(\Omega_0)$ *for each* $a \in \mathbb{R}^+, \theta \geq 0$ *whenever* $\Omega(t)$ *is a mild solution to (E) with initial data* Ω_0*.*

Let (E$_0$) be invariant with respect to $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$ and $\{\mathcal{T}_\theta\}_{\theta\in\mathbb{R}}$. Then the above definition is expressed by the relations such as

$$R_\lambda e^{\lambda t\mathcal{A}} = e^{t\mathcal{A}}R_\lambda, \tag{2.11}$$

$$\tau_{a,t+\theta}e^{t\mathcal{A}} = e^{t\mathcal{A}}\tau_{a,\theta}. \tag{2.12}$$

When (E$_0$) is invariant with respect to $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$ induced by $\mathcal{R} = \{R_\lambda\}_{\lambda\in\mathbb{R}^\times}$ we introduce the "similarity transform"

$$\Theta(t) = R_{e^t}e^{(e^t-1)\mathcal{A}} = e^{(1-e^{-t})\mathcal{A}}R_{e^t}, \qquad t \geq 0. \tag{2.13}$$

Then we have

Lemma 1. [6, Lemma 2.1] *The one parameter family* $\{\Theta(t)\}_{t\geq 0}$ *defined by (2.13) is a strongly continuous semigroup in* X*.*

The generator of $\Theta(t)$ is denoted by A. Then it follows that $\mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(B) \subset \mathrm{Dom}(A)$ and

$$Af = \mathcal{A}f + Bf, \quad \text{for} \quad f \in \mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(B). \tag{2.14}$$

2.1. **Several assumptions.** In this section we collect several assumptions on (E) and operators stated in [6].

2.1.1. *Assumptions on* (E_0). We first state the assumptions on (E_0). As stated in the previous section, the operator $\mathcal{A}$ is assumed to generate a strongly continuous semigroup $e^{t\mathcal{A}}$ in X.

(E1) *There is a scaling* $\mathcal{R} = \{R_\lambda\}_{\lambda\in\mathbb{R}^\times}$ *such that* (E_0) *is invariant with respect to* $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$.

(E2) *There are finite numbers of strongly continuous one-parameter families of translations* $\{\mathcal{T}_\theta^{(j)}\}_{\theta\in\mathbb{R}}$, $1 \le j \le n$, *such that they are independent and* (E_0) *is invariant with respect to* $\{\mathcal{T}_\theta^{(j)}\}_{\theta\in\mathbb{R}}$ *for each* j.

Let B, $D_\theta^{(j)}$, and $\Gamma_{a,\theta}^{(j)}$ be the generator of R_λ, the generator of $\mathcal{T}_\theta^{(j)}$, and the derivative of $\tau_{a,\theta}^{(j)}$ with respect to θ defined by (2.5), respectively. For a pair of linear operators $L_1,\ L_2$ its commutator is defined by $[L_1, L_2] = L_1L_2 - L_2L_1$. The next assumption represents the relation between the scaling $\mathcal{R}$ and translations $\{\mathcal{T}_\theta^{(j)}\}_{\theta\in\mathbb{R}}$.

(T1) *For all* $a, \theta \in \mathbb{R}$ *and* $j = 1, \cdots, n$ *the inclusion*

$$\tau_{a,\theta}^{(j)}\big(\mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(B) \cap \mathrm{Dom}(D_\theta^{(j)}) \cap \mathrm{Dom}(\Gamma_{a,\theta}^{(j)})\big) \subset \mathrm{Dom}(B)$$

holds, and there is a $\mu_j > 0$ *such that*

$$[B, \tau_{a,\theta}^{(j)}]f + \theta\Gamma_{a,\theta}^{(j)}f = -a\mu_j D_\theta^{(j)}\tau_{a,\theta}^{(j)}f \tag{2.15}$$

holds for $f \in \mathrm{Dom}(\mathcal{A}) \cap \mathrm{Dom}(B) \cap \mathrm{Dom}(\Gamma_{a,\theta}^{(j)}) \cap \mathrm{Dom}(D_\theta^{(j)})$.

(T2) *For any nontrivial* f *belonging to* $\mathrm{Dom}(\mathcal{A})\cap\mathrm{Dom}(B)\cap\cap_{j=1}^n\mathrm{Dom}(D_1^{(j)})$ *the functions* $\{Bf, D_1^{(1)}f, \cdots, D_1^{(n)}f\}$ *are linearly independent.*

The values μ_j in **(T1)** are related with the eigenvalues of A and they play important roles in our arguments. We set

$$\mu^* = \max\{\mu_1, \cdots, \mu_n, 1\}, \qquad \mu_* = \min\{\mu_1, \cdots, \mu_n, 1\}. \tag{2.16}$$

2.1.2. *Assumptions on* A. We recall that A is the generator of the strongly continuous semigroup $\Theta(t) = R_{e^t}e^{(e^t-1)\mathcal{A}}$. Let $\sigma(A)$ be the spectrum of A and let $r_{\mathrm{ess}}(e^{tA})$ be the radius of the essential spectrum of e^{tA}; see [2, Chapter IV] for definitions.

(A1) *There is a positive number* ϱ *such that* $\sigma(A) \subset \{0\}\cup\{\mu \in \mathbb{C} \mid \mathrm{Re}(\mu) \le -\varrho\}$. *Moreover,* 0 *is a simple eigenvalue of* A *in* X.

(A2) *There is a number* ζ *such that* $\zeta > \max\{\varrho, \mu^*\}$ *and* $r_{\mathrm{ess}}(e^{tA}) \le e^{-\zeta t}$.

Let w_0 be the eigenfunction to the eigenvalue 0 of A normalized to be 1 in X. Then we introduce the eigenprojections $\mathbf{P}_{0,0}$ and $\mathbf{Q}_{0,0}$, which are defined by

$$\mathbf{P}_{0,0}f = < f, w_0^* > w_0, \qquad \mathbf{Q}_{0,0}f = f - \mathbf{P}_{0,0}f \tag{2.17}$$

where $<,>$ is a dual coupling of X and its dual space X^*, and w_0^* is the eigenfunction to the eigenvalue 0 of the adjoint operator A^* in X^* with $< w_0, w_0^* >= 1$. From **(A2)** the set $\{\mu \in \sigma(A) \mid \mathrm{Re}(\mu) > -\zeta\}$ consists of isolated eigenvalues with finite algebraic multiplicities; see [2, Corollary IV-2-11].

2.1.3. *Assumptions on $\mathcal{N}$.* Finally we give the assumptions on the nonlinear operator $\mathcal{N}$. For a linear operator T we denote by $\|\cdot\|_{\mathrm{Dom}(T)}$ the graph norm of T, i.e., $\|f\|_{\mathrm{Dom}(T)} = \|f\|_X + \|Tf\|_X$.

(N1) *$\mathcal{N}$ maps $\mathrm{Dom}(A)$ into $\mathbf{Q}_{0,0}X$ and there is $q > 0$ such that $\|\mathcal{N}(f)\|_X \le C\|f\|_{\mathrm{Dom}(A)}^{1+q}$ holds for any $\|f\|_{\mathrm{Dom}(A)} \le 1$.*

(N2) *There are $\alpha \in (0,1]$, $\beta \in [0,1)$, and $\epsilon_0 \in [0,\varrho)$ such that for each $t > 0$ the operator*

$$N(t,\cdot) = e^{tA}\mathcal{N}(\cdot) \tag{2.18}$$

is a $C^{1+\alpha}$ map from $\mathrm{Dom}(A)$ into $\mathbf{Q}_{0,0}X$ satisfying the estimates

$$\|N'(t,f)h - N'(t,g)h\|_X \le C(\frac{1+t}{t})^{\beta} e^{-(\varrho-\epsilon_0)t}\|f-g\|_X^{\alpha}\|h\|_X, \tag{2.19}$$

for all $f,\ g,\ h \in \mathrm{Dom}(A)$ with a constant $C > 0$ depending only on α, β, ϱ, ϵ_0, and $M > 0$ when $\|f\|_X + \|g\|_X + \|h\|_X \le M$. Here $N'(t,f)$ is a Fréchet derivative of $N(t,\cdot)$ at f.

(N3) *There is a dense set $\mathcal{W}$ in X such that $\lambda R_\lambda \mathcal{N} = \mathcal{N} R_\lambda$ and $\tau_{a,\theta}^{(j)}\mathcal{N} = \mathcal{N}\tau_{a,\theta}^{(j)}$ hold in $\mathcal{W}$ for any $\lambda \in \mathbb{R}^\times$, $a \in \mathbb{R}^+$, $\theta \in \mathbb{R}$, and j.*

2.2. **Results of** [6]**.** Now let us state the results in [6]. The first result gives the existence of self-similar solutions to (E). Let $w_0,\ w_0^*,\ \mathbf{P}_{0,0},\ \mathbf{Q}_{0,0}$ be the functions and the projections in (2.17).

Theorem 4 (Theorem 2.1 [6])**.** *Assume that* **(E1)***,* **(A1)***,* **(N1)***,* **(N2)***, and* **(N3)** *hold. Let $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$ be the scaling induced by $\mathcal{R}$ in* **(E1)***, and q, α be the numbers in* **(N1)***,* **(N2)***. Then there is a number $\delta_0 > 0$ such that the following statement holds. There is a family of self-similar solutions $\{R_{\frac{1}{t}} U_\delta\}_{|\delta|\le\delta_0}$ to (E) with respect to $\{\Theta_\lambda\}_{\lambda\in\mathbb{R}^\times}$ such that U_δ is $C^{1+\alpha}$ in X with respect to δ and written in the form $U_\delta = \delta w_0 + v_\delta$ for some $v_\delta \in \mathbf{Q}_{0,0}X$ with $\|v_\delta\|_{\mathrm{Dom}(A)} \le C|\delta|^{1+q}$.*

The second result is on the existence of time global solutions to (E) and their self-similar asymptotics at time infinity.

Theorem 5 (Theorem 2.2 [6]). *Assume that* **(E1)**, **(A1)**, **(A2)**, **(N1)**, **(N2)**, *and* **(N3)** *hold. Let* ϱ, ϵ_0 *be the numbers in* **(A1)**, **(N2)**. *If* $\|\Omega_0\|_X$ *is sufficiently small, then there is a unique mild solution* $\Omega(t) \in C([0,\infty); X)$ *to (E) with initial data* Ω_0 *such that*

$$\|R_{1+t}\Omega(t) - U_\delta\|_X \le C(1+t)^{-\frac{\varrho}{2}}\|\Omega_0 - U_\delta\|_X, \quad t > 0. \tag{2.20}$$

Here $\delta = <\Omega_0, w_0^*>$ *and* U_δ *is the function in Theorem 4.*

The estimate (2.20) in Theorem 5 implies that solutions are approximated by the self-similar solution in large time with accuracy up to $O(t^{-\frac{\varrho}{2}})$. In view of **(A1)** and **(A2)**, the rate $O(t^{-\frac{\varrho}{2}})$ could be improved but in general at most up to $O(t^{-\varrho+\epsilon})$ for any $\epsilon > 0$. The aim in [6] was to present an abstract method to capture more precise asymptotic profiles of solutions by making use of symmetries of equations, translation and scaling invariances. Especially, in many applications our method gives a suitable shift of the self-similar solution as an asymptotic approximation with accuracy beyond $O(t^{-\varrho})$.

For $y = (y_1, \cdots, y_{n+1})^\top \in \mathbb{R}^{n+1}$ we define the shift operator

$$S(y; f) = \tau^{(1)}_{y_1, 1+y_{n+1}} \cdots \tau^{(n)}_{y_n, 1+y_{n+1}} R_{\frac{1}{1+y_{n+1}}} f. \tag{2.21}$$

Note that if $\mathcal{O} \subset \mathbb{R}^{n+1}$ is a sufficiently small open ball centered at the origin, then $S(y; f)$ is a continuous map from $\mathcal{O}$ to X. The following lemma represents the relations between symmetries of (E_0) and the operator A.

Lemma 2 (Lemma 2.3 [6]). *Assume that* **(E1)**, **(E2)**, **(T1)**, **(T2)**, *and* **(A1)** *hold. Let* w_0 *be the eigenfunction for the eigenvalue* 0 *of* A *in* **(A1)** *with* $\|w_0\|_X = 1$. *Suppose that* $S(\cdot\, ; w_0) : \mathcal{O} \to X$ *is* C^1. *Then* Bw_0 *and* $D_1^{(j)} w_0$ *are eigenfunctions of* A *for the eigenvalues* -1 *and* $-\mu_j$, *respectively. Moreover,* w_0, Bw_0 *and* $D_1^{(j)} w_0$, $j = 1, \cdots, n$, *are linearly independent.*

If in addition **(A2)** holds, then the set $\{\mu \in \sigma(A) \mid \mathrm{Re}(\mu) \ge -\mu^*\}$ with $\mu^* = \max\{1, \mu_1, \cdots, \mu_n\}$ consists of finite numbers of eigenvalues with finite algebraic multiplicities (note that the relation $\mu^* \ge \varrho$ holds by **(A1)** and Lemma 2). Let $\mathbf{E}_0$ be the total eigenprojection to the eigenvalues $\{\mu \in \sigma(A) \mid \mathrm{Re}(\mu) \ge -\mu^*\}$, that is,

$$\mathbf{E}_0 = \frac{1}{2\pi i}\int_{\tilde{\gamma}} (\lambda - A)^{-1} d\lambda, \tag{2.22}$$

where $\tilde{\gamma}$ is a suitable curve around $\{\mu \in \sigma(A) \mid \mathrm{Re}(\mu) \ge -\mu^*\}$.

Set

$$e_{0,0} = w_0, \quad e_{0,n+1} = c_{0,n+1} B w_0, \quad e_{0,j} = c_{0,j} D_1^{(j)} w_0, \qquad j = 1, \cdots, n. \tag{2.23}$$

Here $c_{0,j}$ is taken as $\|e_{0,j}\|_X = 1$. Then $\{e_{0,j}\}_{j=0}^{n+1}$ forms a part of the basis of the generalized eigenspace $\mathbf{E}_0 X = \{\mathbf{E}_0 f \mid f \in X\}$. So there are $\{e^*_{0,j}\}_{j=0}^{n+1} \subset X^*$ which forms a part of the basis of the generalized eigenspace associated with the eigenvalues $\{\mu \in \sigma(A^*) \mid \mathrm{Re}(\mu) \geq -\mu^*\}$ to the adjoint operator A^* and satisfies the relations $< e_{0,j}, e^*_{0,k} >= \delta_{jk}$, where $<,>$ is a dual coupling of X and its dual space X^*, and δ_{jk} is kronecker's delta. By (**A1**) at least $e^*_{0,0}(= w^*_0)$ is the eigenfunction for the simple eigenvalue 0 of A^*. We set the projections as

$$\mathbf{P}_{0,j} f =< f, e^*_{0,j} > e_{0,j}, \qquad \mathbf{Q}_{0,j} f = f - \mathbf{P}_{0,j} f, \qquad 0 \leq j \leq n+1, \tag{2.24}$$

$$\mathbf{P}_0 f = \sum_{j=0}^{n+1} \mathbf{P}_{0,j} f, \qquad \mathbf{Q}_0 f = f - \mathbf{P}_0 f. \tag{2.25}$$

Let $-\nu_0$ be the growth bound of $e^{t\mathbf{Q}_0 A \mathbf{Q}_0}$, that is,

$$-\nu_0 = \inf\{\mu \in \mathbb{R} \mid \exists C_\mu > 0 \text{ s.t. } \|e^{t\mathbf{Q}_0 A \mathbf{Q}_0} f\|_X \leq C_\mu e^{\mu t} \|f\|_X, \ \forall f \in \mathbf{Q}_0 X\}. \tag{2.26}$$

Note that we always have $\varrho \leq \nu_0 \leq \zeta$, where ϱ and ζ are the numbers in **(A1)** and **(A2)**. Next we set

$$H(y_0, y; U_\delta) = S(y; U_{\delta+y_0}). \tag{2.27}$$

Then $H(y_0, y; U_\delta)$ is continuous from $(-\delta_0 + \delta, \delta_0 - \delta) \times \mathcal{O} \subset \mathbb{R}^{n+2}$ to X for each $\delta \in (-\delta_0, \delta_0)$. Moreover, from the proof of Theorem 4 we will see that $H(y_0, y; U_\delta)$ is $C^{1+\alpha}$ with respect to y_0 in X and

$$\partial_{y_0} H(y_0, y; U_\delta)|_{\delta=y_0=0} = S(y; w_0). \tag{2.28}$$

The main contribution of [6] is as follows. Set

$$\mu_0 = 0, \qquad \mu_{n+1} = 1. \tag{2.29}$$

Theorem 6. [6, Theorem 2.3] *Assume that* **(E1)**, **(E2)**, **(T1)**, **(T2)**, **(A1)**, **(A2)**, *and* **(N1)** - **(N3)** *hold. Suppose that $S(y; w_0)$ is C^1 near $y = 0$ and $H(y_0, y; U_\delta)$ is $C^{1+\gamma}$ near $(y_0, y) = (0, 0)$ for some $\gamma > 0$. Let $\Omega(t)$ be the mild solution in Theorem 5 with $\delta \neq 0$ and let ν_0 be the number in (2.26). Assume that $\nu_0 \geq \mu^*$ and $\{-\mu_j\}_{j=1}^{n+1}$ are semisimple eigenvalues of $A - \mathcal{N}'(U_\delta)$. Then there exist $\eta(\delta) \in \mathbb{R}$ and $y^* \in \mathbb{R}^{n+1}$ such that for any $\epsilon > 0$ the following estimate holds for $t \gg 1$:*

$$\|R_{1+t}\Omega(t) - S\Big(\frac{y^*_1}{(1+t)^{\mu_1}}, \cdots, \frac{y^*_n}{(1+t)^{\mu_n}}, \frac{y^*_{n+1}}{1+t}; U_\delta\Big)\|_X \leq C_\epsilon (1+t)^{-\nu_0+\eta(\delta)+\epsilon}. \tag{2.30}$$

Here $\eta(\delta)$ satisfies $\lim_{\delta\to 0} \eta(\delta) = 0$ and C_ϵ is independent of $t \gg 1$. Especially, if $\nu_0 > \mu^*$ and $|\delta|$ is sufficiently small, then each of $\{-\mu_j\}_{j=1}^{n+1}$ is semisimple, and thus (2.30) holds in this case.

Remark 2. The value of $\eta(\delta)$ in Theorem 6 is determined by the spectrum of the linearized operator $L_\delta = A - \mathcal{N}'(U_\delta)$. Indeed, in [6, Lemma 6.2] it is proved that there exists an $\eta(\delta) \in \mathbb{R}$ with $\lim_{\delta\to 0} \eta(\delta) = 0$ such that the spectrum of L_δ is included in the set

$$\{-\mu_j\}_{j=0}^{n+1} \cup \{\mu \in \mathbb{C} \mid \mathrm{Re}(\mu) \leq -\nu_0 + \eta(\delta)\}. \tag{2.31}$$

The number $\eta(\delta)$ in Theorem 6 is nothing but $\eta(\delta)$ in (2.31).

Remark 3. Let ζ be the number in **(A2)**. As stated in [6, Remark 6.4], for the linearized operator $L_\delta = A - \mathcal{N}'(U_\delta)$ there is a number $\eta'(\delta)$ satisfying $\lim_{\delta\to 0}\eta'(\delta) = 0$ such that the spectrum $\{\mu \in \sigma(L_\delta) \mid \mathrm{Re}(\mu) > -\zeta + \eta'(\delta)\}$ consists of isolated eigenvalues with finite multiplicity. Especially, if $\zeta > \nu_0$ and all eigenvalues in $\{\mu \in \sigma(A - \mathcal{N}'(U_\delta)) \mid \mathrm{Re}(\mu) \geq -\nu_0 + \eta(\delta)\}$ are semisimple in Theorem 6, then we can take $\epsilon = 0$ in (2.30); see [6, Remark 6.4] for details. This fact will be used to obtain (1.6).

In Theorem 6 we consider the shifts of U_δ with respect to both translations and scaling. We can also consider the shifts of U_δ with respect to only translations under weaker assumptions on A. Set

$$\tilde{\mu}^* = \max\{\mu_1, \cdots, \mu_n\}, \tag{2.32}$$

and

$$\tilde{\mathbf{P}}_0 f = \sum_{j=0}^{n} \mathbf{P}_{0,j} f, \qquad \tilde{\mathbf{Q}}_0 f = f - \tilde{\mathbf{P}}_0 f. \tag{2.33}$$

Let $-\tilde{\nu}_0$ be the growth bound of $e^{t\tilde{\mathbf{Q}}_0 A \tilde{\mathbf{Q}}_0}$, that is,

$$-\tilde{\nu}_0 = \inf\{\mu \in \mathbb{R} \mid \exists C_\mu > 0 \text{ s.t. } \|e^{t\tilde{\mathbf{Q}}_0 A \tilde{\mathbf{Q}}_0} f\|_X \leq C_\mu e^{\mu t}\|f\|_X,\ \forall f \in \tilde{\mathbf{Q}}_0 X\}. \tag{2.34}$$

In this case the assumption **(A2)** is weaken to

(A2)' *There is a number* ζ *such that* $\zeta > \max\{\varrho, \tilde{\mu}^*\}$ *and* $r_{\mathrm{ess}}(e^{tA}) \leq e^{-\zeta t}$.

For $\tilde{y} = (y_1, \cdots, y_n)^\top \in \mathbb{R}^n$ let us define the shift operator $\tilde{S}(\tilde{y}; f)$ by

$$\tilde{S}(\tilde{y}; f) = S(\tilde{y}, 0; f) = \tau^{(1)}_{y_1,1} \cdots \tau^{(n)}_{y_n,1} f. \tag{2.35}$$

Then we have

Theorem 7. [6, Theorem 2.4] *Assume that* **(E1)**, **(E2)**, **(T1)**, **(T2)**, **(A1)**, **(A2)'**, *and* **(N1)** - **(N3)** *hold. Suppose that* $S(y; w_0)$ *is* C^1 *near* $y = 0$ *and* $H(y_0, y; U_\delta)$ *is* $C^{1+\gamma}$ *near* $(y_0, y) = (0, 0)$ *for some* $\gamma > 0$. *Let* $\Omega(t)$ *be the mild solution in Theorem 5 with* $\delta \neq 0$ *and let* $\tilde{\nu}_0$ *be the number in (2.34). Assume that* $\tilde{\nu}_0 \geq \tilde{\mu}^*$ *and* $\{-\mu_j\}_{j=1}^n$ *are semisimple eigenvalues of* $A - \mathcal{N}'(U_\delta)$. *Then there exist* $\eta(\delta) \in \mathbb{R}$ *and* $\tilde{y}^* = (y_1^*, \cdots, y_n^*)$ *such that for any* $\epsilon > 0$ *the following estimate holds for all* $t \gg 1$*:*

$$\|R_{1+t}\Omega(t) - \tilde{S}\Big(\frac{y_1^*}{(1+t)^{\mu_1}}, \cdots, \frac{y_n^*}{(1+t)^{\mu_n}}; U_\delta\Big)\|_X \leq C_\epsilon (1+t)^{-\tilde{\nu}_0 + \tilde{\eta}(\delta) + \epsilon}. \tag{2.36}$$

Here $\tilde{\eta}(\delta)$ *satisfies* $\lim_{\delta\to 0}\tilde{\eta}(\delta) = 0$ *and* C_ϵ *is independent of* $t \gg 1$. *Especially, if* $\tilde{\nu}_0 > \tilde{\mu}^*$ *and* $|\delta|$ *is sufficiently small, then each of* $\{-\mu_j\}_{j=1}^n$ *is semisimple, and thus (2.36) holds in this case.*

Remark 4. As in the case of Theorem 6 the number $\tilde{\eta}(\delta)$ in Theorem 7 is related with the spectrum of $L_\delta = A - \mathcal{N}'(U_\delta)$. Under the setting of Theorem 7 the spectrum of L_δ is included in the set

$$\{-\mu_j\}_{j=0}^{n+1} \cup \{\mu \in \mathbb{C} \mid \mathrm{Re}(\mu) \le -\tilde{\nu}_0 + \tilde{\eta}(\delta)\}, \tag{2.37}$$

see [6, Section 6.4]. In particular, the number $\tilde{\eta}(\delta)$ in Theorem 7 is the one in (2.37).

3. Asymptotic behavior of solutions to (1.1)

In this section we apply the results stated in the previous section to the Keller-Segel system (1.1). Let us introduce function spaces. For positive number m and nonnegative integer s let L^2_m and H^s_m be the complex Hilbert spaces defined by

$$\begin{aligned} L^2_m &= \{\phi \in L^2(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} (1+|x|^2)^m |\phi(x)|^2 dx < \infty\}, \\ &\qquad < \phi_1, \phi_2 >_{L^2_m} = \int_{\mathbb{R}^2} \phi_1(x)\overline{\phi_2(x)}(1+|x|^2)^m dx, \\ H^s_m &= \{\phi \in L^2_m \mid \partial_x^l \phi \in L^2_m,\ |l| \le s\}, \end{aligned}$$

where the inner product of H^s_m is defined in a natural way. Let $G(x) = \frac{1}{4\pi} e^{-\frac{|x|^2}{4}}$ be the two dimensional Gaussian. We also introduce a Gaussian weighted L^2 space:

$$\begin{aligned} L^2_\infty &= \{\phi \in L^2(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} |\phi(x)|^2 \frac{dx}{G(x)} < \infty\}, \\ &\qquad < \phi_1, \phi_2 >_{L^2_\infty} = \int_{\mathbb{R}^2} \phi_1(x)\overline{\phi_2(x)} \frac{dx}{G(x)}, \\ H^s_\infty &= \{\phi \in L^2_\infty \mid \partial_x^l \phi \in L^2_\infty,\ |l| \le s\}. \end{aligned}$$

Let $m \in (2, \infty]$ and let X_m be the Hilbert space defined by

$$X_m = L^2_m \times H^2_{m-2}. \tag{3.1}$$

For (E) we take $X = X_m$ and the operators $\mathcal{A}$ and $\mathcal{N}$ are respectively given by

$$\mathcal{A} = \begin{pmatrix} \Delta & 0 \\ I & \Delta \end{pmatrix}, \quad \mathcal{N}(\Omega) = \begin{pmatrix} \nabla \cdot (\Omega^{(1)} \nabla \Omega^{(2)}) \\ 0 \end{pmatrix}.$$

Then the domain of $\mathcal{A}$ is given by

$$\begin{aligned} &\mathrm{Dom}\ (\mathcal{A}) = \{f = (f^{(1)}, f^{(2)})^\top \in X_m \mid \mathcal{A} f \in X_m\} \\ &\subset \{(f^{(1)}, f^{(2)})^\top \in X_m \mid \partial_x^{s_1} f^{(1)} \in L^2_m,\ \partial_x^{s_2} f^{(2)} \in L^2_{m-2},\ |s_1| \le 2,\ |s_2| \le 4\}. \end{aligned}$$

We introduce a scaling $\{R_\lambda\}_{\lambda \in \mathbb{R}^\times}$:

$$R_\lambda \begin{pmatrix} \Omega^{(1)}(x) \\ \Omega^{(2)}(x) \end{pmatrix} := \begin{pmatrix} \lambda \Omega^{(1)}(\lambda^{\frac12} x) \\ \Omega^{(2)}(\lambda^{\frac12} x) \end{pmatrix},$$

and one parameter families of translations $\{\mathcal{T}_\theta^{(j)}\}_{\theta\in\mathbb{R}}$, $j=1,2$:

$$\mathcal{T}_\theta^{(1)} = \mathcal{T}^{(1)} = \{\tau_a^{(1)}\}_{a\in\mathbb{R}^+}, \qquad (\tau_a^{(1)}\Omega)(x) = \Omega(x_1+a, x_2),$$
$$\mathcal{T}_\theta^{(2)} = \mathcal{T}^{(2)} = \{\tau_a^{(2)}\}_{a\in\mathbb{R}^+}, \qquad (\tau_a^{(2)}\Omega)(x) = \Omega(x_1, x_2+a).$$

Then the generators of $\{R_\lambda\}_{\lambda\in\mathbb{R}^\times}$ and $\{\mathcal{T}_\theta^{(j)}\}_{\theta\in\mathbb{R}}$, $j=1,2$, are respectively given by

$$B = \begin{pmatrix} \frac{x}{2}\cdot\nabla + I & 0 \\ 0 & \frac{x}{2}\cdot\nabla \end{pmatrix}, \qquad D_\theta^{(j)} = D^{(j)} = \begin{pmatrix} \partial_j & 0 \\ 0 & \partial_j \end{pmatrix},$$

and $\Gamma_{a,\theta}^{(j)}$ (the derivative of $\{\tau_{a,\theta}^{(j)}\}_{a,\theta\in\mathbb{R}}$ with respect to θ) is zero. Now it is not difficult to see

Proposition 2. *Let $m\in(2,\infty)$ and $X=X_m$. Then for the above $\mathcal{A}$, $\{R_\lambda\}_{\lambda\in\mathbb{R}^\times}$, and $\{\mathcal{T}_\theta^{(j)}\}_{\theta\in\mathbb{R}}$, $j=1,2$, the assumptions* **(E1)**, **(E2)**, **(T1)** *and* **(T2)** *are satisfied with $\mu_j=\frac{1}{2}$, $j=1,2$.*

The proof is omitted. To study the semigroup $e^{tA}=e^{(1-e^{-t})\mathcal{A}}R_{e^t}$ let us introduce the differential operator

$$\mathcal{L} = \Delta + \frac{x}{2}\cdot\nabla + 1, \tag{3.2}$$

whose associated semigroup is given by

$$e^{t\mathcal{L}}g = \frac{e^t}{4\pi a(t)}\int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4a(t)}} g(ye^t)dy, \qquad a(t)=1-e^{-t}. \tag{3.3}$$

One can check that e^{tA} is represented by

$$e^{tA}f = \begin{pmatrix} e^{t\mathcal{L}}f^{(1)} \\ e^{-t}e^{t\mathcal{L}}f^{(2)} + (1-e^{-t})e^{t\mathcal{L}}f^{(1)} \end{pmatrix}.$$

Then we have

Proposition 3. *Let $m\in(2,\infty]$ and $X=X_m$. Then $\sigma(A)=\{-\frac{k}{2}\mid k=0,1,2,\cdots\}\cup\{\mu\in\mathbb{C}\mid \mathrm{Re}(\mu)\le-\frac{m-1}{2}\}$ and 0 is a simple eigenvalue of A in X with the eigenfunction $w_0=(G,G)^\top$. The associated eigenprojection is given by*

$$\mathbf{P}_{0,0}f = c(f)w_0, \qquad c(f)=\int_{\mathbb{R}^2} f^{(1)}(x)dx. \tag{3.4}$$

Moreover, we have $r_{ess}(e^{tA})=e^{-\frac{m-1}{2}t}$, and if $m>k+1$ then $-\frac{k}{2}$ is a semisimple eigenvalue with multiplicity $k+1+\max\{k-1,0\}$. Especially, the multiplicity of $-\frac{1}{2}$ is 2 when $m>2$, and the multiplicity of -1 is 4 when $m>3$.

Proof. We first observe that if $\mu\in\mathbb{C}$ is an eigenvalue of $\mathcal{L}$ in L^2_m then μ is an eigenvalue of A in X_m. Indeed, let $f^{(1)}\in\mathrm{Dom}(\mathcal{L})=\{\phi\in L^2_m\mid \mathcal{L}\phi\in L^2_m\}$ be an eigenfunction to the eigenvalue μ. We note that $\mathrm{Dom}(\mathcal{L})=\{\phi\in H^2_m\mid x\cdot\nabla\phi\in L^2_m\}$ with equivalent norms (for example, see [6] when $m<\infty$)

and by the elliptic regularity we can also show that $\mathrm{Dom}(\mathcal{L}^2) \hookrightarrow H^4_m$. Then it is straightforward to see that $f = (f^{(1)}, f^{(1)})^\top \in H^4_m \times H^4_m$ satisfies $Af = \mu f$. Let $2 < m < \infty$. In [5, Theorem A.1] it is proved that $\mu \in \mathbb{C}$ with $\mathrm{Re}(\mu) < -\frac{m-1}{2}$ is an eigenvalue of $\mathcal{L}$ in L^2_m. It is also shown that if $k \in \mathbb{N} \cup \{0\}$ satisfies $m > k+1$ then $-\frac{k}{2}$ is a semisimple eigenvalue of $\mathcal{L}$ whose eigenspace is spanned by $\{\partial_1^{l_1}\partial_2^{l_2} G\}_{l_1+l_2=k}$. Since the spectrum is closed, we have $\{\mu \in \mathbb{C} \mid \mathrm{Re}(\mu) \le -\frac{m-1}{2}\} \cup \{-\frac{k}{2} \mid k \in \mathbb{N} \cup \{0\}\} \subset \sigma(A)$. Set

$$G_{l_1,l_2} = \alpha_{l_1,l_2}\partial_1^{l_1}\partial_2^{l_2} G, \qquad \alpha_{l_1,l_2} = \|\partial_1^{l_1}\partial_2^{l_2} G\|^{-1}_{L^2_\infty}.$$

For each $k \in \{-1, 0\} \cup \mathbb{N}$ satisfying $m > k+1$ let $\mathcal{P}^{(1)}_k$ be the projection on L^2_m defined by

$$\mathcal{P}^{(1)}_k \phi = \sum_{l_1+l_2=k} < \phi, G_{l_1,l_2} >_{L^2_\infty} G_{l_1,l_2}, \qquad \mathcal{P}^{(1)}_{-1}\phi = 0. \tag{3.5}$$

Let n be the integer such that $n+1 < m \le n+2$. We set for $l = -1, 0, \cdots, n$,

$$\mathcal{Q}^{(1)}_l \phi = (I - \sum_{k=-1}^{l} \mathcal{P}^{(1)}_k)\phi. \tag{3.6}$$

Then by [5, Proposition A.2] we have for any $\epsilon > 0$,

$$\|\partial_1^{j_1}\partial_2^{j_2} e^{t\mathcal{L}} \mathcal{Q}^{(1)}_l \phi\|_{L^2_m} \le \frac{C_\epsilon}{a(t)^{\frac{j_1+j_2}{2}}} e^{-\frac{l+1+\min\{0, m-l-2-\epsilon\}}{2}t} \|f\|_{L^2_m}, \tag{3.7}$$

where $a(t) = 1 - e^{-t}$. Especially, by the relation $\partial_i e^{t\mathcal{L}} = e^{\frac{t}{2}} e^{t\mathcal{L}} \partial_i$ and the semigroup property of $e^{t\mathcal{L}}$, it is easy to verify

$$\|e^{t\mathcal{L}} \mathcal{Q}^{(1)}_n \phi\|_{H^s_m} \le C_\epsilon e^{-\frac{m-1+\epsilon}{2}t} \|f\|_{H^s_m}, \tag{3.8}$$

for any nonnegative integer s. For $m > 2$ and $n \in \mathbb{N}$ with $n+1 < m \le n+2$, we set

$$\mathcal{P}_n f = (I - \mathcal{Q}_n) f, \quad \mathcal{Q}_n f = (\mathcal{Q}^{(1)}_n f^{(1)}, \mathcal{Q}^{(1)}_{n-2} f^{(2)})^\top, \quad f \in X_m. \tag{3.9}$$

Note that $n - 2 \ge -1$ and $\mathcal{Q}^{(1)}_{n-2} f^{(2)}$ is well-defined for $f^{(2)} \in L^2_{m-2}$. Then from (3.8) we have for any $\epsilon > 0$,

$$\|e^{tA} \mathcal{Q}_n f\|_{X_m} \le C_\epsilon e^{-\frac{m-1+\epsilon}{2}t} \|f\|_{X_m}. \tag{3.10}$$

Since $e^{tA}\mathcal{P}_n$ is a finite rank operator, we have from (3.10) that $r_{ess}(e^{tA}) = r_{ess}(e^{tA}\mathcal{Q}_n) \le e^{-\frac{m-1}{2}t}$. On the other hand, by [2, Corollary IV-2-11], for any $w > \frac{1}{t}\log r_{ess}(e^{tA})$, the set $\{\mu \in \sigma(A) \mid \mathrm{Re}(\mu) \ge w\}$ must be finite. This implies $r_{ess}(e^{tA}) \ge e^{-\frac{m-1}{2}t}$ since each $\mu \in \mathbb{C}$ with $\mathrm{Re}(\mu) < -\frac{m-1}{2}$ is already shown to belong to $\sigma(A)$. Hence $r_{ess}(e^{tA}) = e^{-\frac{m-1}{2}t}$ holds. Since e^{tA} is expressed as

$$e^{tA} f = e^{tA}\mathcal{Q}_n f + \begin{pmatrix} e^{t\mathcal{L}} \sum_{k=-1}^{n} \mathcal{P}^{(1)}_k f^{(1)} \\ e^{-t} e^{t\mathcal{L}} \sum_{k=-1}^{n-2} \mathcal{P}^{(1)}_k f^{(2)} + (1 - e^{-t}) e^{t\mathcal{L}} \sum_{k=-1}^{n} \mathcal{P}^{(1)}_k f^{(1)} \end{pmatrix}$$

$$= e^{tA}\mathcal{Q}_n f + \begin{pmatrix} \sum_{k=-1}^{n} e^{-\frac{k}{2}t}\mathcal{P}_k^{(1)} f^{(1)} \\ \sum_{k=-1}^{n-2} e^{-\frac{k+2}{2}t}\mathcal{P}_k^{(1)} f^{(2)} + (1-e^{-t})\sum_{k=-1}^{n} e^{-\frac{k}{2}t}\mathcal{P}_k^{(1)} f^{(1)} \end{pmatrix},$$

one can check that $-\frac{k}{2}$ is a semisimple eigenvalue of A if $k \in \mathbb{N}\cup\{0\}$ satisfies $m > k+1$, and its eigenspace is spanned by

$$\begin{array}{ll} \{(\partial_1^{l_1}\partial_2^{l_2}G, \partial_1^{l_1}\partial_2^{l_2}G)^\top \mid l_1+l_2=k\} & \text{if } k=0,1 \\ \{(\partial_1^{l_1}\partial_2^{l_2}G, \partial_1^{l_1}\partial_2^{l_2}G)^\top \mid l_1+l_2=k\} \cup \{(0,\partial_1^{l_1}\partial_2^{l_2}G)^\top \mid l_1+l_2=k-2\} & \text{if } k\geq 2. \end{array}$$

Especially, the projection $\mathbf{P}_{0,0}$ is given as in (3.4).

When $m=\infty$, it is well-known that $\mathcal{L}$ is self-adjoint in L^2_∞ and its spectrum consists of semisimple eigenvalues $\{-\frac{k}{2} \mid k=0,1,2,\cdots\}$ whose eigenspace is spanned by $\{\partial_1^{l_1}\partial_2^{l_2}G\}_{l_1+l_2=k}$ for each $-\frac{k}{2}$; for example, see [3]. Moreover, instead of (3.7) the estimate

$$\|\partial_1^{j_1}\partial_2^{j_2}e^{t\mathcal{L}}\mathcal{Q}_l^{(1)}\phi\|_{L^2_\infty} \leq \frac{C}{a(t)^{\frac{j_1+j_2}{2}}}e^{-\frac{l+1}{2}t}\|\phi\|_{L^2_\infty}, \tag{3.11}$$

holds for any $t>0$. Indeed, in L^2_∞ by the spectral decomposition theorem we have

$$e^{t\mathcal{L}}\mathcal{Q}_l^{(1)}\phi = \sum_{k=l+1}^{\infty} e^{-\frac{k}{2}t}\sum_{l_1+l_2=k} <\phi, G_{l_1,l_2}>_{L^2_\infty} G_{l_1,l_2}.$$

and so (3.11) holds for $j_1=j_2=0$. From (3.3) it is not difficult to see

$$\|\partial_1^{j_1}\partial_2^{j_2}e^{t\mathcal{L}}\phi\|_{L^2_\infty} \leq \frac{C}{a(t)^{\frac{j_1+j_2}{2}}}\|\phi\|_{L^2_\infty}, \qquad 0<t<1.$$

Thus by the semigroup property we get (3.11). Then instead of (3.8) and (3.10) we have

$$\|e^{t\mathcal{L}}\mathcal{Q}_n^{(1)}\phi\|_{H^s_\infty} \leq Ce^{-\frac{n+1}{2}t}\|\phi\|_{H^s_\infty}, \tag{3.12}$$

and

$$\|e^{tA}\mathcal{Q}_n f\|_{X_\infty} \leq Ce^{-\frac{n+1}{2}t}\|f\|_{X_\infty}. \tag{3.13}$$

As in the case of $m<\infty$, the expansion of the semigroup shows that $-\frac{k}{2}$ is a semisimple eigenvalue of A in X_∞ for each $k\in\mathbb{N}\cup\{0\}$ and its multiplicity is $k+1+\max\{k-2,0\}$. This completes the proof of Proposition 3.

Proposition 3 immediately yields

Corollary 1. *Let $X=X_m$. If $m>2$ then* **(A1)** *and* **(A2)'** *hold with $\varrho=\frac{1}{2}$, $\zeta=\frac{m-1}{2}$. Furthermore, the number $\tilde{\nu}_0$ defined in (2.34) is given by $\tilde{\nu}_0=\min\{1,\frac{m-1}{2}\}$. If $m>3$ then* **(A1)** *and* **(A2)** *hold with the same ϱ and ζ as above. Furthermore, the number ν_0 defined in (2.26) is given by $\nu_0=1$.*

Next we consider the nonlinear term $\mathcal{N}(\Omega)=(\nabla\cdot(\Omega^{(1)}\nabla\Omega^{(2)}),0)^\top$ in (1.1).

Proposition 4. *Let $m \in (2,\infty)$ and $X = X_m$. Then $\mathcal{N}(\Omega)$ in (1.1) satisfies* **(N1)**, **(N2)**, *and* **(N3)** *with $q = \alpha = 2$, $\beta = \frac{3}{4}$, and $\epsilon_0 = 0$.*

Proof. Since it is easy to check **(N3)**, we show only **(N1)** and **(N2)**. We first note that

$$\begin{aligned} \mathrm{Dom}(A) &= \{(f^{(1)}, f^{(2)})^\top \in L^2_m \times H^2_{m-2} \mid \mathcal{L}f^{(1)} \in L^2_m,\ \mathcal{L}f^{(2)} \in H^2_{m-2}\} \\ &\hookrightarrow H^2_m \times H^4_{m-2}. \end{aligned}$$

Set $b(x) = (1+|x|^2)^{\frac{1}{2}}$ and let $m \in (2,\infty)$. Then **(N1)** follows from the definition of $\mathbf{P}_{0,0}$ and

$$\begin{aligned} \|\mathcal{N}(\Omega)\|_{X_m} &= \|\nabla\cdot(\Omega^{(1)}\nabla\Omega^{(2)})\|_{L^2_m} \\ &\le \|b^m\nabla\Omega^{(1)}\cdot\nabla\Omega^{(2)}\|_{L^2} + \|b^m\Omega^{(1)}\Delta\Omega^{(2)}\|_{L^2} \\ &\le \|b^m\nabla\Omega^{(1)}\|_{L^2}\|\nabla\Omega^{(2)}\|_{L^\infty} + \|b^m\Omega^{(1)}\|_{L^2}\|\Delta\Omega^{(2)}\|_{L^\infty} \\ &\le C\|\Omega^{(1)}\|_{H^2_m}\|\Omega^{(2)}\|_{H^4_{m-2}} + C\|\Omega^{(1)}\|_{H^2_m}\|\Omega^{(2)}\|_{H^4_{m-2}} \\ &\le C\|\Omega\|^2_{\mathrm{Dom}(A)}. \end{aligned} \tag{3.14}$$

Here we used the Sobolev embedding theorem. Since $\mathcal{N}$ is bilinear, in order to prove **(N2)**, it suffices to estimate

$$e^{tA}\mathcal{N}(\Omega) = \begin{pmatrix} e^{t\mathcal{L}}\nabla\cdot(\Omega^{(1)}\nabla\Omega^{(2)}) \\ (1-e^{-t})e^{t\mathcal{L}}\nabla\cdot(\Omega^{(1)}\nabla\Omega^{(2)}) \end{pmatrix}.$$

Let $p \in (1,2)$. From [5, Proposition A.5] it follows that

$$\|e^{t\mathcal{L}}\phi\|_{L^2_m} \le \frac{C}{a(t)^{\frac{1}{p}-\frac{1}{2}}}\|b^m\phi\|_{L^p}, \qquad 0 < t \le 1.$$

Then by using the relation $e^{-\frac{t}{2}}\partial_j e^{t\mathcal{L}} = e^{t\mathcal{L}}\partial_j$, (3.7), and the semigroup property, we get

$$\|e^{t\mathcal{L}}\partial_j\phi\|_{L^2_m} \le \frac{C}{a(t)^{\frac{1}{p}}}e^{-\frac{t}{2}}\|b^m\phi\|_{L^p}, \qquad t > 0. \tag{3.15}$$

Thus we have

$$\begin{aligned} \|e^{t\mathcal{L}}\nabla\cdot(\Omega^{(1)}\nabla\Omega^{(2)})\|_{L^2_m} &\le \frac{Ce^{-\frac{t}{2}}}{a(t)^{\frac{1}{p}}}\|b^m\Omega^{(1)}\nabla\Omega^{(2)}\|_{L^p} \\ &\le \frac{Ce^{-\frac{t}{2}}}{a(t)^{\frac{1}{p}}}\|\Omega^{(1)}\|_{L^2_m}\|\nabla\Omega^{(2)}\|_{L^{\frac{2p}{2-p}}} \\ &\le \frac{Ce^{-\frac{t}{2}}}{a(t)^{\frac{1}{p}}}\|\Omega^{(1)}\|_{L^2_m}\|\nabla^2\Omega^{(2)}\|^{2(1-\frac{1}{p})}_{L^2}\|\nabla\Omega^{(2)}\|^{\frac{2}{p}-1}_{L^2}. \end{aligned} \tag{3.16}$$

Here we used the Gagliardo-Nirenberg inequality in the last line. Next we see from (3.7),

$$\begin{aligned}\|(1-e^{-t})\partial_x^s e^{t\mathcal{L}}\nabla\cdot(\Omega^{(1)}\nabla\Omega^{(2)})\|_{L^2_m} &\leq C(1-e^{-t})a(\frac{t}{2})^{-\frac{|s|}{2}}\|e^{\frac{t}{2}\mathcal{L}}\nabla\cdot(\Omega^{(1)}\nabla\Omega^{(2)})\|_{L^2_m}\\ &\leq \frac{Ce^{-\frac{t}{2}}}{a(t)^{\frac{|s|}{2}-1+\frac{1}{p}}}\|\Omega^{(1)}\|_{L^2_m}\|\nabla^2\Omega^{(2)}\|_{L^2}^{2-\frac{2}{p}}\|\nabla\Omega^{(2)}\|_{L^2}^{\frac{2}{p}-1}.\end{aligned}$$

Hence we have

(3.17)

$$\|(1-e^{-t})e^{t\mathcal{L}}\nabla\cdot(\Omega^{(1)}\nabla\Omega^{(2)})\|_{H^2_m}\leq\frac{Ce^{-\frac{t}{2}}}{a(t)^{\frac{1}{p}}}\|\Omega^{(1)}\|_{L^2_m}\|\nabla^2\Omega^{(2)}\|_{L^2}^{2-\frac{2}{p}}\|\nabla\Omega^{(2)}\|_{L^2}^{\frac{2}{p}-1}.$$

Combining these above with $p=\frac{4}{3}$, we get

$$\|e^{tA}\mathcal{N}(\Omega)\|_{X_m}\leq\frac{Ce^{-\frac{t}{2}}}{a(t)^{\frac{3}{4}}}\|\Omega^{(1)}\|_{L^2_m}\|\Omega^{(2)}\|_{H^2_{m-2}}^{\frac{1}{2}}\|\Omega^{(2)}\|_{H^1_{m-2}}^{\frac{1}{2}}. \tag{3.18}$$

This gives **(N2)**. The proof is complete.

From Proposition 2 - Proposition 4 we can apply Theorem 4 and obtain (real-valued) the self-similar solutions $R_{\frac{1}{t}}U_\delta$ with $U_\delta=\delta w_0+v_\delta\in\mathrm{Dom}(A)$ for sufficiently small $|\delta|$. In order to apply Theorem 6 or Theorem 7 we need to show more regularities of U_δ. Recall that $v_\delta=(v_\delta^{(1)},v_\delta^{(2)})^\top$ is the (real-valued) solution to

$$\begin{cases}-\mathcal{L}v_\delta^{(1)}+\nabla\cdot\{(\delta G+v_\delta^{(1)})\nabla(\delta G+v_\delta^{(2)})\}=0, & x\in\mathbb{R}^2,\\ -(\mathcal{L}-I)v_\delta^{(2)}-v_\delta^{(1)}=0, & x\in\mathbb{R}^2.\end{cases} \tag{3.19}$$

From the definition of $\mathbf{P}_{0,0}$ we see that v_δ satisfies $\int_{\mathbb{R}^2}v_\delta^{(1)}(x)dx=0$, and the uniqueness of v_δ such that $\|v_\delta\|_{\mathrm{Dom}(A)}\leq C\delta^2$ also follows by [6, Theorem 4.1].

Let $\varphi=(\varphi^{(1)},\varphi^{(2)})^\top\in H^2_\infty\times H^4_\infty$, $\int_{\mathbb{R}^2}\varphi^{(1)}(x)dx=0$, be the solution to

$$\begin{cases}-\mathcal{L}\varphi^{(1)}+\nabla\cdot(G\nabla G)=0, & x\in\mathbb{R}^2,\\ -(\mathcal{L}-I)\varphi^{(2)}-\varphi^{(1)}=0, & x\in\mathbb{R}^2.\end{cases} \tag{3.20}$$

It is not difficult to see that φ uniquely exists and is radially symmetric. Indeed, since $\mathcal{L}$ is self-adjoint in L^2_∞ and satisfies $-\mathcal{L}\geq\frac{1}{2}$ in $\{\phi\in L^2_\infty\mid\int_{\mathbb{R}^2}\phi(x)dx=0\}$, the terms $\varphi^{(1)}=(-\mathcal{L})^{-1}\nabla\cdot(G\nabla G)\in\mathrm{Dom}(\mathcal{L})$ and $\varphi^{(2)}=(-\mathcal{L}+I)^{-1}\varphi^{(1)}\in\mathrm{Dom}(\mathcal{L}^2)$ make sense. Since the radial symmetry is preserved under the action of $(-\mathcal{L})^{-1}$, we get the radial symmetry of φ. Below we will freely use the relation $\mathrm{Dom}(\mathcal{L})\hookrightarrow H^2_\infty$ and $\mathrm{Dom}(\mathcal{L}^2)\hookrightarrow H^4_\infty$ (in fact, we can show the equality with equivalent norms).

Then we have

Proposition 5. *Let $m \in (2,\infty)$ and $X = X_m$. If $|\delta|$ is sufficiently small, then the function $U_\delta \in X$ in Theorem 4 for (1.1) satisfies*

$$U_\delta = \delta w_0 + \delta^2 \varphi + \delta^3 z_\delta, \tag{3.21}$$

where z_δ is C^2 with respect to δ in $H^2_\infty \times H^4_\infty$ and $\|z_\delta\|_{H^2_\infty \times H^4_\infty} \le C$. Moreover, U_δ is radially symmetric.

Proof. We first observe that $v_\delta^{(1)} \in H^2_m$ satisfies the equation

$$-\mathcal{L} v_\delta^{(1)} + \nabla(\delta G + v_\delta^{(2)}) \cdot \nabla v_\delta^{(1)} + \{\Delta(\delta G + v_\delta^{(2)})\} v_\delta^{(1)} = -\nabla \cdot \{\delta G \nabla(\delta G + v_\delta^{(2)})\}.$$

Since $v_\delta^{(2)} \in H^4_{m-2}$, we have $\|\nabla(\delta G + v_\delta^{(2)})\|_{L^\infty} + \|\Delta(\delta G + v_\delta^{(2)})\|_{L^\infty} < \infty$ and

$$\lim_{R\to\infty} \sup_{|x|\ge R} \big(|\nabla(\delta G + v_\delta^{(2)})(x)| + |\Delta(\delta G + v_\delta^{(2)})(x)|\big) = 0.$$

Now we can apply Proposition 12 in Appendix below and then $v_\delta^{(1)}$ belongs to $\mathrm{Dom}(\mathcal{L}) \hookrightarrow H^2_\infty$; in fact, we can show the Gaussian decay such as $|v_\delta^{(1)}(x)| \le C_\epsilon e^{-\frac{1-\epsilon}{4}|x|^2}$ for any $\epsilon > 0$ by [9, Proposition 1.1, Lemma 1.1]. Again by Proposition 12 we have $v_\delta^{(2)} \in H^2_\infty$, since $v_\delta^{(2)}$ solves $(-\mathcal{L} + I) v_\delta^{(2)} = v_\delta^{(1)}$. Moreover, since $v_\delta^{(2)} = (-\mathcal{L}+I)^{-1} v_\delta^{(1)}$ and $v_\delta^{(1)} \in \mathrm{Dom}(\mathcal{L})$ in L^2_∞, we conclude $v_\delta^{(2)} \in \mathrm{Dom}(\mathcal{L}^2) \subset H^4_\infty$.

We set $z_\delta \in H^2_\infty \times H^4_\infty$ as $v_\delta = \delta^2\varphi + \delta^3 z_\delta$. Then z_δ satisfies the equation

$$-A z_\delta + \mathcal{N}'(\delta G + \delta^2\varphi) z_\delta + \delta^3 \mathcal{N}(z_\delta) = -\mathcal{N}'(G)\varphi - \delta\mathcal{N}(\varphi). \tag{3.22}$$

That is, z_δ is a fixed point of the map

$$\Psi_\delta(f) = -(-A)^{-1}\big\{\mathcal{N}'(\delta G + \delta^2\varphi) f + \delta^3 \mathcal{N}(f)\big\} - (-A)^{-1}\big\{\mathcal{N}'(G)\varphi + \delta\mathcal{N}(\varphi)\big\}. \tag{3.23}$$

Since $\mathcal{N}$ is bilinear, we have $2\mathcal{N}(f) = \mathcal{N}'(f) f$ and thus it suffices to prove the estimate such as

$$\|(-A)^{-1}\mathcal{N}'(f) h\|_{H^2_\infty \times H^4_\infty} \le C \|f\|_{H^2_\infty \times H^4_\infty} \|h\|_{H^2_\infty \times H^4_\infty}. \tag{3.24}$$

Indeed, since $|\delta|$ is sufficiently small, (3.24) leads to $\|z_\delta\|_{H^2_\infty \times H^4_\infty} < \infty$ and to C^2 regularity of z_δ with respect to δ in $H^2_\infty \times H^4_\infty$ by the uniform contraction mapping theorem. Set $z = (-A)^{-1}\mathcal{N}'(f) h$. Then we have

$$\begin{aligned}
& \|z\|_{H^2_\infty \times H^4_\infty} \\
\le\ & \|z^{(1)}\|_{H^2_\infty} + \|z^{(2)}\|_{H^4_\infty} \\
=\ & \|(-\mathcal{L})^{-1}\nabla \cdot (f^{(1)}\nabla h^{(2)} + h^{(1)}\nabla f^{(2)})\|_{H^2_\infty} + \|(-\mathcal{L}+I)^{-1} z^{(1)}\|_{H^4_\infty} \\
\le\ & C\|(-\mathcal{L})^{-1}\nabla \cdot (f^{(1)}\nabla h^{(2)} + h^{(1)}\nabla f^{(2)})\|_{\mathrm{Dom}(\mathcal{L})} + C\|(-\mathcal{L}+I)^{-1} z^{(1)}\|_{\mathrm{Dom}(\mathcal{L}^2)} \\
\le\ & C\|\nabla \cdot (f^{(1)}\nabla h^{(2)} + h^{(1)}\nabla f^{(2)})\|_{L^2_\infty}.
\end{aligned}$$

By arguing as same as in (3.14), the last term is bounded from above by $C\|f\|_{H^2_\infty \times H^4_\infty}\|h\|_{H^2_\infty \times H^4_\infty}$. It is not difficult to show that z_δ is radially symmetric. Indeed, if z_δ is a fixed point of Ψ_δ, then $z_\delta(O\cdot)$ is also a fixed point of Ψ_δ for any orthogonal matrix O. Hence by the uniquness of the fixed point (which follows from the contraction mapping theorem) we get $z_\delta(\cdot) = z_\delta(O\cdot)$, i.e., z_δ is radially symmetric. Then the radial symmetry of U_δ follows from the radial symmetry of w_0, φ, and z_δ. This completes the proof.

By Proposition 5 the function U_δ belongs to $H^2_\infty \times H^4_\infty$ and is C^2 with respect to δ in this space. Recalling the definition $H(y_0, y; U_\delta) = \tau^{(1)}_{y_1}\tau^{(2)}_{y_2} R_{\frac{1}{1+y_3}} U_{\delta+y_0}$, we have

Corollary 2. *Let $m \in (2,\infty)$ and $X = X_m$. If $|\delta|$ is sufficiently small, then $H(y_0, y; U_\delta)$ is C^2 in X near $(y_0, y) = (0,0) \in \mathbb{R}\times\mathbb{R}^3$.*

The proof of this corollary is omitted. We are now in position to prove (1.5) in Theorem 2.

Proof of (1.5) in Theorem 2. Let $m \in (2,\infty)$ and $X = X_m$. We first assume that $\Omega_0 \in X$ and $\|\Omega_0\|_X \ll 1$ with $\int_{\mathbb{R}^2} \Omega_0^{(1)}(x)dx \neq 0$. Then, since $\tilde{\nu}_0 = \min\{1, \frac{m-1}{2}\} > \frac{1}{2} = \mu_1 = \mu_2$ by Corollary 1 and Proposition 2, we can apply Theorem 7 to (1.1) and obtain

$$\|R_{1+t}\Omega(t) - \tilde{S}(\frac{y_1^*}{(1+t)^{\frac{1}{2}}}, \frac{y_2^*}{(1+t)^{\frac{1}{2}}}; U_\delta)\|_{X_m} \leq C_\epsilon (1+t)^{-\min\{1,\frac{m-1}{2}\}+\tilde{\eta}(\delta)+\epsilon},$$

for $t \gg 1$. From the definitions of R_λ and $\tilde{S}$ we have for $1 \leq p \leq 2$,

$$\begin{aligned} &(1+t)^{1-\frac{1}{p}} \|\Omega^{(1)}(t) - \frac{1}{1+t} U_\delta^{(1)}(\frac{\cdot + y^*}{(1+t)^{\frac{1}{2}}})\|_{L^p} \\ \leq\ & C\|R_{1+t}\Omega(t) - \tilde{S}(\frac{y_1^*}{(1+t)^{\frac{1}{2}}}, \frac{y_2^*}{(1+t)^{\frac{1}{2}}}; U_\delta)\|_{X_m}. \end{aligned}$$

Combining these above, we get the desired estimate when $\Omega_0 \in X_m = L^2_m \times H^2_{m-2}$. If $\Omega_0 \in L^2_m \times H^1_{m-2}$ and $\|\Omega_0\|_{L^2_m\times H^1_{m-2}} \ll 1$, then we can show the existence of solution $\Omega \in C([0,\infty); L^2_m \times H^1_{m-2})$ to (1.1) such that $\Omega \in C((0,\infty); L^2_m \times H^2_{m-2})$ with $\|\Omega|_{t=1}\|_{L^2_m\times H^2_{m-2}} \ll 1$; see Appendix below. So the problem is reduced to the case $\Omega_0 \in X_m = L^2_m \times H^2_{m-2}$. This completes the proof of (1.5).

4. Behavior of the eigenvalue -1 to the linearized operator

In this section we will show Theorem 3 which leads to (1.6) in Theorem 2 by applying Theorem 6 and Remark 3 to (1.1). Let A be the generator of the semigroup $e^{(1-e^{-t})\mathcal{A}} R_{e^t}$. As is seen in the previous section, for (1.1) this is given by

$$A = \mathcal{A} + B = \begin{pmatrix} \mathcal{L} & 0 \\ I & \mathcal{L} - I \end{pmatrix},$$

where $\mathcal{L} = \Delta + \frac{x}{2} \cdot \nabla + I$. The Fréchet derivative of $\mathcal{N}$ at ϕ is given by

$$\mathcal{N}'(\phi)f = \begin{pmatrix} \nabla \cdot (\phi^{(1)} \nabla f^{(2)}) + \nabla \cdot (f^{(1)} \nabla \phi^{(2)}) \\ 0 \end{pmatrix}.$$

We first consider the case $2 < m \le 3$. By Remark 4 and Corollary 1 the spectrum of $L_\delta = A - \mathcal{N}'(U_\delta)$ in X_m, $2 < m \le 3$, satisfies

$$\sigma(L_\delta) \subset \{0, -\frac{1}{2}, -1\} \cup \{\mu \in \mathbb{C} \mid \mathrm{Re}(\mu) \le -\min\{1, \frac{m-1}{2}\} + \tilde{\eta}(\delta)\}, \tag{4.1}$$

if $|\delta| \ll 1$, where $\tilde{\eta}(\delta)$ is a number satisfying $\lim_{\delta \to 0} \tilde{\eta}(\delta) = 0$. From [6, Lemma 6.2] we see: 0 is a simple eigenvalue whose eigenfunction is $\partial_\delta U_\delta$; $-\frac{1}{2}$ is a semisimple eigenvalue with multiplicity 2 whose eigenfunctions are $\partial_j U_\delta$, $j = 1, 2$; -1 is an eigenvalue which has the eigenfunction BU_δ. Especially, Theorem 3 holds for $2 < m \le 3$.

In order to prove Theorem 3 for $m > 3$ we note that, by Remark 3, the set $\{\mu \in \sigma(L_\delta) \mid \mathrm{Re}(\mu) > -\frac{m-1}{2} + \eta'(\delta)\}$ consists of isolated eigenvalues with finite multiplicity. Here $\eta'(\delta)$ is a number satisfying $\lim_{\delta \to 0} \eta'(\delta) = 0$. Since $\sigma(A) = \{-\frac{k}{2} \mid k = 0, 1, 2, \cdots\} \cup \{\mu \in \mathbb{C} \mid \mathrm{Re}(\mu) \le -\frac{m-1}{2}\}$ by Proposition 3, from the general perturbation theory the eigenvalues in $\{\mu \in \sigma(L_\delta) \mid \mathrm{Re}(\mu) > -\frac{m-1}{2} + \eta'(\delta)\}$ must be near $\{-\frac{k}{2} \mid k \in \mathbb{N} \cup \{0\},\ k < m - 1\}$ when $|\delta| \ll 1$. Hence if $m > 3$ and $|\delta| \ll 1$ then we have instead of (4.1),

$$\begin{aligned} \sigma(L_\delta) \subset \{0, -\frac{1}{2}, -1\} &\cup \{\mu \in \mathbb{C} \mid \mathrm{Re}(\mu) \le -\min\{\frac{3}{2}, \frac{m-1}{2}\} + \eta'(\delta)\} \\ &\cup \{\text{eigenvalues of } L_\delta \text{ near } -1\}. \end{aligned} \tag{4.2}$$

So when $m > 3$ the main task is to study the behavior of the eigenvalues near -1 to the linearized operator L_δ. From the general perturbation theory, at least for sufficiently small $|\delta|$, the rank of the eigenprojection to all eigenvalues near -1 is 4 since the one to the eigenvalue -1 of A is 4 by Proposition 3. The following proposition reduces the eigenvalue problem of L_δ in X_m to the one in $L^2_\infty \times L^2_\infty$, in which some calculations become simpler.

Proposition 6. *Let $m > 2$. Assume that $\mu \in \mathbb{C}$ satisfies $\mathrm{Re}(\mu) > -\frac{m-1}{2}$. Let $f \in H^2_m \times H^4_{m-2}$ be a solution to*

$$(A - \mathcal{N}'(U_\delta))f = \mu f. \tag{4.3}$$

Then $f \in H^2_\infty \times H^4_\infty$. Especially, μ is an eigenvalue of $L_\delta = A - \mathcal{N}'(U_\delta)$ in $L^2_\infty \times L^2_\infty$.

The proof of Proposition 6 is almost the same as in [9, Proposition 1.1], although real-valued functions are considered there. For convenience to the reader we give the proof of Proposition 6 in Appendix.

4.1. **Spectral analysis in $L^2_\infty \times L^2_\infty$.** In this section functions are assumed to belong to $L^2_\infty \times L^2_\infty$. We first prove

Proposition 7. *Let $f \in H^2_\infty \times H^2_\infty$. Then $\mathcal{N}'(f)$ is relatively bounded with respect to A in $L^2_\infty \times L^2_\infty$. More precisely, we have*

$$\|\mathcal{N}'(f)u\|_{L^2_\infty \times L^2_\infty} \le C\|f\|_{H^2_\infty \times H^2_\infty}\|u\|_{\mathrm{Dom}(A)}, \tag{4.4}$$

for all $u \in \mathrm{Dom}(A) = \{u \in L^2_\infty \times L^2_\infty \mid \mathcal{L}u^{(1)},\ \mathcal{L}u^{(2)} \in L^2_\infty\}$ equipped with the graph norm.

Proof. From the definition of $\mathcal{N}$ and $\mathrm{Dom}(\mathcal{L}) \hookrightarrow H^2_\infty$ in L^2_∞, it suffices to show

$$\|\nabla \cdot (f^{(1)}\nabla u^{(2)})\|_{L^2_\infty} \le C\|f^{(1)}\|_{H^2_\infty}\|u^{(2)}\|_{H^2_\infty}. \tag{4.5}$$

Then we have from the Hölder inequality and the Sobolev embedding theorem,

$$\begin{aligned}
\|\nabla \cdot (f^{(1)}\nabla u^{(2)})\|_{L^2_\infty} &\le \|G^{-\frac{1}{2}}\nabla f^{(1)} \cdot \nabla u^{(2)}\|_{L^2} + \|G^{-\frac{1}{2}} f^{(1)}\Delta u^{(2)}\|_{L^2} \\
&\le \|\nabla f^{(1)}\|_{L^4}\|G^{-\frac{1}{2}}\nabla u^{(2)}\|_{L^4} + \|f^{(1)}\|_{L^\infty}\|G^{-\frac{1}{2}}\Delta u^{(2)}\|_{L^2} \\
&\le C\|f^{(1)}\|_{H^2}\|G^{-\frac{1}{2}}\nabla u^{(2)}\|_{L^4} + C\|f^{(1)}\|_{H^2}\|u^{(2)}\|_{H^2_\infty}.
\end{aligned}$$

Hence it suffices to show $\|G^{-\frac{1}{2}}\nabla u^{(2)}\|_{L^4} \le C\|u^{(2)}\|_{H^2_\infty}$. To see this we note that $\|x\phi\|_{L^2_\infty} \le C\|\phi\|_{H^1_\infty}$, which is verified from the equality

$$\|\nabla\phi\|^2_{L^2_\infty} = \int_{\mathbb{R}^2} |\nabla(G^{-\frac{1}{2}}\phi)|^2 dx + \|x\phi\|^2_{L^2_\infty} + \frac{1}{2}\|\phi\|^2_{L^2_\infty}. \tag{4.6}$$

If we set $\tilde{\phi} = G^{-\frac{1}{2}}\partial_j u^{(2)}$, then we have from the Sobolev embedding theorem,

$$\begin{aligned}
\|\tilde{\phi}\|_{L^4} \le C\|\tilde{\phi}\|_{H^1} &\le C\|\nabla(G^{-\frac{1}{2}}\partial_j u^{(2)})\|_{L^2} + C\|u^{(2)}\|_{H^1_\infty} \\
&\le C\|x\partial_j u^{(2)}\|^2_{L^2_\infty} + C\|u^{(2)}\|^2_{H^2_\infty} \\
&\le C\|u^{(2)}\|^2_{H^2_\infty}.
\end{aligned}$$

This completes the proof.

As a corollary of Proposition 7, we have

$$\|\mathcal{N}'(U_\delta)f\|_{L^2_\infty \times L^2_\infty} \le C\|U_\delta\|_{H^2_\infty \times H^2_\infty}\|f\|_{\mathrm{Dom}(A)} \le C|\delta|\|f\|_{\mathrm{Dom}(A)}, \tag{4.7}$$

from Proposition 5. Since $|\delta|$ is sufficiently small, we see $A - \mathcal{N}'(U_\delta)$ is realized as a closed operator with the domain $\mathrm{Dom}(A - \mathcal{N}'(U_\delta)) = \mathrm{Dom}(A)$ in $L^2_\infty \times L^2_\infty$; see [8, Theorem IV-1-1].

In order to study the eigenvalue problem of L_δ in $L^2_\infty \times L^2_\infty$ we prepare several invariant subspaces as follows. Set

$$G_{j_1,j_2} = \alpha_{j_1,j_2}\partial_1^{j_1}\partial_2^{j_2}G, \qquad \alpha_{j_1,j_2} = \|\partial_1^{j_1}\partial_2^{j_2}G\|^{-1}_{L^2_\infty}. \tag{4.8}$$

Then it is well-known that $\{G_{j_1,j_2}\}_{j_1,j_2\in\mathbb{N}\cup\{0\}}$ forms a complete orthonormal basis of L^2_∞. Moreover, each G_{j_1,j_2} satisfies

$$\mathcal{L}G_{j_1,j_2} = -\frac{j_1+j_2}{2}G_{j_1,j_2}. \tag{4.9}$$

Then we set

$$\begin{aligned} L^2_{\infty,1} &= \text{a closed subspace of } L^2_\infty \text{ spanned by } \{G_{2k,2l}\}_{k,l\in\mathbb{N}\cup\{0\}} \\ L^2_{\infty,2} &= \text{a closed subspace of } L^2_\infty \text{ spanned by } \{G_{2k+1,2l+1}\}_{k,l\in\mathbb{N}\cup\{0\}} \\ Y_1 &= L^2_{\infty,1}\times L^2_{\infty,1} \\ Y_2 &= L^2_{\infty,2}\times L^2_{\infty,2}. \end{aligned}$$

It is clear that Y_1 and Y_2 are orthogonal to each other. Let O be the orthogonal matrix defined by

$$O = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Then we also set

$$Y_3 = \{f \in Y_1 \mid f(O\cdot) \in Y_2\}$$

Proposition 8. *Each Y_i, $i = 1,2,3$, is invariant under the action of L_δ. More precisely, if $f \in \mathrm{Dom}(L_\delta)\cap Y_i$ then $L_\delta f \in Y_i$.*

Proof. The assertion follows from the definitions of A and $\mathcal{N}'(U_\delta)$, and the facts that U_δ is radially symmetric and O is an orthogonal matrix. We omit the details here.

The next proposition is essential to obtain Theorem 3.

Proposition 9. *The eigenvalues of L_δ in Y_1 near -1 consist of three simple eigenvalues $\{-1, \lambda_1(\delta), \lambda_2(\delta)\}$, where*

$$\lambda_1(\delta) = -1 + \frac{1}{16\pi}\delta + o(\delta), \qquad \lambda_2(\delta) = -1 - \frac{1}{2^8 3\pi^2}\delta^2 + o(\delta^2), \tag{4.10}$$

at $|\delta| \ll 1$. Moreover, $\lambda_2(\delta)$ is the eigenvalue of L_δ also in Y_3.

The proof is given in Section 4.1.1 and Section 4.1.2 below based on the reduction process in [8, Section II-2-3].

4.1.1. *Proof of Proposition 9: First order reduction process.* We first note that, from the definition of $L^2_{\infty,1}$, the eigenspace of the eigenvalue -1 of A in $Y_1(= L^2_{\infty,1}\times L^2_{\infty,1})$ is spanned by

$$\mathbf{e}_1 = \begin{pmatrix} \Delta G \\ \Delta G \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} (\partial_1^2-\partial_2^2)G \\ (\partial_1^2-\partial_2^2)G \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ G \end{pmatrix}.$$

Then the associated eigenprojection is

$$\mathbf{P}f = \sum_{i=1}^{3} < f, \mathbf{e}_i^* >_X \mathbf{e}_i, \qquad X = L^2_\infty \times L^2_\infty, \tag{4.11}$$

where

$$\mathbf{e}_1^* = \begin{pmatrix} \Delta G \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2^* = \begin{pmatrix} (\partial_1^2 - \partial_2^2)G \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3^* = \begin{pmatrix} -G \\ G \end{pmatrix}.$$

Note that $< \mathbf{e}_i, \mathbf{e}_j^* >_X = \delta_{ij}$ holds. Let L_δ be the linear operator defined by $L_\delta = A - \mathcal{N}'(U_\delta)$ with the domain $\mathrm{Dom}(L_\delta) = \mathrm{Dom}(A) = \{f \in Y_1 \mid \mathcal{L}f^{(1)} \in L^2_{\infty,1},\ \mathcal{L}f^{(2)} \in L^2_{\infty,1}\}$. We consider the eigenvalue problem

$$L_\delta u = \lambda(\delta) u, \qquad u \in \mathrm{Dom}(L_\delta). \tag{4.12}$$

Since U_δ is C^2 with respect to δ in $H^2_\infty \times H^2_\infty$ by Proposition 5, we see from (4.4) that $\mathcal{N}'(U_\delta)$ is continuously depending on δ as a bounded operator from $\mathrm{Dom}(A)$ to Y_1. Hence the eigenvalue $\lambda(\delta)$ is continuous with respect to δ.

By Proposition 5 $\mathcal{N}'(U_\delta)$ is decomposed as

$$\begin{aligned} \mathcal{N}'(U_\delta)u &= \delta \mathcal{N}'(w_0)u + \delta^2 \mathcal{N}'(\varphi)u + \delta^3 \mathcal{N}'(z_\delta)u \\ &=: \delta B_1 u + \delta^2 B_2 u + \delta B_3(\delta) u \\ &=: \delta B(\delta) u \end{aligned}$$

Note that B_1 and B_2 are independent of δ. Let $\lambda \in \rho(A)$ and set $R(A,\lambda) = (A - \lambda I)^{-1}$. We have from (4.4) that

$$\begin{aligned} \|B(\delta)R(A,\lambda)\|_{\mathcal{B}(Y_1)} &= \|\delta^{-1}\mathcal{N}'(U_\delta)R(A,\lambda)\|_{\mathcal{B}(Y_1)} \\ &\le C|\delta|^{-1}\|U_\delta\|_{H^2_\infty \times H^2_\infty} \le C. \end{aligned}$$

Here the constant C is uniform in $|\delta| \ll 1$. Then the resolvent $R(L_\delta, \lambda) = (L_\delta - \lambda I)^{-1}$ is expanded as

$$\begin{aligned} R(L_\delta,\lambda) &= R(A,\lambda)(I - \delta B(\delta)R(A,\lambda))^{-1} \\ &= R(A,\lambda)\sum_{k=0}^{\infty} \delta^k (B(\delta)R(A,\lambda))^k \\ &= R(A,\lambda) + \delta R(A,\lambda)B_1 R(A,\lambda) \\ &\quad + \delta^2\{R(a,\lambda)B_2 R(A,\lambda) + R(A,\lambda)(B_1 R(A,\lambda))^2\} + \delta^2 O_{1,\delta}. \end{aligned}$$

Here $O_{1,\delta} \in \mathcal{B}(Y_1)$ satisfies $\|O_{1,\delta}\|_{\mathcal{B}(Y_1)} \le C|\delta|$. This expansion leads to the expansion of the projection

$$\mathbf{P}(\delta) = -\frac{1}{2\pi i}\int_{\Gamma_{-1}} R(L_\delta,\lambda)\,d\lambda, \tag{4.13}$$

where Γ_{-1} is a sufficiently small circle around -1 oriented counter clockwise. Since -1 is a semisimple eigenvalue of A, we have $\mathbf{P}(0) = \mathbf{P}$ where $\mathbf{P}$ is given by (4.11), and $\int_{\Gamma_{-1}}(\lambda+1)R(A,\lambda)d\lambda = 0$ holds. Then

$$\begin{aligned}(L_\delta + I)\mathbf{P}(\delta) &= -\frac{1}{2\pi i}\int_{\Gamma_{-1}}(\lambda+1)R(L_\delta,\lambda)d\lambda \\ &= -\frac{1}{2\pi i}\int_{\Gamma_{-1}}(\lambda+1)\big[R(A,\lambda)+\delta R(A,\lambda)B_1R(A,\lambda) \\ &\quad +\delta^2\{R(A,\lambda)B_2R(A,\lambda)+R(A,\lambda)(B_1R(A,\lambda))^2+O_{1,\delta}\}\big]d\lambda\end{aligned}$$

Recall that $R(A,\lambda)$ has a Laurant expansion around -1:

$$R(A,\lambda) = -\frac{\mathbf{P}}{\lambda+1}+\sum_{k=0}^{\infty}(\lambda+1)^k\mathbf{S}^{k+1} \tag{4.14}$$

where $\mathbf{S}$ is the reduced resolvent

$$\mathbf{S} = \lim_{\lambda\to-1}(I-\mathbf{P})R(A,\lambda). \tag{4.15}$$

Then we have from the Cauchy integral theorem,

$$(L_\delta+I)\mathbf{P}(\delta) = \delta A_1(\delta),$$

where
(4.16)
$A_1(\delta) = -\mathbf{P}B_1\mathbf{P}-\delta\{\mathbf{P}B_2\mathbf{P}+\mathbf{S}B_1\mathbf{P}B_1\mathbf{P}+\mathbf{P}B_1\mathbf{S}B_1\mathbf{P}+\mathbf{P}B_1\mathbf{P}B_1\mathbf{S}\}+\delta O_{2,\delta}$,

with $\|O_{2,\delta}\|_{\mathcal{B}(Y_1)} \le C|\delta|$. Hence we have

$$\begin{aligned}0 &= (L_\delta-\lambda(\delta))u = (L_\delta+I)\mathbf{P}(\delta)u-(1+\lambda(\delta))\mathbf{P}(\delta)u \\ &= \delta A_1(\delta)\mathbf{P}(\delta)u-(1+\lambda(\delta))\mathbf{P}(\delta)u,\end{aligned}$$

which gives

$$\mathbf{P}(\delta)A_1(\delta)\mathbf{P}(\delta)u = \frac{1+\lambda(\delta)}{\delta}u =: \mu_1(\delta)u. \tag{4.17}$$

So the behavior of $\lambda(\delta)$ is determined by $\mu_1(\delta)$ which are eigenvalues of $\mathbf{P}(\delta)A_1(\delta)\mathbf{P}(\delta)$. Since $\mathbf{P}(\delta)A_1(\delta)\mathbf{P}(\delta)$ is continuous with respect to δ in $\mathcal{B}(Y_1)$, its eigenvalue $\mu_1(\delta)$ is also continuous. Especially, if $|\delta|\ll 1$, $\mu_1(\delta)$ is near $\mu_1(0)$ which are the eigenvalues of the bounded operator

$$\mathbf{P}(0)A_1(0)\mathbf{P}(0) = -\mathbf{P}B_1\mathbf{P}. \tag{4.18}$$

Let us calculate the eigenvalues of $-\mathbf{P}B_1\mathbf{P}$. Recall that

$$\mathbf{P}u = \sum_{i=1}^{3} < u, \mathbf{e}_i^* >_X \mathbf{e}_i, \qquad X = L^2_\infty\times L^2_\infty,$$

where $\mathbf{e}_i$, $\mathbf{e}_i^*$ appear in (4.11), and

$$B_1u = \mathcal{N}'(w_0)u = \begin{pmatrix} \nabla\cdot(G\nabla u^{(2)})+\nabla\cdot(u^{(1)}\nabla G) \\ 0\end{pmatrix}.$$

Since

$$-\mathbf{P}B_1\mathbf{P}u = -\mathbf{P}B_1 \sum_{i=1}^{3} < u, \mathbf{e}_i^* >_X \mathbf{e}_i = -\sum_{j=1}^{3}\sum_{i=1}^{3} < B_1\mathbf{e}_i, \mathbf{e}_j^* >_X < u, \mathbf{e}_i^* >_X \mathbf{e}_j,$$

the matrix representation $T = (t_{ij})_{1\le i,j\le 3}$ of $-\mathbf{P}B_1\mathbf{P}$ is

$$t_{ij} = - < B_1\mathbf{e}_j, \mathbf{e}_i^* >_X . \tag{4.19}$$

Then the direct calculations by using the integration by parts yield that

$$t_{11} = \int_{\mathbb{R}^2} |\nabla G(x)|^2 dx = \frac{1}{16\pi}, \tag{4.20}$$

$$t_{13} = -\frac{1}{2}\int_{\mathbb{R}^2} |G(x)|^2 dx = -\frac{1}{16\pi}, \tag{4.21}$$

and $t_{ij} = 0$, otherwise. Hence the eigenvalues of T are $t_{11} = \frac{c_1^2}{16\pi} = \frac{1}{16\pi}$ and 0, whose eigenspaces are respectively given by

$$\{(a_1, 0, 0)^\top \mid a_1 \in \mathbb{C}\}, \quad \{(0, a_1, 0)^\top + (a_2, 0, a_2)^\top \mid a_i \in \mathbb{C}\}.$$

So each of the eigenvalues $\frac{1}{16\pi}$ and 0 to T is semisimple. Equivalently, the eigenvalues of $-\mathbf{P}B_1\mathbf{P}$ are $t_{11} > 0$ and 0 which are semisimple, and the associated eigenspaces are

$$\{a_1\mathbf{e}_1 \mid a_1 \in \mathbb{C}\}, \tag{4.22}$$

$$\{a_1\mathbf{e}_2 + a_2(\mathbf{e}_1 + \mathbf{e}_3) \mid a_i \in \mathbb{C}\}, \tag{4.23}$$

respectively. Moreover, the eigenprojection $\mathbf{P}_1$ of the eigenvalue 0 for $-\mathbf{P}B_1\mathbf{P}$ is given by

$$\mathbf{P}_1 u = < u, \mathbf{e}_2^* >_X \mathbf{e}_2 + < u, \mathbf{e}_3^* >_X (\mathbf{e}_1 + \mathbf{e}_3). \tag{4.24}$$

From the above arguments and the fact that -1 is an eigenvalue of L_δ in Y_1 with the eigenfunction BU_δ since U_δ is radially symmetric, the eigenvalues of L_δ in Y_1 near -1 consist of

$$-1, \qquad \lambda_1(\delta) = -1 + \frac{1}{16\pi}\delta + o(|\delta|), \qquad \lambda_2(\delta) = -1 + o(|\delta|). \tag{4.25}$$

It is now important to study the behavior of $\lambda_2(\delta) = -1 + o(|\delta|)$, or equivalently, the behavior of the eigenvalues $\mu_1(\delta)$ of $\mathbf{P}(\delta)A_1(\delta)\mathbf{P}_1(\delta)$ in (4.17), which is of the form $\mu_1(\delta) = o(|\delta|)$.

4.1.2. *Proof of Proposition 9: Second order reduction process.* As in the previous section, by the arguments of reduction process we consider the behavior of the eigenvalues near 0 for the operator

$$L_1(\delta) := \mathbf{P}(\delta)A_1(\delta)\mathbf{P}(\delta). \tag{4.26}$$

Here $A_1(\delta)$ is given by (4.16) and $\mathbf{P}(\delta)$ is the eigenprojection of L_δ defined by (4.13). Set

$$A_1 = A_1(0) = -\mathbf{P}B_1\mathbf{P}. \tag{4.27}$$

Then from $\mathbf{PS} = \mathbf{SP} = 0$ and $\mathbf{P}A_1\mathbf{P} = A_1$ it is not difficult to see that $L_1(\delta)$ is expressed as

$$L_1(\delta) = A_1 - \delta D(\delta) \tag{4.28}$$

where

$$D(\delta) = \mathbf{P}B_2\mathbf{P} + \mathbf{P}B_1\mathbf{S}B_1\mathbf{P} - \mathbf{S}B_1A_1 - A_1B_1\mathbf{S} + O_{3,\delta}$$

with $\|O_{3,\delta}\|_{\mathcal{B}(Y_1)} \le C|\delta|$. Hence we have

$$D(0) = \mathbf{P}B_2\mathbf{P} + \mathbf{P}B_1\mathbf{S}B_1\mathbf{P} - \mathbf{S}B_1A_1 - A_1B_1\mathbf{S}. \tag{4.29}$$

We set

$$\mathbf{P}_1(\delta) = -\frac{1}{2\pi i}\int_{\Gamma_0} R(L_1(\delta),\lambda)d\lambda, \tag{4.30}$$

where Γ_0 is a sufficiently small circle around 0 oriented counter clockwise.

Then as in the previous section, we have from (4.28) the expansion of the resolvent

$$R(L_1(\delta),\lambda) = R(A_1,\lambda) + \delta R(A_1,\lambda)D(0)R(A_1,\lambda) + \delta O_{4,\delta}$$

with $\|O_{4,\delta}\|_{\mathcal{B}(Y_1)} \le C|\delta|$. This gives

$$L_1(\delta)\mathbf{P}_1(\delta) = \delta A_2(\delta),$$

where

$$A_2(\delta) = -\mathbf{P}_1(0)D(0)\mathbf{P}_1(0) + O_{5,\delta}$$

with $\|O_{5,\delta}\|_{\mathcal{B}(Y_1)} \le C|\delta|$. Thus we see

$$\begin{aligned} 0 &= L_1(\delta)u - \lambda_1(\delta)u = L_1(\delta)\mathbf{P}_1(\delta)u - \lambda_1(\delta)u \\ &= \delta A_2(\delta)u - \lambda_1(\delta)u, \end{aligned}$$

which yields

$$\mathbf{P}_1(\delta)A_2(\delta)\mathbf{P}_1(\delta)u = \frac{\lambda_1(\delta)}{\delta}u =: \mu_2(\delta)u. \tag{4.31}$$

Since $A_2(\delta)$ is continuous with respect to δ in $\mathcal{B}(Y_1)$, so is $\mu_2(\delta)$. Especially, $\mu_2(\delta)$ is near $\mu_2(0)$, which are the eigenvalues of the operator

$$\begin{aligned} \mathbf{P}_1(0)A_2(0)\mathbf{P}_1(0) = -\mathbf{P}_1D(0)\mathbf{P}_1 &= -\mathbf{P}_1(\mathbf{P}B_2\mathbf{P} + \mathbf{P}B_1\mathbf{S}B_1\mathbf{P})\mathbf{P}_1 \\ &= -\mathbf{P}_1B_2\mathbf{P}_1 - \mathbf{P}_1B_1\mathbf{S}B_1\mathbf{P}_1. \end{aligned} \tag{4.32}$$

Here we used (4.29), $\mathbf{P}_1\mathbf{P} = \mathbf{P}\mathbf{P}_1 = \mathbf{P}_1$, and $\mathbf{P}_1\mathbf{S} = \mathbf{S}\mathbf{P}_1 = 0$, which are verified from (4.11) and (4.24). For simplicity of notations, we set

$$\epsilon_1 = \mathbf{e}_2, \qquad \epsilon_1^* = \mathbf{e}_2^*, \qquad \epsilon_2 = \mathbf{e}_1 + \mathbf{e}_3, \qquad \epsilon_2^* = \mathbf{e}_3^*. \tag{4.33}$$

Then from (4.24), $\mathbf{P}_1$ is written as

$$\mathbf{P}_1u = \sum_{i=1}^{2} < u, \epsilon_i^* >_X \epsilon_i. \tag{4.34}$$

Let $M = (m_{ij})_{1\le i,j\le 2}$, $N = (n_{ij})_{1\le i,j\le 2}$ be the representation matrices of $-\mathbf{P}_1B_2\mathbf{P}_1$, $-\mathbf{P}_1B_1\mathbf{S}B_1\mathbf{P}_1$, respectively.

(i) Calculations of M

It is easy to see that $m_{ij} = - < B_2\epsilon_j, \epsilon_i^* >_X$. Recall that

$$B_2 u = \mathcal{N}'(\varphi)u = \begin{pmatrix} \nabla\cdot(\varphi^{(1)}\nabla u^{(2)}) + \nabla\cdot(u^{(1)}\nabla\varphi^{(2)}) \\ 0 \end{pmatrix},$$

where $\varphi = (\varphi^{(1)}, \varphi^{(2)})^\top$ is the solution to

$$\begin{cases} -\mathcal{L}\varphi^{(1)} + \nabla(G\cdot\nabla G) = 0, & x\in\mathbb{R}^2, \\ -(\mathcal{L}-I)\varphi^{(2)} - \varphi^{(1)} = 0, & x\in\mathbb{R}^2, \end{cases}$$

satisfying $\int_{\mathbb{R}^2}\varphi^{(1)}(x)dx = 0$. We note that φ is radially symmetric. Hence if u is radially symmetric, so is B_2u by the definition. Especially, $B_2\epsilon_2$ is radially symmetric. Then $m_{12} = 0$ since $\epsilon_1 = \mathbf{e}_2 \in Y_3$ is orthogonal to radially symmetric functions. Moreover, we have

$$\begin{aligned} m_{22} &= -\int_{\mathbb{R}^2}\{\nabla\cdot(\varphi^{(1)}\nabla\Delta G) + \nabla\cdot\big((\Delta G + G)\nabla\varphi^{(2)}\big)\}(-G)\frac{dx}{G} \\ &= 0. \end{aligned}$$

Hence it suffices to calculate m_{11}. Set

$$H_2 = (\partial_1^2 - \partial_2^2)G.$$

We observe that

$$\begin{aligned} m_{11} &= - < B_2\mathbf{e}_2, \mathbf{e}_2^* >_X \\ &= -\{< \nabla\cdot(\varphi^{(1)}\nabla H_2), H_2 >_{L^2_\infty} + < \nabla\cdot(H_2\nabla\varphi^{(2)}), H_2 >_{L^2_\infty}\} \\ &= -I_1^{(11)} - I_2^{(11)}, \end{aligned}$$

By the integration by parts we have

$$\begin{aligned} I_2^{(11)} &= - < H_2^2, \frac{x}{2}\cdot\nabla\varphi^{(2)} >_{L^2_\infty} - < H_2, G^{-1}\nabla\varphi^{(2)}\cdot\nabla H_2 >_{L^2} \\ &= - < H_2^2, \frac{x}{2}\cdot\nabla\varphi^{(2)} >_{L^2_\infty} + \frac{1}{2} < H_2^2, \nabla\cdot(G^{-1}\nabla\varphi^{(2)}) >_{L^2} \\ &= - < H_2^2, \frac{x}{2}\cdot\nabla\varphi^{(2)} >_{L^2_\infty} - \frac{1}{2} < H_2^2, \varphi^{(1)} >_{L^2_\infty} . \end{aligned}$$

In the last line we used the relation $(\Delta + \frac{x}{2}\cdot\nabla)\varphi^{(2)} = -\varphi^{(1)}$.

If a given $\phi \in L^2_\infty$ is radially symmetric, we can check that

$$< H_2^2, \phi >_{L^2_\infty} = \frac{1}{32} < \phi, |x|^4G^2 >_{L^2_\infty} . \tag{4.35}$$

Hence we have

$$\begin{aligned} & I_2^{(11)} \\ = {} & -\frac{1}{32} < \frac{x}{2}\cdot\nabla\varphi^{(2)}, |x|^4G^2 >_{L^2_\infty} - \frac{1}{64} < \varphi^{(1)}, |x|^4G^2 >_{L^2_\infty} \\ = {} & \frac{1}{32}\{-\frac{1}{4} < \varphi^{(2)}, |x|^6G^2 >_{L^2_\infty} + 3 < \varphi^{(2)}, |x|^4G^2 >_{L^2_\infty}\} - \frac{1}{64} < \varphi^{(1)}, |x|^4G^2 >_{L^2_\infty} . \end{aligned}$$

The direct calculations show that

$$\begin{aligned}|x|^2G^2 &= 2\mathcal{L}G^2 + 2G^2,\\ |x|^4G^2 &= 2\mathcal{L}(|x|^2G^2) + 16\mathcal{L}G^2 + 8G^2,\\ |x|^6G^2 &= 2\mathcal{L}(|x|^4G^2) + 28\mathcal{L}(|x|^2G^2) + 160\mathcal{L}G^2 + 48G^2.\end{aligned}$$

Then from the facts that $\mathcal{L}\varphi^{(2)} = \varphi^{(2)} - \varphi^{(1)}$ and $\mathcal{L}$ is self-adjoint in L^2_∞, we have

$$\begin{aligned}< \varphi^{(2)}, |x|^2G^2 >_{L^2_\infty} &= < \varphi^{(2)}, 2\mathcal{L}G^2 + 2G^2 >_{L^2_\infty}\\ &= 2 < \mathcal{L}\varphi^{(2)}, G^2 >_{L^2_\infty} + 2 < \varphi^{(2)}, G^2 >_{L^2_\infty}\\ &= 4 < \varphi^{(2)}, G^2 >_{L^2_\infty} - 2 < \varphi^{(1)}, G^2 >_{L^2_\infty} .\end{aligned}$$

Similarly, we have

$$< \varphi^{(2)}, |x|^4G^2 >_{L^2_\infty} = 32 < \varphi^{(2)}, G^2 >_{L^2_\infty} - 2 < \varphi^{(1)}, |x|^2G^2 >_{L^2_\infty} - 20 < \varphi^{(1)}, G^2 >_{L^2_\infty},$$

$$\begin{aligned}< \varphi^{(2)}, |x|^6G^2 >_{L^2_\infty} = {}& 384 < \varphi^{(2)}, G^2 >_{L^2_\infty} - 2 < \varphi^{(1)}, |x|^4G^2 >_{L^2_\infty}\\ & - 32 < \varphi^{(1)}, |x|^2G^2 >_{L^2_\infty} - 256 < \varphi^{(2)}, G^2 >_{L^2_\infty} .\end{aligned}$$

Thus we get

$$\begin{aligned}& -\frac{1}{4} < \varphi^{(2)}, |x|^6G^2 >_{L^2_\infty} + 3 < \varphi^{(2)}, |x|^4G^2 >_{L^2_\infty}\\ = {}& \frac{1}{2} < \varphi^{(1)}, |x|^4G^2 >_{L^2_\infty} + 2 < \varphi^{(1)}, |x|^2G^2 >_{L^2_\infty} + 4 < \varphi^{(1)}, G^2 >_{L^2_\infty},\end{aligned}$$

which gives

$$I_2^{(11)} = \frac{1}{16} < \varphi^{(1)}, |x|^2G^2 >_{L^2_\infty} + \frac{1}{8} < \varphi^{(1)}, G^2 >_{L^2_\infty} . \tag{4.36}$$

On the other hand, we see

$$I_1^{(11)} = < \nabla \cdot (\varphi^{(1)}\nabla H_2), H_2 >_{L^2_\infty} = -\frac{1}{4} < \varphi^{(1)}, |x|^2G^2 >_{L^2_\infty} + \frac{1}{32} < \varphi^{(1)}, |x|^4G^2 >_{L^2_\infty} .$$

Hence we have

$$\begin{aligned}m_{11} &= -I_1^{(11)} - I_2^{(11)}\\ &= -\frac{1}{32} < \varphi^{(1)}, |x|^4G^2 >_{L^2_\infty} + \frac{3}{16} < \varphi^{(1)}, |x|^2G^2 >_{L^2_\infty} - \frac{1}{8} < \varphi^{(1)}, G^2 >_{L^2_\infty} .\end{aligned}$$

Now by direct calculations we can check that

$$\varphi^{(1)} = G^2 - \frac{1}{8\pi}G \tag{4.37}$$

satisfies $-\mathcal{L}\varphi^{(1)} + \nabla \cdot (G\nabla G) = 0$ and $\int_{\mathbb{R}^2} \varphi^{(1)} dx = 0$.

Thus

$$< \varphi^{(1)}, |x|^4G^2 >_{L^2_\infty} = \left(\frac{2}{3^3} - \frac{1}{2^3}\right)\frac{1}{\pi^2}, \qquad < \varphi^{(1)}, |x|^2G^2 >_{L^2_\infty} = -\frac{1}{2^5 3^2 \pi^2},$$

$$< \varphi^{(1)}, G^2 >_{L^2_\infty} = \frac{1}{2^6 3 \pi^2}.$$

Then we obtain

$$m_{11} = \frac{1}{2^7 3^3 \pi^2}. \tag{4.38}$$

(2) Calculations of N

Since $-\mathbf{P}_1 B_1 \mathbf{S} B_1 \mathbf{P}_1 u = -\sum_{1\le i,j\le 2} < u, \epsilon_i^* >_X < B_1 \mathbf{S} B_1 \epsilon_i, \epsilon_j >_X \epsilon_j$, we see

$$n_{ij} = - < B_1 \mathbf{S} B_1 \epsilon_j, \epsilon_i^* >_X . \tag{4.39}$$

Let us recall that

$$B_1 u = \mathcal{N}'(w_0) u = \begin{pmatrix} \nabla \cdot (G \nabla u^{(2)}) + \nabla \cdot (u^{(1)} \nabla G) \\ 0 \end{pmatrix},$$

and $\mathbf{S}$ is the reduced resolvent of A:

$$\mathbf{S} = \lim_{\lambda \to -1} (I - \mathbf{P}) R(A, \lambda) = \lim_{\lambda \to -1} (I - \mathbf{P})(A - \lambda I)^{-1}.$$

We note that $\epsilon_1 \in Y_3$ and ϵ_2 is radially symmetric, which is preserved under the action of $B_1 \mathbf{S} B_1$. This implies

$$n_{12} = n_{21} = 0, \tag{4.40}$$

since Y_3 is orthogonal to the subspace of all radially symmetric functions in $L^2_\infty \times L^2_\infty$. Moreover, since $< B_1 f, \epsilon_2^* >_X = 0$ for any $f \in H^2_\infty \times H^2_\infty$ by the definition of B_1 and ϵ_2^*, we also have

$$n_{22} = 0. \tag{4.41}$$

Hence it suffices to compute n_{11}. The direct calculations show $\mathbf{P} B_1 \epsilon_1 = 0$. Hence if we set

$$z_1 = \mathbf{S} B_1 \epsilon_1,$$

then z_1 is the solution to

$$\begin{cases} (\mathcal{L} + I) z_1^{(1)} = \{\nabla(G \cdot \nabla H_2) + \nabla(H_2 \cdot \nabla G)\}, & x \in \mathbb{R}^2, \\ \mathcal{L} z_1^{(2)} + z_1^{(1)} = 0, \quad x \in \mathbb{R}^2, \end{cases}$$

and satisfies $\mathbf{P} z_1 = 0$.

It is easy to check

$$\nabla(G \cdot \nabla H_2) + \nabla(H_2 \cdot \nabla G) = \frac{x_1^2 - x_2^2}{4} (|x|^2 - 6) G^2.$$

Set

$$v_1^{(1)} = 2 H_2 G = \frac{x_1^2 - x_2^2}{2} G^2 \in L^2_{\infty,1}, \tag{4.42}$$

and let $v_1^{(2)}$ be the solution to

$$\mathcal{L} v_1^{(2)} + v_1^{(1)} = 0, \qquad \int_{\mathbb{R}^2} v_1^{(2)}(x) dx = 0. \tag{4.43}$$

Then z_1 is given by

$$z_1 = v_1 - \sum_{j=1}^{3} < v_1, \mathbf{e}_j^* >_X \mathbf{e}_j = v_1 - < v_1, \epsilon_1^* >_X \epsilon_1, \tag{4.44}$$

where $v_1 = (v_1^{(1)}, v_1^{(2)})^\top$. Hence we have

$$\begin{aligned} n_{11} &= - < B_1 \mathbf{S} B_1 \epsilon_1, \epsilon_1^* >_X = - < B_1 z_i, \epsilon_1^* >_X \\ &= - < B_1 v_1, \epsilon_1^* >_X + < v_1, \epsilon_1^* >_X < B_1 \epsilon_1, \epsilon_1^* >_X \\ &= - < B_1 v_1, \epsilon_1^* >_X . \end{aligned} \tag{4.45}$$

Here we used the fact that $< B_1 \epsilon_1, \epsilon_1^* >_X = 0$ by the direct calculations. We note that

$$- < B_1 v_1, \epsilon_1^* >_X = - \int_{\mathbb{R}^2} \{ \nabla \cdot (G \nabla v_1^{(2)}) + \nabla \cdot (v_1^{(1)} \nabla G) \} H_2 \frac{dx}{G}, \tag{4.46}$$

and at least the second term can be computed explicitly. Indeed, from (4.42) we have

$$- \int_{\mathbb{R}^2} \nabla \cdot (v_1^{(1)} \nabla G) H_2 \frac{dx}{G} = - \frac{1}{6^3 \pi^2}. \tag{4.47}$$

Hence the problem is to determine the exact value of $\int_{\mathbb{R}^2} \nabla \cdot (G \nabla v_1^{(2)}) H_2 \frac{dx}{G}$. It seems to be difficult to find the exact representation of $v_1^{(2)}$ or to use the argument in the calculations of M. So instead, we use a series expansion here in order to compute the value of the above integral. For this purpose, we first observe that

$$\begin{aligned} - \int_{\mathbb{R}^2} \nabla \cdot (G \nabla v_1^{(2)}) H_2 \frac{dx}{G} = - \int_{\mathbb{R}^2} v_1^{(2)} \mathcal{L} H_2 dx &= \int_{\mathbb{R}^2} v_1^{(2)} H_2 dx \\ &= \frac{1}{2} < v_1^{(2)}, v_1^{(1)} >_X . \end{aligned} \tag{4.48}$$

Here we used $\mathcal{L} H_2 = -H_2$ and (4.42).

Since $v_1^{(1)} \in L^2_{\infty,1}$ it is expanded as

$$v_1^{(1)} = \sum_{k=1}^{\infty} \sum_{l=0}^{k} c_{k,l} G_{2l,2k-2l}, \tag{4.49}$$

where

$$c_{k,l} = < v_1^{(1)}, G_{2l,2k-2l} >_{L^2_\infty} . \tag{4.50}$$

Since $\mathcal{L}$ is self-adjoint in L^2_∞ we have from (4.9),

$$- < v_1^{(2)}, G_{2l,2k-2l} >_{L^2_\infty} = k^{-1} < v_1^{(2)}, \mathcal{L} G_{2l,2k-2l} >_{L^2_\infty} = < \mathcal{L} v_1^{(2)}, G_{2l,2k-2l} >_{L^2_\infty} .$$

Then recalling $\mathcal{L}v_1^{(2)} = -v_1^{(1)}$, we get

$$\begin{aligned}\frac{1}{2} < v_1^{(2)}, v_1^{(1)} >_{L^2_\infty} &= \frac{1}{2}\sum_{k=1}^{\infty}\sum_{l=0}^{k} c_{k,l} < v_1^{(2)}, G_{2l,2k-2l} >_{L^2_\infty} \\ &= \frac{1}{2}\sum_{k=1}^{\infty} k^{-1}\sum_{l=0}^{k} c_{k,l} < v_1^{(1)}, G_{2l,2k-2l} >_{L^2_\infty} \\ &= \frac{1}{2}\sum_{k=1}^{\infty} k^{-1}\sum_{l=0}^{k} c_{k,l}^2. \qquad (4.51)\end{aligned}$$

From the definition of $v_1^{(1)}$ we see

$$\begin{aligned}c_{k,l} &= 2 < (\partial_1^2 - \partial_2^2)G, G_{2l,2k-2l} >_{L^2} \\ &= 2\alpha_{2l,2k-2l}(-1)^{k+1}\big(\|\partial_1^{l+1}\partial_2^{k-l}G\|_{L^2}^2 - \|\partial_1^{l}\partial_2^{k-l+1}G\|_{L^2}^2\big),\end{aligned}$$

where $\alpha_{2l,2k-2l} = \|\partial_1^{2l}\partial_2^{2k-2l}G\|_{L^2\infty}^{-1}$. Now we use

Proposition 10. *For each $k,\ l \in \mathbb{N}\cup\{0\}$ we have*

$$\|\partial_1^k\partial_2^l G\|_{L^2}^2 = \frac{(2k)!(2l)!}{8\pi 8^{k+l}k!l!}, \qquad \|\partial_1^k\partial_2^l G\|_{L^2_\infty}^2 = \frac{k!l!}{2^{k+l}}.$$

The proof will be given in Appendix 5.3. As a corollary of Proposition 10, we have

Corollary 3. *Set ${}_mC_{m-n} = \frac{m!}{n!(m-n)!}$. Then it follows that*

$$c_{k,l}^2 = \frac{1}{64\pi^2}\frac{(k-2l)^2}{16^k}\, {}_{2l}C_l\ {}_{2(k-l)}C_{k-l},$$

Proof. We first observe from Proposition 10 that

$$\begin{aligned}&\|\partial_1^{l+1}\partial_2^{k-l}G\|_{L^2}^2 - \|\partial_1^{l}\partial_2^{k-l+1}G\|_{L^2}^2 \\ =\ & \frac{1}{8\pi 8^{k+1}}\Big\{\frac{(2l+2)!(2k-2l)!}{(l+1)!(k-l)!} - \frac{(2l)!(2k-2l+2)!}{l!(k-l+1)!}\Big\} \\ =\ & \frac{(2l-k)}{16\pi 8^k}\frac{(2l)!(2k-2l)!}{l!(k-l)!}.\end{aligned}$$

Then by applying Proposition 10 again, the assertion follows from the definition $\alpha_{m,n}^2 = \|\partial_1^m\partial_2^n G\|_{L^2_\infty}^{-2}$. This completes the proof of Corollary 3.

From (4.51) and Corollary 3, it remains to compute the value of

$$K_1 = \sum_{k=1}^{\infty}\frac{1}{k16^k}\sum_{l=0}^{k}(k-2l)^2\, {}_{2l}C_l\ {}_{2(k-l)}C_{k-l}. \qquad (4.52)$$

Proposition 11. *Let K_1 be the number defined by (4.52). Then $K_1 = \frac{7}{18}$.*

Proof. To calculate K_1 we set for $0 < r < \frac{1}{4}$,

$$F_1(r) = \sum_{k=0}^{\infty} kr^k \sum_{l=0}^{k} {}_{2l}C_l\ {}_{2(k-l)}C_{k-l}, \qquad F_2(r) = \sum_{k=0}^{\infty} r^k \sum_{l=0}^{k} l\,{}_{2l}C_l\ {}_{2(k-l)}C_{k-l},$$

$$F_3(r) = \sum_{k=1}^{\infty} k^{-1} r^k \sum_{l=0}^{k} l^2\,{}_{2l}C_l\ {}_{2(k-l)}C_{k-l},$$

and

$$H_1(r) = \sum_{k=0}^{\infty} r^k \sum_{l=0}^{k} {}_{2l}C_l\ {}_{2(k-l)}C_{k-l}, \qquad H_3(r) = \sum_{k=0}^{\infty} r^k \sum_{l=0}^{k} l^2\,{}_{2l}C_l\ {}_{2(k-l)}C_{k-l}.$$

These functions converge when $0 < r < \frac{1}{4}$. Then K_1 is decomposed as

$$K_1 = F_1(\frac{1}{16}) - 4F_2(\frac{1}{16}) + 4F_3(\frac{1}{16}). \tag{4.53}$$

By using the equality $(\sum_{k=0}^{\infty} a_k)(\sum_{k=0}^{\infty} b_k) = \sum_{k=0}^{\infty}\sum_{l=0}^{k} a_l b_{k-l}$ for absolutely convergent series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$, we observe that

$$H_1(r) = (\sum_{k=0}^{\infty} r^k\,{}_{2k}C_k)^2, \qquad F_2(r) = (\sum_{k=0}^{\infty} kr^k\,{}_{2k}C_k)(\sum_{k=0}^{\infty} r^k\,{}_{2k}C_k),$$

$$H_3(r) = (\sum_{k=0}^{\infty} k^2 r^k\,{}_{2k}C_k)(\sum_{k=0}^{\infty} r^k\,{}_{2k}C_k).$$

Now we set

$$f_1(r) = \sum_{k=0}^{\infty} r^k\,{}_{2k}C_k, \qquad f_2(r) = \sum_{k=0}^{\infty} kr^k\,{}_{2k}C_k, \qquad f_3(r) = \sum_{k=0}^{\infty} k^2 r^k\,{}_{2k}C_k.$$

Then for $0 \le r < \frac{1}{4}$ we have

$$\begin{aligned} f_1'(r) = \sum_{k=1}^{\infty} kr^{k-1}\,{}_{2k}C_k &= \sum_{k=1}^{\infty} r^{k-1} 2(2k-1)\,{}_{2(k-1)}C_{k-1} \\ &= \sum_{k=0}^{\infty} r^k 2(2k+1)\,{}_{2k}C_k \\ &= 4rf_1'(r) + 2f_1(r). \end{aligned}$$

Solving this differential equation with $f_1(0) = 1$, we get

$$f_1(r) = \frac{1}{(1-4r)^{\frac{1}{2}}}, \qquad 0 \le r < \frac{1}{4}.$$

Moreover, we can check the relations

$$f_2(r) = \frac{2r}{1-4r} f_1(r), \qquad f_3(r) = \frac{2r}{1-4r}\big(3f_2(r) + f_1(r)\big).$$

Hence we have

$$f_2(r) = \frac{2r}{(1-4r)^{\frac{3}{2}}}, \qquad f_3(r) = \frac{r}{(1-4r)^{\frac{3}{2}}}(-1+\frac{3}{1-4r}).$$

This implies

$$H_1(r) = \frac{1}{1-4r}, \quad F_2(r) = \frac{2r}{(1-4r)^2}, \quad H_3(r) = \frac{r}{(1-4r)^2}(-1+\frac{3}{1-4r})$$

Since

$$F_1(r) = rH_1'(r), \qquad F_3'(r) = r^{-1}H_3(r),$$

we have

$$F_1(r) = \frac{4r}{(1-4r)^2}, \qquad F_3(r) = \frac{3}{8(1-4r)^2} - \frac{1}{4(1-4r)} - \frac{1}{8}.$$

Combining these and (4.53), we finally get $K_1 = \frac{7}{18}$. This completes the proof.

From Proposition 11 we have

$$\begin{aligned} \frac{1}{2} < v_1^{(2)}, v_1^{(1)} >_{L^2_\infty} &= \frac{1}{128\pi^2}\sum_{k=1}^{\infty}\frac{1}{k16^k}\sum_{l=0}^{k}(k-2l)^2{}_{2l}C_l\ {}_{2(k-l)}C_{k-l} \\ &= \frac{7}{18\cdot 128\pi^2}. \end{aligned} \tag{4.54}$$

From (4.45), (4.46), (4.47), (4.48), and (4.54), the value of n_{11} is

$$n_{11} = \frac{7}{18\cdot 128\pi^2} - \frac{1}{6^3\pi^2} = -\frac{11}{2^8\cdot 3^3\pi^2}, \tag{4.55}$$

Then, from (4.38) and (4.55) the representation matrix of $\mathbf{P}_1(0)A_1(0)\mathbf{P}_1(0) = -\mathbf{P}_1B_2\mathbf{P}_1 - \mathbf{P}_1B_1\mathbf{S}B_1\mathbf{P}_1$ is

$$M + N = \begin{pmatrix} -\frac{1}{2^8 3\pi^2} & 0 \\ 0 & 0 \end{pmatrix}.$$

That is, from (4.31) the eigenvalues $\mu_2(\delta)$ of $\mathbf{P}_1(\delta)A_2(\delta)\mathbf{P}_1(\delta)$ take the forms $\mu_2(\delta) = -\frac{1}{2^8 3\pi^2}\delta + o(|\delta|)$ or $\mu_2(\delta) = o(|\delta|)$. Hence, by (4.17) the eigenvalues of L_δ in Y_1 around -1 consist of

$$-1, \qquad \lambda_1(\delta) = -1 + \frac{1}{16\pi}\delta + o(|\delta|), \qquad \lambda_2(\delta) = -1 - \frac{1}{2^8 3\pi^2}\delta^2 + o(|\delta|^2). \tag{4.56}$$

Here we used the fact that -1 is an eigenvalue of $L_\delta = A - \mathcal{N}'(U_\delta)$ which reflects the scaling invariance of the equation. Moreover, these three must be simple eigenvalues of L_δ in Y_1 from the general perturbation theory, for -1 is a semisimple eigenvalue of A in Y_1 with multiplicity 3. From the above proof it is not difficult to see that $\lambda_2(\delta)$ is in fact a bifircation from the eigenvalue -1 of A with the eigenfunction $\mathbf{e}_2 \in Y_3$. Especially, $\lambda_2(\delta)$ is an eigenvalue of L_δ in Y_3. This completes the proof of Proposition 9.

As a corollary of Proposition 9, we have

Corollary 4. *The eigenvalues of* L_δ *in* $L^2_\infty \times L^2_\infty$ *near* -1 *consist of three eigenvalues* $\{-1, \lambda_1(\delta), \lambda_2(\delta)\}$ *as in Proposition 9. Moreover,* -1 *and* $\lambda_1(\delta)$ *are simple eigenvalues, and* $\lambda_2(\delta)$ *is a semisimple eigenvalue with multiplicity* 2.

Proof. Since we have already known that the rank of the eigenprojection around the eigenvalues near -1 must be 4, it suffices to show that the multiplicity of $\lambda_2(\delta)$ is 2. Let f be an eigenfunction of the eigenvalue $\lambda_2(\delta)$ in Y_1. Then by Proposition 9 we have $f \in Y_3$. From the definition of Y_3 we have $f(O\cdot) \in Y_2$, and hence, $f(O\cdot)$ and $f(\cdot)$ are linearly independent since Y_1 and Y_2 are orthogonal to each other. Moreover, since O is an orthogonal matrix, we have $L_\delta(f(O\cdot)) = (L_\delta f)(O\cdot) = \lambda_2(\delta) f(O\cdot)$ and so the function $f(O\cdot)$ is also an eigenfunction to the eigenvalue $\lambda_2(\delta)$. Hence there are two linearly independent eigenfunctions to the eigenvalue $\lambda_2(\delta)$, which gives the claim. This completes the proof.

4.2. **Proof of Theorem 3 and (1.6).** Corollary 4 yields Theorem 3, and hence, (1.6) in Theorem 2 as follows.

Let $m > 3$. Then by Proposition 6 the eigenvalues of L_δ in X_m near -1 are those of L_δ in $L^2_\infty \times L^2_\infty$. Hence by (4.2) and Corollary 4 we get (1.8). The asymptotics (1.9) follows from (4.10). This proves Theorem 3.

Next we prove (1.6). Let $m \in (3, \infty)$. As in the proof of (1.5), we may assume that $\Omega_0 \in X_m$. The conditions **(E1)**, **(E2)**, **(A1)**, **(A2)**, **(N1)**-**(N3)**, and the regularities of $S(y; w_0)$ and $H(y_0, y; U_\delta)$ required in Theorem 6 have already been checked by Proposition 2, Corollary 1, Proposition 4, and Corollary 2. The numbers ν_0 and μ^* are $\nu_0 = \mu^* = 1$ in this case. From Theorem 3 the spectrum of L_δ in X_m satisfies

$$\sigma(L_\delta) \subset \{0, -\frac{1}{2}, -1\} \cup \{\mu \in \mathbb{C} \mid \mathrm{Re}(\mu) \leq -1 + \eta(\delta)\}$$

with $\eta(\delta) = \frac{1}{16\pi}\delta + o(\delta)$ if δ is positive and $\eta(\delta) = -\frac{1}{2^8 3\pi^2}\delta^2 + o(\delta^2)$ if δ is negative. Moreover, the eigenvalue -1 is simple, and the set $\{\mu \in \sigma(L_\delta) \mid \mathrm{Re}(\mu) \geq -1 + \eta(\delta)\}$ consists of semisimple eigenvalues. Hence we can apply Theorem 6, Remark 2, and Remark 3, to (1.1) and obtain

$$\|R_{1+t}\Omega(t) - S(\frac{y_1^*}{(1+t)^{\frac{1}{2}}}, \frac{y_2^*}{(1+t)^{\frac{1}{2}}}, \frac{y_3^*}{1+t}; U_\delta)\|_{X_m} \leq C(1+t)^{-1+\eta(\delta)}, \tag{4.57}$$

for all $t \gg 1$ with the above $\eta(\delta)$. Then (1.6) follows by the same arguments as in the proof of (1.5). This completes the proof of (1.6).

5. Appendix

5.1. **Solvability of (1.1)-(1.2) for initial data in $L^2_m \times H^1_{m-2}$.** Let $m > 2$ and $X_m = L^2_m \times H^2_{m-2}$. In this section we prove the unique existence of mild solutions to (1.1)-(1.2) when the initial data Ω_0 belongs to $L^2_m \times H^1_{m-2}$ and satisfies $\|\Omega_0\|_{L^2_m \times H^1_{m-2}} \ll 1$. Moreover, we show the estimate $\|\Omega(1)\|_{X_m} \leq$

$C\|\Omega_0\|_{L^2_m\times H^1_{m-2}}$, which enables us to deal with the problem in X_m as stated in the proof of Theorem 2 and the beginning of Section 4.

By [6, Lemma 3.2] it suffices to solve the integral equation

$$u(t) = e^{tA}\Omega_0 - \int_0^t e^{(t-s)A}\mathcal{N}(u(s))ds, \qquad t > 0, \tag{5.1}$$

which is equivalent with the integral equations for (1.1)-(1.2) through the similarity transforms. Especially, for a solution u to (5.1) the function $\Omega(t) = R_{\frac{1}{1+t}}u(\log(1+t))$ becomes a mild solution to (1.1)-(1.2).

Let $\Omega_0 \in L^2_m \times H^1_{m-2}$ and we solve (5.1) in the closed ball

$$\begin{aligned} B_R &= \{f \in C((0,\infty); L^2_m \times H^2_{m-2}) \mid \\ &\quad \|f\| = \sup_{t>0}\|f^{(1)}(t)\|_{L^2_m} + \sup_{t>0}\|f^{(2)}(t)\|_{H^1_{m-2}} + \sup_{t>0} a(t)^{\frac{1}{2}}\|f^{(2)}(t)\|_{H^2_{m-2}} \le R\}, \end{aligned}$$

where $a(t) = 1 - e^{-t}$. From the representation of e^{tA}

$$e^{tA}f = \begin{pmatrix} e^{t\mathcal{L}}f^{(1)} \\ e^{-t}e^{t\mathcal{L}}f^{(2)} + (1-e^{-t})e^{t\mathcal{L}}f^{(1)} \end{pmatrix},$$

and the estimates for $e^{t\mathcal{L}}$ in (3.7), it is not difficult to see

$$\|e^{\cdot A}\Omega_0\| \le C_0\|\Omega_0\|_{L^2_m\times H^1_{m-2}}. \tag{5.2}$$

On the other hand, we have from (3.18) that

$$\begin{aligned} &\|\int_0^t e^{(t-s)A}\mathcal{N}(f(s))ds\|_{X_m} \\ \le\ & C\int_0^t e^{-\frac{t-s}{2}}a(t-s)^{-\frac{3}{4}}\|f^{(1)}(s)\|_{L^2_m}\|f^{(2)}(s)\|^{\frac{1}{2}}_{H^2_{m-2}}\|f^{(2)}(s)\|^{\frac{1}{2}}_{H^1_{m-2}}ds \\ \le\ & C\|f\|^2. \end{aligned} \tag{5.3}$$

Hence if we set the right hand side of (5.1) by $\Phi(u)(t)$, we get from (5.2) and (5.3),

$$\|\Phi(u)\| \le C_0\|\Omega_0\|_{L^2_m\times H^1_{m-2}} + C\|u\|^2.$$

From the bilinear structure of $\mathcal{N}$ it is not difficult to see

$$\|\Phi(u_1) - \Phi(u_2)\| \le C(\|u_1\| + \|u_2\|)\|u_1 - u_2\|$$

Hence $\Phi(u)$ is a contraction mapping on B_R if

$$R = 2C_0\|\Omega_0\|_{L^2\times H^1_{m-2}} \ll 1.$$

So there is a unique fixed point of Φ in B_R, which is a solution to (5.1). Moreover, the fixed point u satisfies the estimate $\|u(\log 2)\|_{X_m} \le C\|\Omega_0\|_{L^2_m\times H^1_{m-2}}$ from the construction. The continuity of $u(t)$ at $t = 0$ in $L^2_m \times H^1_{m-2}$ follows from the density arguments, but we omit the details here. This completes the proof of the assertion.

5.2. **Proof of Proposition 6.** Since $U_\delta \in H^2_\infty \times H^4_\infty$ by Proposition 5, we have $\|\nabla U^{(2)}_\delta\|_{L^\infty} + \|\Delta U^{(2)}_\delta\|_{L^\infty} < \infty$ and

$$\lim_{R\to\infty} \sup_{|x|\geq R} (|\nabla U^{(2)}_\delta(x)| + |\Delta U^{(2)}_\delta(x)|) = 0. \tag{5.4}$$

If $f \in H^2_m \times H^4_{m-2}$ is a solution to (4.3), then $f^{(1)}$ and $f^{(2)}$ solve the equations

$$\begin{aligned} &-\mathcal{L}f^{(1)} + \nabla U^{(2)}_\delta \cdot \nabla f^{(1)} + \Delta U^{(2)}_\delta f^{(1)} + \mu f^{(1)} = -\nabla \cdot (U^{(1)}_\delta \nabla f^{(2)}), \\ &-\mathcal{L}f^{(2)} + (1+\mu) f^{(2)} = f^{(1)}. \end{aligned}$$

Note that $\nabla \cdot (U^{(1)}_\delta \nabla f^{(2)}) \in L^2_\infty$ by the Sobolev embedding theorem. We first prove

Proposition 12. *Let $\mu \in \mathbb{C}$ and $g \in L^2_\infty$. Let $B \in (L^\infty(\mathbb{R}^2))^2$ and $d \in L^\infty(\mathbb{R}^2)$ be functions satisfying*

$$\lim_{R\to\infty} \sup_{|x|\geq R} |B(x)| = \lim_{R\to\infty} \sup_{|x|\geq R} |d(x)| = 0. \tag{5.5}$$

Assume that $\phi \in H^2_m$ is a solution to

$$-\mathcal{L}\phi - B \cdot \nabla\phi - d\phi + \mu\phi = g, \qquad x \in \mathbb{R}^2. \tag{5.6}$$

If $\mathrm{Re}(\mu) > -\frac{m-1}{2}$, then $\phi \in \mathrm{Dom}(\mathcal{L})$ in L^2_∞. Especially, $\phi \in H^2_\infty$.

Proof. We first show that $e^{-\frac{1-\epsilon}{8}|x|^2} u(x) \in L^2(\mathbb{R}^n)$ for all $\epsilon > 0$. For $k \geq 1$, $\epsilon > 0$, $l \geq 1$, and $\theta \in [0, m]$, we set

$$\rho_{k,\epsilon}(x) = e^{\frac{(1-\epsilon)k|x|^2}{4k+|x|^2}}, \qquad \zeta_{l,\theta}(x) = \frac{l}{l+|x|^2}(1+|x|^2)^\theta. \tag{5.7}$$

These test functions are used in [9], which were originally motivated by Fukuizumi-Ozawa [4]. Then we multiply both sides of (5.6) by $\zeta_{l,\theta}\rho_{k,\epsilon}\bar{\phi}$ and get from the integration by parts,

$$\begin{aligned} &\int_{\mathbb{R}^2} \zeta_{l,\theta}\rho_{k,\epsilon}|\nabla\phi|^2 dx + \mathrm{Re}\int_{\mathbb{R}^2} \bar{\phi}\nabla\phi \cdot \nabla(\zeta_{l,\theta}\rho_{k,\epsilon}) dx + \frac{1}{4}\int_{\mathbb{R}^2} |\phi|^2 x \cdot \nabla(\zeta_{l,\theta}\rho_{k,\epsilon}) dx \\ = \ &\mathrm{Re}\int_{\mathbb{R}^2} \zeta_{l,\theta}\rho_{k,\epsilon}\bar{\phi} B \cdot \nabla\phi dx + \int_{\mathbb{R}^2} \zeta_{l,\theta}\rho_{k,\epsilon}\big(\mathrm{Re}(d-\mu) + \frac{1}{2}\big)|\phi|^2 dx + \mathrm{Re}\int_{\mathbb{R}^2} \zeta_{l,\theta}\rho_{k,\epsilon} g\bar{\phi} dx. \end{aligned}$$

Since

$$\nabla\zeta_{l,\theta} = 2x\Big(\frac{\theta}{1+|x|^2} - \frac{1}{l+|x|^2}\Big)\zeta_{l,\theta}, \qquad \nabla\rho_{k,\epsilon} = \frac{8(1-\epsilon)k^2\rho_{k,\epsilon}x}{(4k+|x|^2)^2},$$

we have for $\eta_1 > 0$,

$$
\begin{aligned}
&\mathrm{Re}\int_{\mathbb{R}^2} \bar{\phi}\nabla\phi\cdot\nabla(\zeta_{l,\theta}\rho_{k,\epsilon})dx \\
\geq\ & -\int_{\mathbb{R}^2}|\phi|^2 x\cdot\nabla\{(\frac{\theta}{1+|x|^2}-\frac{1}{l+|x|^2})\zeta_{l,\theta}\rho_{k,\epsilon}\}dx - C\int_{\mathbb{R}^2}\frac{\zeta_{l,\theta}\rho_{k,\epsilon}}{1+|x|^2}|\phi|^2dx \\
&+\mathrm{Re}\int_{\mathbb{R}^2}\frac{8(1-\epsilon)k^2\zeta_{l,\theta}\rho_{k,\epsilon}}{(4k+|x|^2)^2}\bar{\phi}x\cdot\nabla\phi dx \\
\geq\ & -\int_{\mathbb{R}^2}|\phi|^2\frac{\theta\zeta_{l,\theta}}{1+|x|^2}x\cdot\nabla\rho_{k,\epsilon}dx - \int_{\mathbb{R}^2}|\phi|^2\rho_{k,\epsilon}x\cdot\nabla\frac{\theta\zeta_{l,\theta}}{1+|x|^2}dx \\
&+\int_{\mathbb{R}^2}|\phi|^2x\cdot\nabla\frac{\zeta_{l,\theta}\rho_{k,\epsilon}}{l+|x|^2}dx - C\int_{\mathbb{R}^2}\frac{\zeta_{l,\theta}\rho_{k,\epsilon}}{1+|x|^2}|\phi|^2dx \\
&-\int_{\mathbb{R}^2}\frac{2(1-\epsilon)k\zeta_{l,\theta}\rho_{k,\epsilon}}{4k+|x|^2}|\phi x||\nabla\phi|dx \\
\geq\ & -8(1-\epsilon)\theta\int_{\mathbb{R}^2}\frac{k^2\zeta_{l,\theta}\rho_{k,\epsilon}|x\phi|^2}{(4k+|x|^2)^2(1+|x|^2)}dx - \int_{\mathbb{R}^2}(\frac{C\zeta_{l,\theta}\rho_{k,\epsilon}}{1+|x|^2}+x\cdot\nabla(\frac{\zeta_{l,\theta}\rho_{k,\epsilon}}{l+|x|^2}))|\phi|^2dx \\
&-(1-\eta_1)\int_{\mathbb{R}^2}\zeta_{l,\theta}\rho_{k,\epsilon}|\nabla\phi|^2dx - \frac{(1-\epsilon)^2}{1-\eta_1}\int_{\mathbb{R}^2}\frac{k^2\zeta_{l,\theta}\rho_{k,\epsilon}|\phi x|^2}{(4k+|x|^2)^2}dx,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{4}\int_{\mathbb{R}^2}|\phi|^2x\cdot\nabla(\zeta_{l,\theta}\rho_{k,\epsilon})dx \\
=\ & \frac{1}{2}\int_{\mathbb{R}^2}\zeta_{l,\theta}\rho_{k,\epsilon}|x\phi|^2(\frac{\theta}{1+|x|^2}-\frac{1}{l+|x|^2})dx + 2(1-\epsilon)\int_{\mathbb{R}^2}\frac{k^2\zeta_{l,\theta}\rho_{k,\epsilon}|x\phi|^2}{(4k+|x|^2)^2}dx.
\end{aligned}
$$

Here the constant $C > 0$ does not depend on l, k, and ϵ. Set $\zeta_\theta = (1+|x|^2)^\theta$. We observe that we can take the limit $l \to \infty$ in each term above by the Lebesgue convergence theorem, and obtain for $\eta_2, \eta_3 > 0$,

(5.8)

$$
\begin{aligned}
&(\eta_1-\eta_2)\int_{\mathbb{R}^2}\zeta_\theta\rho_{k,\epsilon}|\nabla\phi|^2dx + \int_{\mathbb{R}^2}\frac{(1-\epsilon)k^2\zeta_\theta\rho_{k,\epsilon}|x\phi|^2}{(4k+|x|^2)^2}(\frac{1-2\eta_1+\epsilon}{1-\eta_1}-\frac{8\theta}{1+|x|^2})dx \\
\leq\ & \int_{\mathbb{R}^2}\zeta_\theta\rho_{k,\epsilon}(\frac{C}{1+|x|^2}+\mathrm{Re}(d-\mu)+\frac{1}{2}+\frac{|B|^2}{4\eta_2}+\eta_3-\frac{\theta}{2})|\phi|^2dx + \frac{1}{4\eta_3}\int_{\mathbb{R}^2}\zeta_\theta\rho_{k,\epsilon}|g|^2dx.
\end{aligned}
$$

Now we take $\eta_1 = \eta_2 = \frac{1}{2}$ and $\theta = m$ in (5.8). Then, from (5.5) and $\mathrm{Re}(\mu) > -\frac{m-1}{2}$ there is an $R > 0$ independent of $k \geq 1$ such that if $\eta_3 > 0$ is sufficiently small, then we have

$$
(1-\epsilon)\epsilon\int_{\mathbb{R}^2}\frac{k^2\zeta_\theta\rho_{k,\epsilon}|x\phi|^2}{(4k+|x|^2)^2}dx \leq C\int_{|x|\leq R}\zeta_\theta\rho_{k,\epsilon}|\phi|^2dx + \frac{1}{4\eta_3}\int_{\mathbb{R}^2}\zeta_\theta\rho_{k,\epsilon}|g|^2dx,
$$

where C is independent of $k \geq 1$. Hence by the Fatou lemma we get

$$
\begin{aligned}
&(1-\epsilon)\epsilon \int_{\mathbb{R}^2} (1+|x|^2)^m e^{\frac{1-\epsilon}{4}|x|^2} |x\phi|^2 dx \\
\leq\ & C(R) \int_{|x|\leq R} |\phi|^2 dx + \frac{1}{4\eta_3} \int_{\mathbb{R}^2} (1+|x|^2)^m e^{\frac{1-\epsilon}{4}|x|^2} |g|^2 dx,
\end{aligned}
$$

which gives $e^{\frac{1-\epsilon}{8}|x|^2}\phi(x) \in L^2(\mathbb{R}^2)$ for all $\epsilon > 0$. Next we take $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{1}{8}$, $\eta_3 = 1$, and $\theta = 0$ in (5.8). Then by the Lebesgue convergence theorem we have

$$
\begin{aligned}
&\frac{1}{8}\int_{\mathbb{R}^2} e^{\frac{1-\epsilon}{4}|x|^2} |\nabla\phi|^2 dx + \frac{1-\epsilon}{24}\int_{\mathbb{R}^2} e^{\frac{1-\epsilon}{4}|x|^2} |x\phi|^2 dx \\
\leq\ & C\int_{\mathbb{R}^2} e^{\frac{1-\epsilon}{4}|x|^2} |\phi|^2 dx + \frac{1}{4}\int_{\mathbb{R}^2} e^{\frac{1-\epsilon}{4}|x|^2} |g|^2 dx,
\end{aligned}
$$

where C does not depend on $\epsilon > 0$. This inequality yields that

$$
\begin{aligned}
&\frac{1}{8}\int_{\mathbb{R}^2} e^{\frac{1-\epsilon}{4}|x|^2} |\nabla\phi|^2 dx + \frac{1-\epsilon}{48}\int_{\mathbb{R}^2} e^{\frac{1-\epsilon}{4}|x|^2} |x\phi|^2 dx \\
\leq\ & C\int_{|x|\leq R'} e^{\frac{1-\epsilon}{4}|x|^2} |\phi|^2 dx + \frac{1}{4}\int_{\mathbb{R}^2} e^{\frac{1-\epsilon}{4}|x|^2} |g|^2 dx,
\end{aligned}
$$

for some $R' > 0$ independent of $\epsilon > 0$. Taking the limit $\epsilon \to 0$, we obtain $\phi \in H^1_\infty$. Then by Eq. (5.6) we have $\mathcal{L}\phi \in L^2_\infty$. Hence $\phi \in \mathrm{Dom}(\mathcal{L})$. This completes the proof of Proposition 12.

Now it is easy to show Proposition 6. Indeed, by Proposition 12 we first observe that $f^{(1)} \in \mathrm{Dom}(\mathcal{L}) \hookrightarrow H^2_\infty$, and then, again by Proposition 12 we also have $f^{(2)} \in \mathrm{Dom}(\mathcal{L}) \hookrightarrow H^2_\infty$ since $f^{(2)}$ is assumed to belong to H^2_{m-2} and $1 + \mathrm{Re}(\mu) > 1 - \frac{m-1}{2} = -\frac{m-2-1}{2}$. Since $f^{(2)} = (-\mathcal{L} + I)^{-1} f^{(1)}$ and $f^{(1)} \in \mathrm{Dom}(\mathcal{L})$, we have $f^{(2)}_\delta \in \mathrm{Dom}(\mathcal{L}^2) \hookrightarrow H^4_\infty$. The proof of Proposition 6 is complete.

5.3. **Proof of Proposition 10.** Let $\Lambda(s)$ be the one dimensional Gaussian, i.e., $\Lambda(s) = \frac{1}{\sqrt{4\pi}} e^{-\frac{s^2}{4}}$. Then since $G(x) = \Lambda(x_1)\Lambda(x_2)$ we have

$$
\begin{aligned}
\|\partial_1^k \partial_2^l G\|^2_{L^2} &= \|\frac{d^k}{ds^k}\Lambda\|^2_{L^2(\mathbb{R})} \|\frac{d^l}{ds^l}\Lambda\|^2_{L^2(\mathbb{R})}, \\
\|\partial_1^k \partial_2^l G\|^2_{L^2_\infty} &= \|\frac{d^k}{ds^k}\Lambda\|^2_{L^2(\Lambda^{-1}ds)} \|\frac{d^l}{ds^l}\Lambda\|^2_{L^2(\Lambda^{-1}ds)}.
\end{aligned}
$$

Hence it suffices to show

(5.9)

$$
\Pi_{1,k} := \|\frac{d^k}{ds^k}\Lambda\|^2_{L^2(\mathbb{R})} = \frac{(2k)!}{2\sqrt{2\pi}8^k k!}, \qquad \Pi_{2,k} := \|\frac{d^k}{ds^k}\Lambda\|^2_{L^2(\Lambda^{-1}ds)} = \frac{k!}{2^k}.
$$

By the Plancherel equality, we have

$$\Pi_{1,k+1} = \int_{\mathbb{R}} \tilde{s}^{2(k+1)} \hat{\Lambda}(\tilde{s})^2 d\tilde{s}.$$

Since $\hat{\Lambda}(\tilde{s}) = ce^{-\tilde{s}^2}$ for some constant c, we have

$$\begin{aligned} \Pi_{1,k+1} = -\frac{c^2}{4} \int_{\mathbb{R}} \tilde{s}^{2k} \tilde{s} (e^{-2\tilde{s}^2})' d\tilde{s} &= \frac{(2k+1)c^2}{4} \int_{\mathbb{R}} \tilde{s}^{2k} e^{-2\tilde{s}^2} d\tilde{s} \\ &= \frac{2k+1}{4} \Pi_{1,k}. \end{aligned}$$

This implies $\Pi_{1,k} = \frac{(2k)!}{2\sqrt{2\pi}8^k k!}$. To calculate $\Pi_{2,k}$, set $\mathcal{L}^{(1)} = \frac{d^2}{ds^2} + \frac{s}{2}\frac{d}{ds} + \frac{1}{2}$. Then it follows that $\mathcal{L}^{(1)} \frac{d^k}{ds^k}\Lambda = -\frac{k}{2}\frac{d^k}{ds^k}\Lambda$. Hence we have from the integration by parts,

$$\begin{aligned} \Pi_{2,k} &= < \frac{d^k}{ds^k}\Lambda, \frac{d^k}{ds^k}\Lambda >_{L^2(\Lambda^{-1}ds)} \\ &= -\frac{2}{k} < \frac{d^k}{ds^k}\Lambda, \mathcal{L}^{(1)} \frac{d^k}{ds^k}\Lambda >_{L^2(\Lambda^{-1}ds)} \\ &= -\frac{2}{k}\Big(- < \frac{d^{k+1}}{ds^{k+1}}\Lambda, \frac{d^{k+1}}{ds^{k+1}}\Lambda >_{L^2(\Lambda^{-1}ds)} + \frac{1}{2} < \frac{d^k}{ds^k}\Lambda, \frac{d^k}{ds^k}\Lambda >_{L^2(\Lambda^{-1}ds)} \Big) \\ &= \frac{2}{k}\Pi_{2,k+1} - \frac{1}{k}\Pi_{2,k}, \end{aligned}$$

which gives $\Pi_{2,k+1} = \frac{k+1}{2}\Pi_{2,k}$, and thus, $\Pi_{2,k} = \frac{k!}{2^k}$ holds. This completes the proof.

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