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ON THE CLASSIFICATION OF RANK 2 ALMOST FANO BUNDLES ON PROJECTIVE SPACE

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ABSTRACT. An almost Fano bundle is a vector bundle on a smooth projective variety that its projectivization is an almost Fano variety. In this paper, we prove that almost Fano bundles exist only on almost Fano manifolds and study rank 2 almost Fano bundles over projective spaces.

INTRODUCTION

An almost Fano variety is a smooth projective variety whose anti-canonical line bundle is nef and big. This is a natural generalization of Fano varieties and often appears in the study of deformation of a Fano variety ([12], [15]). Almost Fano surfaces were completely classified by Demazure [3]. Recently Jahnke, Peternel and Radlo classified almost Fano threefolds with picard number 2 whose pluri-anti-canonical morphism is divisorial in [8]. In [19], Takeuchi studied almost Fano threefolds with del pezzo fibration structure whose pluri-anti-canonical morphism is small. In higher dimensional case, Jahnke and Peternell classified almost del Pezzo varieties, which are almost Fano n -folds with index $n - 1$ i.e. its anti-canonical line bundle is divisible by $n - 1$ in the Picard group.

The aim of this paper is to study ruled almost Fano varieties M of dimension $n \geq 3$ over nonsingular variety S i.e. there is a vector bundle \mathcal{E} on S such that M is isomorphic to its projectivization $\mathbb{P}_S(\mathcal{E})$.

Szurek and Wisniewski introduced the notion of Fano bundle in [16]. As an almost Fano version, we introduce the notion of almost Fano bundle as below.

DEFINITION 0.1. Let \mathcal{E} be a vector bundle on a smooth complex projective variety M . We say that \mathcal{E} is almost Fano if its projectivization $\mathbb{P}_M(\mathcal{E})$ is an almost Fano variety.

Such bundles always exist on an almost Fano variety M . In fact, we notice that the trivial rank r vector bundle is almost Fano since $\mathbb{P}_M(\mathcal{O}_M^{\oplus r}) \cong M \times \mathbb{P}^{r-1}$ is also an almost Fano variety. In [16, Theorem 1.6], it is shown that Fano bundles are only on Fano manifolds. We consider the almost Fano case and obtain the following theorem.

THEOREM A. If \mathcal{E} is an almost Fano bundle over a smooth complex projective variety M , then M is an almost Fano variety.

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On projective spaces, rank 2 Fano bundles are completely classified in [1], [16] and [17]. Using their methods, we study the classification of rank 2 almost Fano bundles on projective spaces and have the list mentioned below.

THEOREM B. Let \mathcal{E} be a rank 2 normalized (i.e. $c_1(\mathcal{E}) = 0$ or -1) almost Fano bundle on \mathbb{P}^n . Assume that \mathcal{E} is not Fano. Then, \mathcal{E} is isomorphic to one of the following :

- (1) $\mathcal{O}_{\mathbb{P}^n}(\lfloor \frac{n}{2} \rfloor) \oplus \mathcal{O}_{\mathbb{P}^n}(\lfloor -\frac{n}{2} \rfloor)$, where $\lfloor \frac{n}{2} \rfloor$ is the largest integer $\leq \frac{n}{2}$.
- (2) a stable bundle on \mathbb{P}^3 with $c_1 = 0$, $c_2 = 2$.
- (3) a stable bundle on \mathbb{P}^3 with $c_1 = 0$, $c_2 = 3$.
- (4) a vector bundle on \mathbb{P}^2 determined by the exact sequence : $0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{E} \rightarrow \mathcal{I}_p(-1) \rightarrow 0$, where \mathcal{I}_p is the ideal sheaf of a point p .
- (5) a stable bundle on \mathbb{P}^2 with $c_1 = -1$, $2 \leq c_2 \leq 5$.
- (6) a stable bundle on \mathbb{P}^2 with $c_1 = 0$, $4 \leq c_2 \leq 6$.

Moreover, we show that all cases stated above really exist, except the case when $c_2 = 6$ in (6). Note that these varieties are of index 1 or 2. On three dimensional projective space, the most difficult part is a construction of almost Fano bundles satisfying the condition in (3). To obtain this, we use Maruyama's theory of elementary transformation of vector bundles. On projective plane, the case $c_1 = -1$ was classified in [7]. Therefore we treat the case $c_1 = 0$ i.e. $\mathbb{P}(\mathcal{E})$ is of index 1. In particular, we study almost Fano threefolds of index 1 whose pluri-anti-canonical morphism is small, having \mathbb{P}^1 -bundle structure over \mathbb{P}^2 .

Ruled varieties play an important role in the classification theory of projective varieties. So we may expect that our results also have applications.

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NOTATION

Throughout this paper \mathcal{E} is a vector bundle on a smooth complex projective variety M and $\xi_{\mathcal{E}}$ is the tautological line bundle on $X = \mathbb{P}_M(\mathcal{E})$. By π we denote the projection $\pi : \mathbb{P}_M(\mathcal{E}) \rightarrow M$ and by H the pull-back of hyperplane if $M = \mathbb{P}^n$ (i.e. $\mathcal{O}_{\mathbb{P}_M(\mathcal{E})}(H) \cong \pi^* \mathcal{O}_{\mathbb{P}^n}(1)$). For a curve C in M , we denote by $[C]$ the numerical equivalence class of C in M .

1. PROOF OF THEOREM A

In this section we prove Theorem A. Before starting the proof, we prepare some facts.

DEFINITION 1.1. Let X be a normal projective variety and Δ an effective \mathbb{Q} -divisor on X . Let $\varphi : Y \rightarrow X$ be a log resolution of (X, Δ) . We set

$$K_Y = \varphi^*(K_X + \Delta) + \sum a_i E_i,$$

where E_i is a prime divisor. The pair (X, Δ) is called kawamata log terminal (klt, for short) if $a_i > -1$ for all i .

DEFINITION 1.2. Let X be a normal projective variety and Δ an effective \mathbb{Q} -divisor on X . We say that the pair (X, Δ) is a log Fano variety if (X, Δ) is klt and $-(K_X + \Delta)$ is an ample \mathbb{Q} -divisor.

LEMMA 1.3. *If X is an almost Fano manifold, there is an effective \mathbb{Q} -divisor Δ such that (X, Δ) is a log Fano variety.*

PROOF. For any ample divisor A , there are an integer m and an effective divisor E such that $-mK_X = A + E$ by [9, Lemma 2.60]. Put $\Delta = \frac{1}{l}E$ for $l \gg 0$, then (X, Δ) is klt from [9], corollary 2.35 and

$$-lm(K_X + \Delta) = m(l - m)(-K_X) + mA$$

is ample. □

Using this lemma, we get the following results by [9] and [21].

THEOREM 1.4. *Let X be an almost Fano manifold. Then,*

(1) *(Basepoint-free Theorem)*

Any nef divisor D on X is semiample (i.e. bD is basepoint free for $b \gg 0$).

(2) *(Cone Theorem)*

There are finitely many rational curves C_j on X such that

$$\overline{NE(X)} = \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_j].$$

(3) *X is rationally connected i.e. for any two points in X there exists a rational curve which passes through them.*

The next lemma is also needed.

LEMMA 1.5. (c.f. [20, Lemma 3.3]). *Let $\pi : X = \mathbb{P}_M(\mathcal{E}) \rightarrow M$ be the projectivization of a rank r almost Fano bundle \mathcal{E} and C an extremal rational curve on X not contracted by π . Then, we have $0 \leq -K_X.C \leq -K_M.\pi(C)$.*

PROOF. Let C be an extremal rational curve on X not contracted by π and φ_C the corresponding elementary contraction map. Then φ_C satisfies the assumption in [20, Lemma 3.3]. Hence we obtain the inequality in the lemma. □

Proof of Theorem A. Put $X = \mathbb{P}_M(\mathcal{E})$. From Theorem 1.4, we can find finitely many extremal rational curves C_0, C_1, \dots, C_ρ in X which generate the Kleiman-Mori cone $\overline{NE}(X)$. Let C_0 be contained in a fiber of the projection π . Then we see that $\overline{NE}(M) = \sum_{i=1}^{\rho} \mathbb{R}_{\geq 0}[\pi(C_i)]$. From Lemma 1.5, it follows that

$$-K_M \cdot \pi(C_i) \geq -K_X \cdot C_i \geq 0$$

for $1 \leq i \leq \rho$. Therefore $-K_M$ is nef. Next we show the bigness of $-K_M$. Applying Theorem 1.4 (1) to $D := \pi^*(-K_M)$, we know D is semiample. Because π is projective space bundle, $-K_M$ is also semiample. Let $\varphi = \varphi_{|-lK_M|} : M \rightarrow W$ be a morphism induced by $-lK_M$ for $l \gg 0$. Suppose that $\dim M > \dim W$. Take the Stein factorization, we may assume a fiber of φ is connected. We denote its general fiber by F . Then F is smooth and we see that $-K_M|_F = -K_F$ holds. From this,

$$\begin{aligned} -K_X|_{\pi^{-1}(F)} &= (r\xi_{\mathcal{E}} - \pi^*(K_M + c_1(\mathcal{E})))|_{\pi^{-1}(F)} \\ &= r\xi_{\mathcal{E}|_F} - \pi^*(K_F + c_1(\mathcal{E}|_F)) = -K_{\mathbb{P}_F(\mathcal{E}|_F)}. \end{aligned}$$

Therefore we may only consider $\varphi(M)$ is a point. In this case, Kodaira dimension $\kappa(M)$ of M is equal to 0. On the other hand, X is rationally connected due to Theorem 1.4 (3). Since π is surjective, M is also rationally connected. Hence we have $\kappa(M) = -\infty$. This is a contradiction. \square

REMARK 1.6. (1) This theorem is proved in [2] if $\dim X = 2$ and $\text{rank } \mathcal{E} = 2$.

(2) Recently Fujino and Gongyo prove if X is almost Fano and $f : X \rightarrow Y$ is a smooth morphism, then Y should be almost Fano [5].

2. PROOF OF THEOREM B

In this section, we study the structure of almost Fano bundles on projective space.

First we consider almost Fano bundles which are decomposed into a direct sum of line bundles. In this case, we can characterize almost Fano bundles for any rank.

PROPOSITION 2.1. *Let $\mathcal{E} \cong \mathcal{O} \oplus \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \dots \oplus \mathcal{O}(a_r)$ be a vector bundle on \mathbb{P}^n , where $0 \leq a_1 \leq a_2 \leq \dots \leq a_r$. Then, \mathcal{E} is almost Fano if and only if $0 \leq c_1(\mathcal{E}) = \sum_{i=1}^r a_i \leq n + 1$. Moreover \mathcal{E} is not Fano if and only if $c_1(\mathcal{E}) = n + 1$.*

PROOF. Put $X = \mathbb{P}_{\mathbb{P}^n}(\mathcal{E})$. Then, we have $-K_X = r\xi_{\mathcal{E}} - (n + 1 - c_1(\mathcal{E}))H$. From the choice of \mathcal{E} , we can check naturally that \mathcal{E} is Fano if and only if $0 \leq c_1(\mathcal{E}) \leq n$. Next we will establish the latter part. It is easy to see that $-K_M$ is nef but not ample if and only if $c_1(\mathcal{E}) = n + 1$. Therefore it is sufficient to show that $-K_M$ is big if $c_1(\mathcal{E}) = n + 1$. In this case, $H^0(\xi_{\mathcal{E}} - H) \cong H^0(\mathcal{E}(-1)) \neq 0$. By Kodaira's lemma, $-K_M = r\xi_{\mathcal{E}} = ((r - 1)\xi_{\mathcal{E}} + H) + (\xi_{\mathcal{E}} - H)$ is big. \square

From now on, we give a proof of Theorem B. The proof is divided into three parts, (I) $n \geq 4$, (II) $n = 3$ and (III) $n = 2$.

(I) $n \geq 4$.

At first, we consider the case where $n \geq 4$. The claim is as follows.

PROPOSITION 2.2. *Let \mathcal{E} be an almost Fano 2-bundle on \mathbb{P}^n , $n \geq 4$. Then \mathcal{E} is a direct sum of two line bundles.*

These bundles are classified in Proposition 2.1. To show this, we need the next two lemmata.

LEMMA 2.3. *Let \mathcal{E} be a normalized rank 2 almost Fano bundle on \mathbb{P}^n . If $n \geq 4$, then $\mathcal{E}(n)$ is generated by its global sections.*

PROOF. The proof is in the similar fashion as in [1, Proposition 2.6]. We give an outline of the proof in the case where n is even and $c_1 = -1$. Put $n = 2k$ and $X = \mathbb{P}_{\mathbb{P}^n}(\mathcal{E})$, then we have

$$-K_X = 2\xi_{\mathcal{E}} + (2k + 2)H = 2(\xi_{\mathcal{E}} + (k + 1)H)$$

is nef and big. Therefore $\mathcal{E}(k+2)$ is ample vector bundle. By Le Potier vanishing theorem,

$$H^i(\mathcal{E}(k+2+j) \otimes K_{\mathbb{P}^n}) = H^i(\mathcal{E}(j-k+1)) = 0$$

for any $i \geq 2$ and $j \geq 0$. Especially letting $j = 3k - i - 1$, we have $H^i(\mathcal{E}(n-i)) = 0$ for $i \geq 2$. Moreover

$$H^1(\mathbb{P}_{\mathbb{P}^n}(\mathcal{E}), 3(\xi_{\mathcal{E}} + (k+1)H) + (k-2)H + K_{\mathbb{P}_{\mathbb{P}^n}(\mathcal{E})}) = H^1(\mathbb{P}^n, \mathcal{E}(n-1)) = 0$$

from Kawamata-Vieweg vanishing theorem. Combining above, then we see that $H^i(\mathcal{E}(n-i)) = 0$ for $i \geq 1$ namely \mathcal{E} is n -regular. By means of Castelnuovo-Mumford lemma, $\mathcal{E}(n)$ is generated by its global sections. Other cases are proved in the same way. \square

LEMMA 2.4. [1] *Let \mathcal{E} be a globally generated 2-bundle on \mathbb{P}^n . Then we have*

(1) *If \mathcal{E} is not stable and $c_2(\mathcal{E}) < (n-1)(c_1(\mathcal{E}) - n + 2)$, then \mathcal{E} is split into a direct sum of two line bundles.*

(2) *If $n \geq 6$ and $c_1(\mathcal{E})^2 < 4c_2(\mathcal{E})$, then we have $c_1(\mathcal{E}) \geq 2n + 3$.*

Proof of Proposition 2.2. Applying Lemma 2.4 to $\mathcal{E}(n)$, we can show immediately that \mathcal{E} is split except for $n = 4$ and 5 essentially in the same as in the proof of Proposition 3.1 and Proposition 5.1 in [1]. If $n = 4$ (resp. $n = 5$), then $\mathcal{E}(3)$ (resp. $\mathcal{E}(4)$) is nef. From [1, Proposition 9.2] (resp. [1, Proposition 9.4]), \mathcal{E} is split. \square

(II) $n = 3$.

Next, we consider the case where $n = 3$. To start with, we demonstrate rank 2 almost Fano bundle on \mathbb{P}^3 is one of vector bundles below.

PROPOSITION 2.5. *Let \mathcal{E} be a normalized almost Fano 2-bundle on \mathbb{P}^3 . Then \mathcal{E} is isomorphic to a direct sum of two line bundles or one of the following :*

- (1) *stable vector bundle with $c_1 = 0$, $c_2 = 1$.*
- (2) *stable vector bundle with $c_1 = 0$, $c_2 = 2$.*
- (3) *stable vector bundle with $c_1 = 0$, $c_2 = 3$.*

PROOF. We shall discuss the two cases $c_1 = 0$ and $c_1 = -1$ separately.

First we treat $c_1 = -1$. Since $-K_X = 2\xi_{\mathcal{E}} + 5H$ is nef and big, we have that $\mathcal{E}(3)$ is ample. We can apply the argument in [16, Theorem 2.2], to this case and we can show that \mathcal{E} is decomposed into a direct sum of two line bundles.

Next we treat $c_1 = 0$. In this case, $\mathcal{E}(2)$ is nef. If $H^0(\mathcal{E}(-2)) \neq 0$, then we can take a non-zero section $s \in H^0(\mathcal{E}(-2))$. If $Z := \{s = 0\} = \emptyset$, then \mathcal{E} is decomposed into a direct sum of line bundles. If $Z \neq \emptyset$, then for a line L meeting Z in a finite number of points we would have

$$\mathcal{E}(-2)|_L \cong \mathcal{O}_L(d) \oplus \mathcal{O}_L(-4-d), (d \geq 1)$$

which contradicts to the nefness of $\mathcal{E}(2)$. If $H^0(\mathcal{E}(-2)) = 0$ and $H^0(\mathcal{E}(-1)) \neq 0$, then we can take a non-zero section $s \in H^0(\mathcal{E}(-1))$. If $Z := \{s = 0\} = \emptyset$, then \mathcal{E} is decomposed into a direct sum of line bundles. If $Z \neq \emptyset$, then Z is a curve. Suppose that $\deg Z \geq 2$, we can take a line L intersecting with Z at least two points. Then

$$\mathcal{E}(-1)|_L \cong \mathcal{O}_L(d) \oplus \mathcal{O}_L(-2-d), (d \geq 2)$$

and contradict to the nefness of $\mathcal{E}(2)$. If $\deg Z = 1$, then Z is a line. But,

$$\deg K_Z = \deg(K_{\mathbb{P}^3} + c_1(\mathcal{E}(-1)))|_Z = -6.$$

This is a contradiction. If $H^0(\mathcal{E}(-1)) = 0$ and $H^0(\mathcal{E}) \neq 0$, then we can take a non-zero section $s \in H^0(\mathcal{E})$. If $Z := \{s = 0\} = \emptyset$, then \mathcal{E} is decomposed into a direct sum of two line bundles. If $Z \neq \emptyset$, then Z is a curve and $\deg Z = c_2 \geq 1$. On the other hand, $\xi_{\mathcal{E}}(-K_X)^3 = 8 - 6c_2 \geq 0$. Therefore $\deg Z = 1$ i.e. Z is a line. But,

$$\deg K_Z = \deg(K_{\mathbb{P}^3} + c_1(\mathcal{E}))|_Z = -4.$$

This is a contradiction. Finally we assume $H^0(\mathcal{E}) = 0$ i.e. \mathcal{E} is stable. In this case, $c_1^2 < 4c_2$ and $(-K_X)^4 = 128(4 - c_2) > 0$ hold. Hence $1 \leq c_2 \leq 3$. \square

It is shown [16] that all stable bundles satisfying $c_1 = 0$, $c_2 = 1$ are Fano. If $c_2 = 2$, then \mathcal{E} is 2-regular by [6]. Therefore $-K_X = 2(\xi_{\mathcal{E}} + 2H)$ is nef and big i.e. \mathcal{E} is almost Fano. Such \mathcal{E} is not Fano bundle [16]. The case $c_2 = 3$ is more complicated. First we show such an almost Fano bundle really exists.

PROPOSITION 2.6. *There is an almost Fano stable bundle on \mathbb{P}^3 with $c_1 = 0$, $c_2 = 3$.*

To show this, we use the following result.

THEOREM 2.7. [13, Proposition 6] *There is a nonsingular elliptic curve C on a smooth quartic surface $S \subset \mathbb{P}^3$ and a very ample divisor A on S such that*

- (1) $\text{Pic}(S) \cong \mathbb{Z}[A] \oplus \mathbb{Z}[C]$.
- (2) $A^2 = 4$, $A.C = 7$, $C^2 = 0$.
- (3) C is base point free.
- (4) S does not contain any rational curve.

Proof of Proposition 2.6. Let (S, C) be a pair in Theorem 2.7. Using the theory of elementary transformation [10] and [11], we can construct a rank 2 regular vector bundle \mathcal{F} on \mathbb{P}^3 where $c_1(\mathcal{E}) = S$, $c_2(\mathcal{E}) = C$ modulo numerical equivalence. We will prove that $\mathcal{E} := \mathcal{F}(-2)$ is the bundle what we want. Since \mathcal{F} has a global sections, we have a following exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{F} \rightarrow \mathcal{I}_C(4) \rightarrow 0.$$

Twist by $\mathcal{O}_{\mathbb{P}^3}(-2)$, we obtain

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow \mathcal{F}(-2) \rightarrow \mathcal{I}_C(2) \rightarrow 0.$$

Because C is not contained in any quadric surface, we see that $H^0(\mathcal{I}_C(2)) = 0$. Therefore \mathcal{F} is stable since $H^0(\mathcal{F}(-2)) = 0$ and $c_1(\mathcal{F}(-2)) = 0$. Next we show \mathcal{F} is nef. Note that \mathcal{F} has 2 global sections which induce the generically surjective morphism $\varphi : \mathcal{O}^{\oplus 2} \rightarrow \mathcal{F}$ where φ is isomorphic outside S by the construction. Consequently \mathcal{F} is nef over curves not contained in S . Over S , we get an exact sequence $0 \rightarrow \mathcal{O}_S(C) \rightarrow \mathcal{F}|_S \rightarrow \mathcal{O}_S(4A - C) \rightarrow 0$. From the choice of C , $\mathcal{O}_S(C)$ is nef. We have only to check the nefness of $\mathcal{O}_S(4A - C)$. Since $(aA + bC)(4A - C) = 9a + 28b$, we must prove that $9a + 28b \geq 0$ if $aA + bC$ is effective. But this is true since $(28A - 9C)^2 = -17 < 0$ and in view of Kleiman-Mori cone. Therefore $-K_X = 2\xi_{\mathcal{F}}$ is nef and big. Namely \mathcal{F} is almost Fano. Hence $\mathcal{E} = \mathcal{F}(-2)$ is a stable 2-bundle with $c_1 = 0$, $c_2 = 3$ which is almost Fano. \square

Let $\mathcal{M}(0, 3)$ be the moduli space of stable rank 2 vector bundles on \mathbb{P}^2 with $c_1 = 0$ and $c_2 = 3$. From Theorem in [4], we see that $\mathcal{M}(0, 3)$ has two irreducible components $\mathcal{M}_0(0, 3)$ and $\mathcal{M}_1(0, 3)$ where $\mathcal{M}_\alpha(0, 3)$ is the moduli space of vector bundles \mathcal{E} satisfying the condition $\dim H^1(\mathcal{E}(-2)) \equiv \alpha \pmod{2}$. The dimension of each components are 21. Almost Fano bundles constructed in Proposition 2.6 are contained in $\mathcal{M}_0(0, 3)$. The author does not know whether $\mathcal{M}_1(0, 3)$ contains almost Fano bundles or not.

Next we show that each components contain the member which is not almost Fano.

EXAMPLE 2.8. From Proposition in [14], we see that vector bundles in $\mathcal{M}_0(0, 3)$ which have a maximal order jumping line is of dimension 20. Such a bundle \mathcal{E} is decomposed into $\mathcal{O}_L(3) \oplus \mathcal{O}_L(-3)$ over some line L . These bundles cannot be almost Fano since $\mathcal{E}(2)$ is not nef.

EXAMPLE 2.9. Let Y be a disjoint union of a plane cubic and a nonsingular space elliptic curve in \mathbb{P}^3 . By Serre construction, we can construct a rank 2 bundle \mathcal{F} on \mathbb{P}^3

with $c_1 = 4$, $c_2 = 7$. Then, we can check $H^0(\mathcal{F}(-2)) = 0$ due to the exact sequence $0 \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{F} \rightarrow \mathcal{I}_Y(4) \rightarrow 0$. Hence \mathcal{F} is stable. Since every nonsingular space elliptic curve is a complete intersection of two quadrics, we have $H^0(\mathcal{I}_Y(3)) = H^0(\mathcal{F}(-1)) \neq 0$. From easy computation, $(\xi_{\mathcal{F}} - H) \cdot (-K_{\mathbb{P}(\mathcal{F})})^3 = -1$. Thus $\mathcal{E} := \mathcal{F}(-2)$ is a stable vector bundle with $c_1 = 0$, $c_2 = 3$ which is not almost Fano. We can check

$$\dim H^1(E(-2)) = \dim H^1(\mathcal{I}_Y) = \dim H^0(\mathcal{O}_Y) - 1 = 1$$

using the exact sequence $0 \rightarrow \mathcal{I}_Y \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{O}_Y \rightarrow 0$. Hence \mathcal{E} is contained in $\mathcal{M}_1(0, 3)$.

(III) $n = 2$.

Finally, we consider the case where $n = 2$. If $c_1 = -1$, then $X = \mathbb{P}_{\mathbb{P}^2}(\mathcal{E})$ is an almost del Pezzo 3-fold and completely classified in [7]. So we may only study bundles with $c_1 = 0$. The statement is as follows.

PROPOSITION 2.10. *Let \mathcal{E} be a rank 2 almost Fano bundle on \mathbb{P}^2 with $c_1 = 0$. Then, \mathcal{E} is isomorphic to one of the following*

- (1) $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$,
- (2) $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}$,
- (3) \mathcal{E} is determined by $0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{E} \rightarrow \mathcal{I}_p \rightarrow 0$, where \mathcal{I}_p is the ideal sheaf of a point,
- (4) stable vector bundle with $2 \leq c_2 \leq 6$.

PROOF. In this case, $\mathcal{E}(2)$ is ample. If $H^0(\mathcal{E}(-1)) \neq 0$, we take a non-zero section $s \in H^0(\mathcal{E}(-1))$. If $Z := \{s = 0\} = \emptyset$, then \mathcal{E} is decomposed into a direct sum of line bundles. If $Z \neq \emptyset$, then for a line L meeting Z in a finite number of points we would have

$$\mathcal{E}(-1)|_L \cong \mathcal{O}_L(d) \oplus \mathcal{O}_L(-2-d), \quad d \geq 1.$$

This contradicts to the ampleness of $\mathcal{E}(2)$.

If $H^0(\mathcal{E}(-1)) = 0$ and $H^0(\mathcal{E}) \neq 0$, take a non-zero section $s \in H^0(\mathcal{E})$. If $Z := \{s = 0\} = \emptyset$, then \mathcal{E} is decomposed into a direct sum of line bundles. If $Z \neq \emptyset$ and $\deg Z \geq 2$, then for a line L intersecting with Z at least two points we would have

$$\mathcal{E}|_L \cong \mathcal{O}_L(d) \oplus \mathcal{O}_L(-d), \quad d \geq 2.$$

This is a contradiction.

If $\deg Z = 1$, \mathcal{E} has an exact sequence $0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{E} \rightarrow \mathcal{I}_p \rightarrow 0$, where \mathcal{I}_p is the ideal sheaf of a point p . In this case \mathcal{E} is Fano bundle by [17, Proposition 2.3]. Finally we consider the case $H^0(\mathcal{E}) = 0$ i.e. \mathcal{E} is stable. Then $2 \leq c_2 \leq 6$ since $(-K_X)^3 = 54 - 8c_2 > 0$. \square

We have some comments of Fano bundles with $c_1 = 0$. If $c_2 = 2$, all stable bundles are Fano bundle from [17]. In the situation $c_2 = 3$, there is a stable Fano bundles by [17]. Moreover, we have the following result.

PROPOSITION 2.11. *If \mathcal{E} is a stable almost Fano bundle on a projective plane with $c_1 = 0$, $c_2 = 3$. Then \mathcal{E} is Fano bundle.*

PROOF. Let \mathcal{E} be a stable almost Fano bundle on a projective plane with $c_1 = 0$, $c_2 = 3$. Using Riemann-Roch theorem, we have $\dim H^0(\mathcal{E}(1)) > 0$. Therefore we get an exact sequence $0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{E}(1) \rightarrow \mathcal{I}_Z(2) \rightarrow 0$ where Z is 4 points in \mathbb{P}^2 and \mathcal{I}_Z is the ideal sheaf of Z . If \mathcal{E} is not Fano, then the linear system $|\xi_{\mathcal{E}} + H|$ has one dimensional base locus B by [17], Claim 2.7. By virtue of Claim 2.10 and 2.11 in [17], we have $H.B \leq 2$ and $(\xi_{\mathcal{E}} + H).B \leq -1$. Since $0 \leq -K_{\mathbb{P}^2(\mathcal{E}).B} = 2(\xi_{\mathcal{E}} + H).B + H.B \leq 0$, we obtain $H.B = 2$ and $(\xi_{\mathcal{E}} + H).B = -1$. If $\pi(B)$ is a line L , we have $\mathcal{E}(1)|_L \cong \mathcal{O}(d) \oplus \mathcal{O}(2-d)$, $d \geq 3$. This contradicts the ampleness of $\mathcal{E}(2)$. If $\pi(B)$ is a two line, take a irreducible component L . In this case $\mathcal{E}(1)$ is not nef over $\pi(B)$, so over L . Therefore we have $\mathcal{E}(1)|_L \cong \mathcal{O}(d) \oplus \mathcal{O}(2-d)$, $d \geq 3$. This contradicts the ampleness of $\mathcal{E}(2)$. Finally we consider the case where $\pi(B)$ is nonsingular conic C . Since $(\xi_{\mathcal{E}} + H).B = -1$, we obtain the splitting $\mathcal{E}(1)|_C \cong \mathcal{O}_C(d) \oplus \mathcal{O}_C(4-d)$, $d \geq 5$. This is impossible because Z is only 4 points. \square

COROLLARY 2.12. *Let \mathcal{E} be a stable vector bundle on a projective plane with $c_1 = 0$, $c_2 = 3$. If $\mathcal{S}^2(\mathcal{E})(3)$ is nef, then $\mathcal{E}(1)$ is generated by global sections.*

PROOF. If $\mathcal{S}^2(\mathcal{E})(3)$ is nef, then \mathcal{E} is almost Fano. From Proposition 2.11, \mathcal{E} is Fano bundle. By means of Proposition 2.6 in [17], $\mathcal{E}(1)$ is generated by global sections. \square

When $c_2 = 4$, no stable 2-bundle is Fano [17]. We can construct almost Fano 2 bundle with $c_1 = 0$, $c_2 = 4$ as follows.

EXAMPLE 2.13. Let Y be 5 points in general position and C is a smooth conic containing Y . Then the pair (C, Y) yields us a rank 2 regular vector bundle \mathcal{F} with $c_1 = C$, $c_2 = Y$ by virtue of an elementary transform by [10] and [11]. We have a following exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{F} \rightarrow \mathcal{I}_Y(2) \rightarrow 0$$

where \mathcal{I}_Y is the ideal sheaf of Y . Twist by $\mathcal{O}_{\mathbb{P}^2}(-1)$, we get

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow \mathcal{F}(-1) \rightarrow \mathcal{I}_Y(1) \rightarrow 0.$$

Because there is no line containing Y , we have $H^0(\mathcal{I}_Y(1)) = 0$. Therefore \mathcal{F} is stable since $H^0(\mathcal{F}(-1)) = 0$ and $c_1(\mathcal{F}(-1)) = 0$. We check $-K_{\mathbb{P}^2(\mathcal{F})} = 2\xi_{\mathcal{F}} + H$ is nef. First we remark that \mathcal{F} has 2 global sections which induce the generically surjective morphism $\varphi : \mathcal{O}^{\oplus 2} \rightarrow \mathcal{F}$ where φ is isomorphic outside C by the construction. Hence we notice that $2\xi_{\mathcal{F}} + H$ is nef outside $\pi^{-1}(C)$. On C , we have that $\mathcal{F}|_C \cong \mathcal{O}_C(5) \oplus \mathcal{O}_C(-1)$ from the theory of elementary transformation. From this fact, we can check that $(2\xi_{\mathcal{F}} + H).D \geq 0$ for every curves D contained in Hirzebruch surface $\mathbb{P}^2(\mathcal{F}|_C)$. The equality holds only for the minimal section associated with the quotient line bundle $\mathcal{F}|_C \rightarrow \mathcal{O}_C(-1) \rightarrow 0$.

Therefore $-K_{\mathbb{P}^2(\mathcal{F})}$ is nef and big. Hence $\mathcal{E} := \mathcal{F}(-1)$ is a stable almost Fano bundle with $c_2 = 4$.

There exists an almost Fano stable bundle with $c_1 = 0$, $c_2 = 5$ from Theorem 0.19(C) in [18]. Finally we construct stable vector bundles with $c_1 = 0$, $3 \leq c_2 \leq 6$ which are not almost Fano.

EXAMPLE 2.14. Let $Y_k = \{p_0, p_1, \dots, p_k\}$ be the $k + 1$ points ($4 \leq k \leq 7$) in \mathbb{P}^2 . We assume that p_0, p_1, p_2 are lying in a line L and other points are not on L . By Serre construction, we have rank 2 vector bundles \mathcal{E}_k on \mathbb{P}^2 with $c_1 = 2$, $c_2 = k + 1$. \mathcal{E}_k has an exact sequence $0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{E}_k \rightarrow \mathcal{I}_{Y_k}(2) \rightarrow 0$. Because there is no line containing Y_k , we see $\dim H^0(\mathcal{E}_k(-1)) = \dim H^0(\mathcal{I}_{Y_k}) = 0$. Combining with $c_1(\mathcal{E}_k(-1)) = 0$, \mathcal{E}_k is stable bundle. Restricting each bundles into L , we get $\mathcal{E}_k|_L \cong \mathcal{O}_L(3) \oplus \mathcal{O}_L(-1)$. Since $\mathcal{E}_k(1)$ is not ample, \mathcal{E}_k is not almost Fano.

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