MI Preprint Series

Kyushu University
The Global COE Program
Math-for-Industry Education & Research Hub

The value distribution of the Gauss map of improper affine spheres

Yu Kawakami & Daisuke Nakajo

MI 2010-16

(Received April 9, 2010)

Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

THE VALUE DISTRIBUTION OF THE GAUSS MAP OF IMPROPER AFFINE SPHERES

YU KAWAKAMI AND DAISUKE NAKAJO

ABSTRACT. We give the best possible upper bounds for the number of exceptional values and totally ramified value number of the Lagrangian Gauss map of complete improper affine maps in the affine three-space. Moreover, by applying the Fujimoto argument, we also obtain the sharp estimate for them of weakly complete improper affine maps. As an application, from the viewpoint of the value distribution of Lagrangian Gauss map, we provide a new proof of the classification of affine complete improper affine spheres. Furthermore, we get a ramification estimate for the ratio of canonical forms of weakly complete flat fronts in hyperbolic three-space.

INTRODUCTION

The study of improper affine spheres has been related to other subjects in differential geometry. For instance, Calabi [Ca2] proved that there exists a local correspondence between solutions of the equation of improper affine spheres in the affine three-space $\mathbf{R}^3$ and solutions of the equation of minimal surfaces in Euclidean three-space. Recently, Martínez [Ma1] discovered the correspondence between improper affine spheres and smooth special Lagrangian immersions in the complex two-space $\mathbf{C}^2$. Moreover, from the viewpoint of this correspondence, he introduced the notion of improper affine maps, that is, a class of (locally strongly convex) improper affine spheres with some admissible singularities and gave a holomorphic representation formula for them. In the same paper, he defined the Lagrangian Gauss map of improper affine maps in $\mathbf{R}^3$ and obtained the characterization of a complete (in the sense of [Ma1], see also Section 1 of this paper) improper affine map whose Lagrangian Gauss map is constant. We remark that the second author [Na] constructed a representation formula for indefinite improper affine spheres with some admissible singularities.

On the other hand, the study of value distribution property of the Gauss map of minimal surfaces in Euclidean three-space has accomplished many significant results. This study

---

Date: 9th April, 2010.

2000 *Mathematics Subject Classification.* Primary 53A15 ; Secondary 30D35, 53C42, 53A35.

Key words and phrases. improper affine sphere, Lagrangian Gauss map, complete, weakly complete, totally ramified value number, flat front.

The first author is partially supported by the Grants-in-Aid for Young Scientists (B) No. 21740053, Japan Society for the Promotion of Science and Global COE program (Kyushu university) "Education and Research Hub for Mathematics-for-Industry".

is a generalization of the classical Bernstein theorem [Be] and initiated by Osserman [Os1, Os2, Os3]. In particular, Fujimoto [Fu1, Fu2] showed that the best possible upper bound for the number D_g of exceptional values and the totally ramified value number δ_g of the Gauss map g of complete nonflat minimal surfaces in Euclidean three-space is four. Moreover, Osserman [Os2, Os3] proved that the Gauss map of a nonflat algebraic minimal surface can omit at most three values (By an algebraic minimal surface, we mean a complete minimal surface with finite total curvature). Furthermore, the first author, Kobayashi and Miyaoka [KKM] gave an effective estimate for D_g and δ_g of a wider class of complete minimal surfaces that includes algebraic minimal surfaces (this class is called pseudo-algebraic). It also provided new proofs of the Fujimoto and the Osserman theorems in this class and revealed the geometric meaning behind them. We got the same estimate for the hyperbolic Gauss map of pseudo-algebraic constant mean curvature one (CMC-1, for short) surfaces in hyperbolic three-space $\mathbf{H}^3$ [Ka2]. These estimates correspond to the defect relation in Nevanlinna theory ([JR], [Ko], [NO] and [Ru]).

The aim of this paper is to study the value distribution of the Lagrangian Gauss map of improper affine maps in $\mathbf{R}^3$. The organization of this paper is as follows: In Section 1, we recall some definitions and basic facts about improper affine maps in $\mathbf{R}^3$ which are used throughout this paper. In particular, we review the definitions of completeness in the sense of [Ma1] and weakly completeness in the sense of [UY]. In Section 2, we give the upper bound for the totally ramified value number δ_ν of the Lagrangian Gauss map ν of complete improper affine maps (Theorem 2.2). This estimate is effective in the sense that the upper bound which we obtained is described in terms of geometric invariants and sharp for some topological cases. Moreover, as a corollary of this estimate, we also obtain the best possible upper bound for the number D_ν of exceptional values of the Lagrangian Gauss map in this class (Corollary 2.4). We remark that this class corresponds to that of algebraic minimal surfaces in Euclidean three-space. In Section 3, by applying the Fujimoto argument, we give the best possible estimates for δ_ν and D_ν of weakly complete improper affine maps (Theorem 3.2 and Corollary 3.3). We remark that the best possible upper bound for δ_ν of this class is “three”, while the best possible upper bound for δ_g of complete minimal surfaces obtained by Fujimoto is “four”. As an application of this estimate, from the viewpoint of the value distribution of Lagrangian Gauss map, we provide a new proof of the well-known result (see [Ca1], [Ca3], [Jo], [LSZ], [TW]) that any affine complete improper affine sphere must be an elliptic paraboloid (Corollary 3.7). In Section 4, after reviewing some definitions and fundamental properties on flat fronts in $\mathbf{H}^3$, we study the value distribution of the ratio of canonical forms of weakly complete flat fronts in $\mathbf{H}^3$. Flat surfaces (resp. fronts) in $\mathbf{H}^3$ are closely related to improper affine spheres (resp. map) in $\mathbf{R}^3$ (See [Ma2] and also [IM]). As an application of our results, we have the classification of weakly complete flat surfaces in $\mathbf{H}^3$ (Corollary 4.6). By Corollary

3.7 and 4.6, we understand that classifications for complete surfaces in these classes follow from the result of boundedness of their Gauss maps.

Finally, the authors would like to particular thank to Professors Wayne Rossman, Masaaki Umehara and Kotaro Yamada for their useful advice. The authors also thank to Professors Ryoichi Kobayashi, Masatoshi Kokubu, Reiko Miyaoka, Junjiro Noguchi and Yoshihiro Ohnita for their encouragement of our study.

1. Preliminaries

We first briefly recall some definitions and basic facts on affine differential geometry. Details can be found, for instance, in [LSZ] and [NS].

Let Σ be an oriented two-manifold, and (ψ, ξ) a pair of an immersion $\psi\colon \Sigma \to \mathbf{R}^3$ into the affine three-space $\mathbf{R}^3$ and a vector field ξ on Σ along ψ which is transversal to $\psi_*(T\Sigma)$. Then the Gauss-Weingarten equations of (ψ, ξ) are as follows:

$$\left\{\begin{array}{l} D_X\psi_*Y = \psi_*(\nabla_X Y) + g(X,Y)\xi, \\ D_X\xi = -\psi_*(SX) + \tau(X)\xi, \end{array}\right. \tag{1.1}$$

where D is the standard flat connection on $\mathbf{R}^3$. Here, g is called the *affine metric* of the pair (f, ξ). Indeed, we can easily show that the rank of g is invariant under the change of the transversal vector field ξ. In particular, we call ψ a *locally strongly convex immersion* when g is positive definite. From now on, we only consider the locally strongly convex case. Given an immersion $\psi\colon \Sigma \to \mathbf{R}^3$, we can uniquely choose the transversal vector field ξ which satisfies the following conditions:

(i) $\tau \equiv 0$ (or equivalently $D_X\xi \in \psi_*(T\Sigma)$ for all $X \in \mathfrak{X}(\Sigma)$) ,
(ii) $\mathrm{vol}_g(X_1, X_2) = \det(\psi_*X_1, \psi_*X_2, \xi)$ for all $X_1, X_2 \in \mathfrak{X}(\Sigma)$,

where vol_g is the volume form of the Riemannian metric g and det is the standard volume element of $\mathbf{R}^3$. The transversal vector field ξ which satisfies the two conditions above is called a *Blaschke normal* (or *affine normal*), and a pair (ψ, ξ) of an immersion and its Blaschke normal is called a *Blaschke immersion.* A Blaschke immersion (f, ξ) with $S = 0$ in (1.1) is called an *improper affine sphere.* In this case, the transversal vector field ξ is constant because $\tau \equiv 0$. Thus a transversal vector field ξ of an improper affine sphere is given by $\xi = (0, 0, 1)$ after a suitable affine transformation of $\mathbf{R}^3$. The *conormal map* $N\colon \Sigma \to (\mathbf{R}^3)^*$ into the dual space of the affine three-space $(\mathbf{R}^3)^*$ for a given Blaschke immersion (f, ξ) is defined as the immersion which satisfy the following conditions:

(i) $N(f_*X) = 0$ for all $X \in \mathfrak{X}(\Sigma)$,
(ii) $N(\xi) = 1$.

For an improper affine sphere with Blaschke normal $(0, 0, 1)$, we can write $N = (n, 1)$ with a smooth map $n\colon \Sigma \to \mathbf{R}^2$.

Next, using the notations defined as above, we introduce the notion of improper affine maps, which is a generalization of improper affine spheres with some admissible singularities. This class was first defined in [Ma1].

Definition 1.1 ([Ma1, Definition 1])**.** A map $\psi = (x, \varphi)\colon \Sigma \to \mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$ is called an *improper affine map* if ψ is expressed as

$$\psi = \left(x, -\int \langle n, dx \rangle\right)$$

by a special Lagrangian immersion $L_\psi = x + \sqrt{-1}n\colon \Sigma \to \mathbf{C}^2$ with respect to double the symplectic structure $2\omega' = \sqrt{-1}(d\zeta_1 \wedge d\bar{\zeta_1} + d\zeta_2 \wedge d\bar{\zeta_2})$ and the calibration $\Re(\sqrt{-1}\Omega') = \Re(\sqrt{-1}d\zeta_1 \wedge d\zeta_2)$. Here, $\mathbf{C}^2$ is the complex two-space with coordinates $\zeta = (\zeta_1, \zeta_2)$, where $\zeta = x + \sqrt{-1}y$ $(x, y \in \mathbf{R}^2)$.

Nonregular points of ψ correspond with degenerate points of $ds^2 := \langle dx, dx \rangle$. Following the terminology of [Ma1], we call ds^2 the *flat fundamental form* of ψ. We remark that at the nondegenerate points of ds^2, the induced metric $d\tau^2 := \langle dx, dx \rangle + \langle dn, dn \rangle$ is conformal to the affine metric $g := -\langle dx, dn \rangle$ [Ma1, Theorem 1 and Theorem 2].

For any improper affine map $\psi\colon \Sigma \to \mathbf{R}^3$, considering the conformal structure given by the induced metric $d\tau^2$ of its associated special Lagrangian immersion L_ψ, we regard Σ as a Riemann surface.

Since every special Lagrangian immersion in $\mathbf{C}^2$ is a complex curve up to a change of the complex sturucture ([CM], [HL]), we see that there exists a complex regular curve $\alpha\colon \Sigma \to \mathbf{C}^2$, $\alpha := (F, G)$, such that if we identify vectors of $\mathbf{R}^2$ with complex numbers in the standard way:

$$(r, s) = r + \sqrt{-1}s, \quad r,\ s \in \mathbf{R}$$

then we can write

$$x = G + \bar{F}, \quad n = \bar{F} - G \tag{1.2}$$

and since the inner product of two vectors $\zeta_i = r_i + \sqrt{-1}s_i$ $(i = 1, 2)$ is given by $\langle \zeta_1, \zeta_2 \rangle = \Re(\zeta_1\bar{\zeta_2})$, then the flat fundamental form ds^2, the induced metric $d\tau^2$ and the affine metric g are given, respectively, by

$$\begin{aligned} ds^2 &= |dF + dG|^2 = |dF|^2 + |dG|^2 + dGdF + \overline{dFdG} \\ d\tau^2 &= 2(|dF|^2 + |dG|^2) \\ g &= |dG|^2 - |dF|^2\,. \end{aligned} \tag{1.3}$$

Moreover, the nontrivial part of the Gauss map of L_ψ (see [CM]) is the holomorphic map $\nu\colon \Sigma \to \mathbf{C} \cup \{\infty\}$ given by

$$\nu := \frac{dF}{dG} \tag{1.4}$$

which be called the *Lagrangian Gauss map* of ψ.

Then Martínez [Ma1] gave the following representation formula for improper affine maps in terms of two holomorphic functions. This generalized a formula in [FMM].

Fact 1.2 ([Ma1, Theorem 3]). *Let* $\psi=(x,\varphi)\colon \Sigma\to \mathbf{R}^3=\mathbf{C}\times\mathbf{R}$ *be an improper affine map. Then there exists a regular complex curve* $\alpha:=(F,G)\colon\Sigma\to\mathbf{C}^2$ *such that*

$$\psi:=\left(G+\bar{F},\frac{|G|^2-|F|^2}{2}+\Re\left(GF-\int FdG\right)\right). \tag{1.5}$$

Here, the conormal map of ψ *becomes*

$$N=(\bar{F}-G,1)\,.$$

Conversely, given a Riemann surface Σ *and a regular complex curve* $\alpha:=(F,G)\colon\Sigma\to\mathbf{C}^2$*, then (1.5) gives an improper affine map which is well defined if and only if* $\int FdG$ *does not have real periods.*

We call the pair (F,G) the *Weierstrass data* of ψ. Note that the singular points of ψ correspond with the points where $|dF|=|dG|$, that is, $|\nu|=1$ ([Ma1], see also [Na]).

An improper affine map $\psi\colon\Sigma\to\mathbf{R}^3$ is said to be *complete* if there exists a symmetric two-tensor T such that $T=0$ outside a compact set $C\subset\Sigma$ and ds^2+T is a complete Riemannian metric on Σ, where ds^2 is the flat fundamental form of ψ. This definition is similar to the definition of completeness for fronts [KUY2]. We remark that an improper affine map in $\mathbf{R}^3$ is a front ([Na], [UY]).

Fact 1.3 (Huber, Martínez). *A complete improper affine map* $\psi\colon\Sigma\to\mathbf{R}^3$ *satisfies the following two conditions:*

- (i) Σ *is biholomorphic to* $\overline{\Sigma}_\gamma\setminus\{p_1,\dots,p_k\}$*, where* $\overline{\Sigma}_\gamma$ *is a closed Riemann surface of genus* γ *and* $p_j\in\overline{\Sigma}_\gamma$ $(j=1,\dots,k)$*,* [Hu].
- (ii) *The Weierstrass data* (F,G) *of* ψ *can be extended meromorphically to* $\overline{\Sigma}_\gamma$*. In particular, its Lagrangian Gauss map can also be a meromorphic function on* $\overline{\Sigma}_\gamma$*,* [Ma1].

Each puncture point p_j $(j=1,\dots,k)$ is called an *end* of ψ. From Fact 1.3, complete improper affine maps have some properties similar to algebraic minimal surfaces in $\mathbf{R}^3$. For example, an analogue of the Osserman inequality [Ma1, Theorem 4] holds for complete improper affine maps. On the other hand, an improper affine map is said to be *weakly complete* if the induced metric $d\tau^2$ as in (1.3) is complete. We remark that the universal cover of a weakly complete improper affine map is also weakly complete, but completeness is not preserved when lifting to the universal cover. The relationship between completeness and weakly completeness in this class is as follows:

Fact 1.4 ([UY]). *An improper affine map in* $\mathbf{R}^3$ *is complete if and only if it is weakly complete, the singular set is compact and all ends are biholomorphic to a puncture disk.*

Finally, we give two examples in [Ma1, Section 4] which play important roles in the following sections.

Example 1.5 (Elliptic paraboloids)**.** An elliptic paraboloid can be obtained by taking $\Sigma = \mathbf{C}$ and Weierstrass data (cz, z), where c is constant. It is complete, and its Lagrangian Gauss map is constant. Note that, if $|c| = 1$, then an improper affine map constructed from this data is a line in $\mathbf{R}^2$.

Example 1.6 (Rotational improper affine maps)**.** A rotational improper affine map is obtained by considering $\Sigma = \mathbf{C}\backslash\{0\}$ and Weierstrass data $(z, \pm r^2/z)$, where $r \in \mathbf{R}\backslash\{0\}$. It is complete and its Lagrangian Gauss map $\nu = \mp z^2/r^2$. In particular, ν omits two values, 0, ∞.

In complete class, elliptic paraboloids are characterized by the Lagrangian Gauss map as follows:

Fact 1.7 ([Ma1])**.** *A complete improper affine map is an elliptic paraboloid if and only if its Lagrangian Gauss map is constant.*

2. On the upper bound for the totally ramified value number of the Lagrangian Gauss map of complete improper affine maps

We first define the totally ramified value number δ_ν of ν.

Definition 2.1 (Nevanlinna [Ne])**.** We call $b \in \mathbf{C} \cup \{\infty\}$ a *totally ramified value* of ν when ν branches at any inverse image of b. We regard exceptional values also as totally ramified values. Let $\{a_1, \dots, a_{r_0}, b_1, \dots, b_{l_0}\} \subset \mathbf{C} \cup \{\infty\}$ be the set of totally ramified values of ν, where a_j's are exceptional values. For each a_j, set $m_j = \infty$, and for each b_j, define m_j to be the minimum of the multiplicities of ν at points $\nu^{-1}(b_j)$. Then we have $m_j \geq 2$. We call

$$\delta_\nu = \sum_{a_j, b_j} \left(1 - \frac{1}{m_j}\right) = r_0 + \sum_{j=1}^{l_0} \left(1 - \frac{1}{m_j}\right)$$

the *totally ramified value number* of ν.

We next give the upper bound for δ_ν of complete improper affine maps. Here, we denote by D_ν the number of exceptional values of ν. By definition, it follows immediately that $D_\nu \leq \delta_\nu$.

Theorem 2.2. *Let $\psi\colon \Sigma = \overline{\Sigma}_\gamma \backslash \{p_1, \dots, p_k\} \to \mathbf{R}^3$ be a complete improper affine map and $\nu\colon \Sigma \to \mathbf{C} \cup \{\infty\}$ be the Lagrangian Gauss map of ψ. Suppose that ν is nonconstant and d is the degree of ν considered as a map on $\overline{\Sigma}_\gamma$. Then we have*

$$D_\nu \leq \delta_\nu \leq 2 + \frac{2}{R}, \qquad \frac{1}{R} = \frac{\gamma - 1 + k/2}{d} < \frac{1}{2}. \tag{2.1}$$

In particular, $D_\nu \leq \delta_\nu < 3$.

Remark 2.3. The geometric meaning of “2” in the upper bound of (2.1) is the Euler number of the Riemann sphere. The geometric meaning of the ratio R is given in [KKM, Section 6].

Proof. By Fact 1.4, if ds^2 is complete, then $d\tau^2$ is a complete Riemannian metric. Then the metric $d\tau^2$ is represented as

$$d\tau^2 = |dF|^2 + |dG|^2 = \left(1 + \left|\frac{dF}{dG}\right|^2\right)|dG|^2 = (1 + |\nu|^2)|dG|^2 \,. \tag{2.2}$$

Since $d\tau^2$ is nondegenerate on Σ, the poles of ν of order k coincide exactly with the zeros of dG of order k. By the completeness of $d\tau^2$, dG has a pole of order $\mu_j \geq 1$ at p_j [Os3]. Moreover, we show that $\mu_j \geq 2$ for each p_j because G is single-valued on $\overline{\Sigma}_\gamma$. Applying the Riemann-Roch theorem to dG on $\overline{\Sigma}_\gamma$, we have

$$d - \sum_{j=1}^{k} \mu_j = 2\gamma - 2 \,. \tag{2.3}$$

Thus we get

$$d = 2\gamma - 2 + \sum_{j=1}^{k} \mu_j \geq 2(\gamma - 1 + k) > 2\left(\gamma - 1 + \frac{k}{2}\right), \tag{2.4}$$

and

$$R^{-1} < \frac{1}{2} \,. \tag{2.5}$$

Assume that ν omits $r_0 = D_\nu$ values. Let n_0 be the sum of the branching orders at the image of these exceptional values of ν. Then we have

$$k \geq dr_0 - n_0 \,. \tag{2.6}$$

Let $b_1, \ldots, b_{l_0}$ be the totally ramified values which are not exceptional values and n_r the sum of branching order at the inverse image of b_i $(i = 1, \ldots, l_0)$ of ν. For each b_i, we denote

$$m_i = \min_{\nu^{-1}(b_i)}\{\text{multiplicity of } \nu(z) = b_i\} \,,$$

then the number of points in the inverse image $\nu^{-1}(b_i)$ is less than or equal to d/m_i. Thus we get

$$dl_0 - n_r \leq \sum_{i=1}^{l_0} \frac{d}{m_i} \,. \tag{2.7}$$

This implies

$$l_0 - \sum_{i=1}^{l_0} \frac{1}{m_i} \leq \frac{n_r}{d} \,. \tag{2.8}$$

Let n_ν be the total branching order of ν on $\overline{\Sigma}_\gamma$. Then applying the Riemann-Hurwitz formula to the meromorphic function ν on $\overline{\Sigma}_\gamma$, we have

$$n_\nu = 2(d+\gamma-1)\,. \tag{2.9}$$

Therefore, we obtain

$$\delta_\nu = r_0 + \sum_{j=1}^{l_0}\left(1-\frac{1}{m_j}\right) \leq \frac{n_0+k}{d} + \frac{n_r}{d} \leq \frac{n_\nu+k}{d} = 2+\frac{2}{R}\,.$$

□

The system of inequalities (2.1) is sharp in the following cases:
(i) When $(\gamma, k, d) = (0, 1, n)$ $(n \in \mathbf{N})$, we have

$$\delta_\nu \leq 2 - \frac{1}{n}\,.$$

In this case, we can set $\Sigma = \mathbf{C}$. Since Σ is simply connected, we have no period condition. We define a Weierstrass data on Σ, by

$$(F, G) = \left(\frac{z^{n+1}}{n+1},\, z\right). \tag{2.10}$$

Then, by Fact 1.2, we can construct a complete improper affine map $\psi\colon \Sigma \to \mathbf{R}^3$ whose Weierstrass data is (2.10). In particular, its Lagrangian Gauss map ν has $\delta_\nu = 2-(1/n)$. In fact, $\nu = z^n$, and it has one exceptional value and another totally ramified value of multiplicity n at $z=0$. Thus (2.1) is sharp in this case.
(ii) When $(\gamma, k, d) = (0, 2, 2)$, we have

$$D_\nu \leq \delta_\nu \leq 2\,.$$

In this case, we can set $\Sigma = \mathbf{C}\backslash\{0\}$. On the other hand, a rotational improper affine map (Example 1.6) has $D_\nu = \delta_\nu = 2$. Thus (2.1) is also sharp in this case.

As a corollary of Theorem 2.2, we obtain the maximal number of the exceptional values of the Lagrangian Gauss map of complete improper affine maps.

Corollary 2.4. *Let ψ be a complete improper affine map. If its Lagrangian Gauss map ν is nonconstant, then ν can omit at most two values.*

The number "two" is sharp, because the Lagrangian Gauss map of a rotational improper affine map omits two values. Hence we provide the best possible upper bound for D_ν in complete case.

3. A RAMIFICATION ESTIMATE FOR THE LAGRANGIAN GAUSS MAP OF WEAKLY COMPLETE IMPROPER AFFINE MAPS

In this section, we study the value distribution of the Lagrangian Gauss map of weakly complete improper affine maps. We begin to consider the case where the Lagrangian Gauss map is constant.

Proposition 3.1. *Let* $\psi\colon \Sigma \to \mathbf{R}^3$ *be a weakly complete improper affine map. If its Lagrangian Gauss map* ν *is constant, then* ψ *is an elliptic paraboloid.*

Proof. Since the metric $d\tau^2$ is represented as (2.2), if ν is constant, then the Gaussian curvature $K_{d\tau^2}$ of $d\tau^2$ vanishes identically on Σ. By the Huber theorem, Σ is a closed Riemann surface of genus γ with k points removed, that is, $\Sigma = \overline{\Sigma}_\gamma \backslash \{p_1, \dots, p_k\}$. Moreover, we obtain the formula (see [Fa, Corollary 1] or [Sh])

$$\frac{1}{2\pi}\int_{\Sigma}(-K_{d\tau^2})dA = -\chi(\overline{\Sigma}_\gamma) - \sum_{j=1}^{k} \mathrm{ord}_{p_j}(d\tau^2),$$

where dA denotes the area element of $d\tau^2$ and $\chi(\overline{\Sigma}_\gamma)$ the Euler number of $\overline{\Sigma}_\gamma$. Since the metric $d\tau^2$ is complete, $\mathrm{ord}_{p_j} d\tau^2 \leq -1$ holds for each end p_j. Thus if ν is constant, then we get $\gamma = 0$ and

$$\sum_{j=1}^{k} \mathrm{ord}_{p_j}(d\tau^2) = -2\,. \tag{3.1}$$

Since $d\tau^2$ is well-defined on Σ, the following two cases are considered:

(a) The improper affine map ψ has two ends p and q, and $\mathrm{ord}_p d\tau^2 = \mathrm{ord}_q d\tau^2 = -1$,
(b) The improper affine map ψ has one end p, and $\mathrm{ord}_p d\tau^2 = -2$.

In this class, the case of (a) cannot occur because F and G are single-valued on Σ. Thus we have only to consider the case of (b). Then we may assume that $p = \infty$ after a suitable Möbius transformation of the Riemann sphere $\overline{\Sigma}_0$. Since ν is constant, dF and dG are well-defined on $\overline{\Sigma}_0$, and it holds that

$$\mathrm{ord}_\infty dF = \mathrm{ord}_\infty dG = -2\,.$$

Hence, for some constant c, we have $dF = c dz$ and $dG = dz$, that is, $F(z) = cz$ and $G(z) = z$. Therefore, the result follows from Example 1.5. $\square$

We next give a ramification estimate for the Lagrangian Gauss map of weakly complete improper affine maps.

Theorem 3.2. *Let* $\psi\colon \Sigma \to \mathbf{R}^3$ *be a weakly complete improper affine map and* $\nu\colon \Sigma \to \mathbf{C} \cup \{\infty\}$ *its Lagrangian Gauss map. Assume that there exist distinct values* $\alpha_1, \dots, \alpha_q$

and positive integers $m_1, \dots, m_q$ *such that the multiplicities of* ν *on the inverse image* $\nu^{-1}(\alpha_j)$ *are greater than or equal to* m_j *for each* j*, and*

$$\gamma := \sum_{j=1}^{q}\left(1 - \frac{1}{m_j}\right) > 3\,.$$

Then ν *is constant, that is,* ν *must be an elliptic paraboloid.*

As a corollary, we can prove the following:

Corollary 3.3. *Let* ψ *be a weakly complete improper affine map in* $\mathbf{R}^3$*. If its Lagrangian Gauss map* ν *is nonconstant, then* ν *can omit at most three values.*

The number "three" is sharp because there exist the following examples.

Example 3.4 (Voss type of improper affine maps)**.** We consider the Lagrangian Gauss map ν and the holomorphic one-form dG on $\Sigma = \mathbf{C}\backslash\{a_1, a_2\}$ for distinct points $a_1, a_2 \in \mathbf{C}$, by

$$\nu = z, \quad dG = \frac{dz}{\prod_j (z - a_j)}\,. \tag{3.2}$$

As F and G are not well-defined on Σ, we obtain an improper affine map $\psi\colon \mathbf{D} \to \mathbf{R}^3$ on the universal covering disk $\mathbf{D}$ of Σ. Since the metric $d\tau^2$ is complete, we can get a weakly complete improper affine map whose Lagrangian Gauss map omits three values, a_1, a_2 and ∞.

Before proceeding to the proof of Theorem 3.2, we recall two function-theoretical lemmas. For two distinct values α, $\beta \in \mathbf{C} \cup \{\infty\}$, we set

$$|\alpha, \beta| := \frac{|\alpha - \beta|}{\sqrt{1 + |\alpha|^2}\sqrt{1 + |\beta|^2}}$$

if $\alpha \neq \infty$ and $\beta \neq 0$, and $|\alpha, \infty| = |\infty, \alpha| := 1/\sqrt{1 + |\alpha|^2}$. Note that, if we take v_1, $v_2 \in \mathbf{S}^2$ with $\alpha = \varpi(v_1)$ and $\beta = \varpi(v_2)$, we have that $|\alpha, \beta|$ is a half of the chordal distance between v_1 and v_2, where ϖ denotes the stereographic projection of $\mathbf{S}^2$ onto $\mathbf{C} \cup \{\infty\}$.

Lemma 3.5 ([Fu2, Corollary 1.4.15])**.** *Let* ν *be a nonconstant meromorphic function on* $\Delta_R = \{z \in \mathbf{C}\,;\, |z| < R\}$ $(0 < R \leq +\infty)$*. Assume that there exist distinct values* $\alpha_1, \dots, \alpha_q$ *and positive integers* $m_1, \dots, m_q$ *such that the multiplicities of* ν *on the inverse image* $\nu^{-1}(\alpha_j)$ *are greater than or equal to* m_j *for each* j*, and*

$$\gamma := \sum_{j=1}^{q}\left(1 - \frac{1}{m_j}\right) > 2\,.$$

Then, for arbitrary positive constants η *and* δ *with* $\gamma - 2 > \gamma\eta + \gamma\delta$, *it holds that*

$$\frac{|\nu'|}{1+|\nu|^2}\frac{1}{(\prod_{j=1}^{q}|\nu, \alpha_j|^{1-1/m_j})^{1-\eta-\delta}} \leq C\frac{2R}{R^2-|z|^2}\,, \tag{3.3}$$

where C *is some constant depending only on* γ, η, δ *and* $L := \min_{i<j}|\alpha_i, \alpha_j|$.

Lemma 3.6 ([Fu2, Lemma 1.6.7])**.** *Let* $d\sigma^2$ *be a conformal flat metric on an open Riemann surface* Σ. *Then, for each point* $p \in \Sigma$, *there exists a local diffeomorphism* Φ *of a disk* $\Delta_{R_0} = \{z \in \mathbf{C}\,;\, |z| < R_0\}$ $(0 < R_0 \leq +\infty)$ *onto an open neighborhood of* p *with* $\Psi(0) = p$ *such that* Ψ *is a local isometry, namely, the pull-back* $\Psi^*(d\sigma^2)$ *is equal to the standard Euclidean metric on* Δ_{R_0} *and, for a point* a_0 *with* $|a_0| = 1$, *the* Ψ*-image* Γ_{a_0} *of the curve* $L_{a_0} = \{w := a_0 s\,;\, 0 < s < R_0\}$ *is divergent in* Σ.

Proof of Theorem 3.2. This is proved by contradiction. For our purpose, we may assume $\alpha_q = \infty$ and that Σ is biholomorphic to the unit disk because Σ can be replaced by its universal covering surface and Theorem 3.2 is obvious in the case where $\Sigma = \mathbf{C}$ by Nevanlinna theory [Ne]. We choose some δ such that $\gamma - 3 > 2\gamma\delta > 0$ and set

$$\eta := \frac{\gamma - 3 - 2\gamma\delta}{\gamma}, \quad \lambda := \frac{1}{1+\gamma\delta}\,. \tag{3.4}$$

Then, if we choose a sufficiently small positive δ depending only on γ, for constant $\varepsilon_0 = (\gamma - 3)/\gamma$ we have

$$0 < \lambda < 1, \quad \frac{\varepsilon_0\lambda}{1-\lambda}\left(= \frac{\gamma-3}{\gamma^2\delta}\right) > 1\,. \tag{3.5}$$

Now we define a new metric

$$d\sigma^2 = |G'_z|^{2/(1-\lambda)}\left(\frac{1}{|\nu'_z|}\prod_{j=1}^{q-1}\left(\frac{|\nu - \alpha_j|}{\sqrt{1+|\alpha_j|}}\right)^{\eta_j(1-\eta-\delta)}\right)^{2\lambda/(1-\lambda)}|dz|^2 \tag{3.6}$$

on the set $\Sigma' := \{z \in \Sigma\,;\, \nu'_z(z) \neq 0 \text{ and } \nu(z) \neq \alpha_j \text{ for all } j\}$, where $dG = G'_z dz$, $\nu'_z = d\nu/dz$ and $\eta_j = 1 - 1/m_j$. Take a point $p \in \Sigma'$. Since the metric $d\sigma^2$ is flat on Σ', by Lemma 3.6, there exists a local isometry Φ satisfying $\Phi(0) = p$ from a disk $\Delta(R) = \{z \in \mathbf{C}\,;\, |z| < R\}$ $(0 < R \leq +\infty)$ with the standard Euclidean metric onto an open neighborhood of p in Σ' with the metric $d\sigma^2$, such that, for a point a_0 with $|a_0| = 1$, the Ψ-image Γ of the curve $L_{a_0} = \{w := a_0 s\,;\, 0 < s < R\}$ is divergent in Σ'. For brevity, we denote the function $\nu \circ \Psi$ on $\Delta(R)$ by ν in the followings. By Lemma 3.5, we get

$$R \leq 2C\frac{1+|\nu(0)|^2}{\nu'_z(0)}\prod_{j=1}^{q}|\nu(0),\, \alpha_j|^{\eta_j(1-\eta-\delta)} < +\infty\,. \tag{3.7}$$

Hence,

$$L_{d\sigma}(\Gamma) = \int_\Gamma d\sigma = R < +\infty\,,$$

where $L_{d\sigma}(\Gamma)$ denotes the length of Γ with respect to the metric $d\sigma^2$.

Then the image $\Psi(\Gamma)$ of Γ diverges in Σ as $t \to 1$. Because, if not, $\Psi(\Gamma)$ is divergent in Σ' and $L_{d\sigma}(\Gamma) < +\infty$, it must tend to a point p_0 where $\nu'_z(p_0) = 0$ or $\nu(p_0) = \alpha_j$ for some j. Taking a local complex coordinate ζ in a neighborhood of p_0 with $\zeta(p_0) = 0$, we can write the metric $d\sigma^2$ as $d\sigma^2 = |\zeta|^{2a\lambda/(1-\lambda)} w |d\zeta|^2$ with some positive smooth function w and some real number a. If $\nu - \alpha_j$ has a zero of order $m (\geq m_j \geq 2)$ at p_0 for some $j \leq q-1$, then ν'_z has a zero of order $m-1$ at p_0 and $G'_z(p_0) \neq 0$. Then we have

$$\begin{aligned} a &= m\Big(1 - \frac{1}{m_j}\Big)(1 - \eta - \delta) - (m-1) \\ &= 1 - \frac{m}{m_j} - \frac{m}{m_j}(m_j - 1)(\eta + \delta) \\ &\leq -(\eta + \delta) \leq -\varepsilon_0 \,. \end{aligned}$$

For the case where ν has a pole of order $m (\geq m_q)$, ν'_z has a pole of order $m+1$, G'_z has a zero of order $2m$ at p_0 and each component $\nu - \alpha_j$ in the right hand side of (3.6) has a pole of order m at p_0. Using the identity $\eta_1 + \cdots + \eta_{q-1} = \gamma - \eta_q$ and (3.5), we get

$$\begin{aligned} a &= \frac{m}{\lambda} + m + 1 - m(\gamma - \eta_q)(1 - \eta - \delta) \\ &= m\eta_q(1 - \eta - \delta) - (m-1) \leq -\varepsilon_0 \,. \end{aligned}$$

Moreover, for the case where $\nu'_z(p_0) = 0$ and $\nu(p_0) \neq \alpha_j$ for all j, then we see $a \leq -1$. In any case, $a\lambda/(1-\lambda) \leq -1$ by (3.5), and there exists a positive constant C' such that

$$d\sigma \geq C' \frac{|d\zeta|}{|\zeta|}$$

in a neighborhood of p_0. Thus we have

$$R = \int_\Gamma d\sigma \geq C' \int_\Gamma \frac{|d\zeta|}{|\zeta|} = +\infty \,,$$

which contradicts (3.7).

On the other hand, since $d\sigma^2 = |dz|^2$, we obtain by (3.6)

$$|G'_z| = \Bigg(|\nu'_z| \prod_{j=1}^{q-1} \Bigg(\frac{\sqrt{1 + |\alpha_j|^2}}{|\nu - \alpha_j|}\Bigg)^{\eta_j(1-\eta-\delta)}\Bigg)^{\lambda} . \tag{3.8}$$

By Lemma 3.5, we have

$$\begin{aligned} \Psi^* d\tau &= |G'_z| \sqrt{1 + |\nu|^2} |dz| \\ &= \Bigg(|\nu'_z| (1 + |\nu|^2)^{1/2\lambda} \prod_{j=1}^{q-1} \Bigg(\frac{\sqrt{1 + |\alpha_j|^2}}{|\nu - \alpha_j|}\Bigg)^{\eta_j(1-\eta-\delta)}\Bigg)^{\lambda} |dz| \\ &= \Bigg(\frac{|\nu'_z|}{1 + |\nu|^2} \frac{1}{\prod_{j=1}^{q} |\nu, \alpha_j|^{\eta_j(1-\eta-\delta)}}\Bigg)^{\lambda} |dz| \\ &\leq C^{\lambda} \Bigg(\frac{2R}{R^2 - |z|^2}\Bigg)^{\lambda} |dz| \,. \end{aligned}$$

Thus, if we denote the distance $d(p)$ from a point $p \in \Sigma$ to the boundary of Σ as the greatest lower bound of the lengths with respect to the metric $d\tau^2$ of all divergent paths in Σ, then we have

$$d(p) \leq \int_{\Psi(\Gamma)} d\tau = \int_{\Gamma} \psi^* d\tau \leq C^\tau \int_{\Gamma} \left(\frac{2R}{R^2 - |z|^2} \right)^{\lambda} |dz| < +\infty\,.$$

However, it contradicts the assumption that $d\tau^2$ is complete. $\square$

As a corollary of Theorem 3.2, we provide a new proof of the classification of affine complete improper affine spheres from the viewpoint of the value distribution of Lagrangian Gauss map.

Corollary 3.7. *Any affine complete improper affine sphere must be an elliptic paraboloid.*

Proof. Because an improper affine sphere has no singularities, the complement of the image of its Lagrangian Gauss map ν contains at least the circle $\{|\nu| = 1\} \subset \mathbf{C} \cup \{\infty\}$. Thus, by exchanging roles of dF and dG if necessarily, it holds that $|\nu| < 1$, that is, $|dF| < |dG|$. On the other hand, we have

$$g = |dG|^2 - |dF|^2 < 2(|dF|^2 + |dG|^2) = d\tau^2.$$

Thus if an improper affine sphere is affine complete, then it is also weakly complete. Therefore, by Theorem 3.2, it is an elliptic paraboloid. $\square$

4. THE VALUE DISTRIBUTION OF THE RATIO OF CANONICAL FORMS FOR WEAKLY COMPLETE FLAT FRONTS IN HYPERBOLIC THREE-SPACE

We first summarize here definitions and basic facts on weakly complete flat fronts in $\mathbf{H}^3$ which we shall need. For more details, we refer the reader to [GMM], [KUY2], [KRUY1], [KRUY2] and [SUY].

Let $\mathbf{L}^4$ be the Lorentz-Minkowski four-space with inner product of signature $(-,+,+,+)$. Then the hyperbolic three-space is given by

$$\mathbf{H}^3 = \{(x_0, x_1, x_2, x_3) \in \mathbf{L}^4 \mid -(x_0)^2 + (x_1)^2 + (x_2)^2 + (x_3)^2 = -1, x_0 > 0\} \tag{4.1}$$

with the induced metric from $\mathbf{L}^4$, which is a simply connected Riemannian three-manifold with constant sectional curvature minus one. Identifying $\mathbf{L}^4$ with the set of 2×2 Hermitian matrices Herm(2)= $\{X^* = X\}$ $(X^* := {}^t\overline{X}\,)$ by

$$(x_0, x_1, x_2, x_3) \longleftrightarrow \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \tag{4.2}$$

where $i = \sqrt{-1}$, we can write

$$\begin{aligned} \mathbf{H}^3 &= \{X \in \mathrm{Herm}(2)\,;\, \det X = 1, \operatorname{trace} X > 0\} \\ &= \{aa^*\,;\, a \in SL(2,\mathbf{C})\} \end{aligned} \tag{4.3}$$

with the metric

$$\langle X, Y\rangle = -\frac{1}{2}\,\mathrm{trace}\,(X\widetilde{Y}), \quad \langle X, X\rangle = -\det(X)\,,$$

where $\widetilde{Y}$ is the cofactor matrix of Y. The complex Lie group $PSL(2,\mathbf{C}) := SL(2,\mathbf{C})/\{\pm\mathrm{id}\}$ acts isometrically on $\mathbf{H}^3$ by

$$\mathbf{H}^3 \ni X \longmapsto aXa^*\,, \tag{4.4}$$

where $a \in PSL(2,\mathbf{C})$.

Let Σ be an oriented two-manifold. A smooth map $f\colon \Sigma \to \mathbf{H}^3$ is called a *front* if there exists a Legendrian immersion

$$L_f\colon \Sigma \to T_1^*\mathbf{H}^3$$

into the unit cotangent bundle of $\mathbf{H}^3$ whose projection is f. Identifying $T_1^*\mathbf{H}^3$ with the unit tangent bundle $T_1\mathbf{H}^3$, we can write $L_f = (f, n)$, where $n(p)$ is a unit vector in $T_{f(p)}\mathbf{H}^3$ such that $\langle df(p), n(p)\rangle = 0$ for each $p \in M$. We call n a *unit normal vector field* of the front f. A point $p \in \Sigma$ where $\mathrm{rank}\,(df)_p < 2$ is called a *singularity* or *singular point.* A point which is not singular is called *regular point*, where the first fundamental form is positive definite.

The *parallel front* f_t of a front f at distance t is given by $f_t(p) = \mathrm{Exp}_{f(p)}(tn(p))$, where "Exp" denotes the exponential map of $\mathbf{H}^3$. In the model for $\mathbf{H}^3$ as in (4.1), we can write

$$f_t = (\cosh t)f + (\sinh t)n, \quad n_t = (\cosh t)n + (\sinh t)f\,, \tag{4.5}$$

where n_t is the unit normal vector field of f_t.

Based on the fact that any parallel surface of a flat surface is also flat at regular points, we define flat fronts as follows: A front $f\colon \Sigma \to \mathbf{H}^3$ is called a *flat front* if, for each $p \in M$, there exists a real number $t \in \mathbf{R}$ such that the parallel front f_t is a flat immersion at p. By definition, $\{f_t\}$ forms a family of flat fronts. We remark that an equivalent definition of flat fronts is that the Gaussian curvature of f vanishes at all regular points. However, there exists a case where this definition is not suitable. For details, see [KUY2, Remark 2.2].

We assume that f is flat. Then there exists a (unique) complex structure on Σ and a holomorphic Legendrian immersion

$$\mathcal{E}_f\colon \widetilde{\Sigma} \to SL(2,\mathbf{C}) \tag{4.6}$$

such that f and L_f are projections of $\mathcal{E}_f$, where $\widetilde{\Sigma}$ is the universal covering of Σ. Here, holomorphic Legendrian map means that $\mathcal{E}_f^{-1}d\mathcal{E}_f$ is off-diagonal (see [GMM], [KUY1], [KUY2]). We call $\mathcal{E}_f$ the *holomorphic Legendrian lift* of f. The map f and its unit normal vector field n are

$$f = \mathcal{E}_f\mathcal{E}_f^*, \quad n = \mathcal{E}_f e_3 \mathcal{E}_f^*, \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.7}$$

If we set

$$\mathcal{E}_f^{-1}d\mathcal{E}_f = \begin{pmatrix} 0 & \theta \\ \omega & 0 \end{pmatrix}, \tag{4.8}$$

the first and second fundamental forms $ds^2 = \langle df, df\rangle$ and $dh^2 = -\langle df, dn\rangle$ are given by

$$\begin{aligned} ds^2 &= |\omega + \bar{\theta}|^2 = Q + \bar{Q} + (|\omega|^2 + |\theta|^2), \quad Q = \omega\theta \\ dh^2 &= |\theta|^2 - |\omega|^2 \end{aligned} \tag{4.9}$$

for holomorphic one-forms ω and θ on $\widetilde{\Sigma}$, with $|\omega|^2$ and $|\theta|^2$ well-defined on Σ itself. We call ω and θ the *canonical forms* of f. The holomorphic two-differential Q appearing in the $(2,0)$-part of ds^2 is defined on Σ, and is called the *Hopf differential* of f. By definition, the umbilic points of f coincide with the zeros of Q. Defining a meromorphic function on $\widetilde{\Sigma}$ by the ratio of canonical forms

$$\rho = \frac{\theta}{\omega}, \tag{4.10}$$

then $|\rho|\colon \Sigma \to [0, +\infty]$ is well-defined on Σ, and $p \in \Sigma$ is a singular point if and only if $|\rho(p)| = 1$.

Note that the $(1,1)$-part of the first fundamental form

$$ds^2_{1,1} = |\omega|^2 + |\theta|^2 \tag{4.11}$$

is positive definite on Σ because it is the pull-back of the canonical Hermitian metric of $SL(2, \mathbf{C})$. Moreover, $2ds^2_{1,1}$ coincides with the pull-back of the Sasakian metric on $T_1^*\mathbf{H}^3$ by the Legendrian lift L_f of f (which is the sum of the first and third fundamental forms in this case, see [KUY2, Section 2] for details). The complex structure on Σ is compatible with the conformal metric $ds^2_{1,1}$. Note that any flat front is orientable ([KRUY1, Theorem B]). In this section, for each flat front $f\colon \Sigma \to \mathbf{H}^3$, we always regard Σ as a Riemann surface with this complex structure.

The two *hyperbolic Gauss maps* are defined by

$$G = \frac{E_{11}}{E_{21}}, \quad G_* = \frac{E_{12}}{E_{22}}, \quad \text{where} \quad \mathcal{E}_f = (E_{ij}). \tag{4.12}$$

By identifying the ideal boundary $\mathbf{S}^2_\infty$ of $\mathbf{H}^3$ with the Riemann sphere $\mathbf{C} \cup \{\infty\}$, the geometric meaning of G and G_* is given as follows ([KRUY2, Appendix A], [Ro]): The hyperbolic Gauss maps G and G_* represent the intersection points in $\mathbf{S}^2_\infty$ for the two oppositely-oriented normal geodesics emanating from f. In particular, G and G_* are meromorphic functions on Σ and parallel fronts have the same hyperbolic Gauss maps. We have already obtained an estimate for the totally ramified value numbers of the hyperbolic Gauss maps of complete flat fronts in $\mathbf{H}^3$ [Ka3]. This estimate is similar to the case of the Gauss map of pseudo-algebraic minimal surfaces in Euclidean four-space [Ka1]. Let z be a local complex coordinate on Σ. Then we have the following identities (see [KUY2]):

$$s(\omega)-S(G)=2Q, \quad s(\theta)-S(G_*)=2Q, \tag{4.13}$$

where $S(G)$ is the Schwarzian derivative of G with respect to z as in

$$S(G)=\left\{\left(\frac{G''}{G'}\right)'-\frac{1}{2}\left(\frac{G''}{G'}\right)^2\right\}dz^2 \qquad \left({}'=\frac{d}{dz}\right), \tag{4.14}$$

and $s(\omega)$ and $s(\theta)$ is the Schwarzian derivative of the integral of ω and θ, respectively.

Here, we remark on the interchangeability of the canonical forms and the hyperbolic Gauss maps. The canonical forms (ω,θ) have the $U(1)$-ambiguity $(\omega,\theta)\mapsto(e^{is}\omega,e^{-is}\theta)$ $(s\in\mathbf{R})$, which corresponds to

$$\mathcal{E}_f\longmapsto\mathcal{E}_f\begin{pmatrix} e^{is/2} & 0\\ 0 & e^{-is/2}\end{pmatrix}. \tag{4.15}$$

For a second ambiguity, defining the *dual* $\mathcal{E}_f^\natural$ of $\mathcal{E}_f$ by

$$\mathcal{E}_f^\natural=\mathcal{E}_f\begin{pmatrix} 0 & i\\ i & 0\end{pmatrix},$$

then $\mathcal{E}_f^\natural$ is also Legendrian with $f=\mathcal{E}_f^\natural\mathcal{E}_f^{\natural *}$. The hyperbolic Gauss maps $G^\natural$, $G_*^\natural$ and canonical forms $\omega^\natural$, $\theta^\natural$ of $\mathcal{E}_f^\natural$ satisfy

$$G^\natural=G_*, \quad G_*^\natural=G, \quad \omega^\natural=\theta, \quad \theta^\natural=\omega\,.$$

Namely, the operation $\natural$ interchanges the roles of ω and θ and also G and G_*.

A flat front $f\colon\Sigma\to\mathbf{H}^3$ is said to be *weakly complete* (resp. *of finite type*) if the metric $ds^2_{1,1}$ as in (4.11) is complete (resp. of finite total curvature). We remark that the universal cover of a weakly complete flat front is also weakly complete, but completeness is not preserved when lifting to the universal cover.

Fact 4.1 ([KRUY1, Proposition 3.2])**.** *If a flat front $f\colon\Sigma\to\mathbf{H}^3$ is weakly complete and of finite type, then Σ is biholomorphic to $\overline{\Sigma}_\gamma\backslash\{p_1,\dots,p_k\}$, where $\overline{\Sigma}_\gamma$ is a closed Riemann surface of genus γ and $p_j\in\overline{\Sigma}_\gamma$ $(j=1,\dots,k)$.*

Each puncture point p_j $(j=1,\dots,k)$ is called an *WCF-end* of f. We can assume that a neighborhood of p_j is biholomorphic to the puncture disk $\mathbf{D}^*=\{z\in\mathbf{C}\,;\,0<|z|<1\}$.

Fact 4.2 ([GMM], [KUY2], [KRUY1, Proposition 3.2])**.** *Let $f\colon\mathbf{D}^*\to\mathbf{H}^3$ be a WCF-end of a flat front. Then the canonical form ω and θ are expressed*

$$\omega=z^{\mu}\hat{\omega}(z)dz, \quad \theta=z^{\mu_*}\hat{\theta}(z)dz, \quad (\mu,\mu_*\in\mathbf{R},\ \mu+\mu_*\in\mathbf{Z}),$$

where $\hat{\omega}$ and $\hat{\theta}$ are holomorphic functions in z which do not vanish at the origin. In particular, the function $|\rho|\colon\mathbf{D}^*\to[0,\infty]$ as in (4.10) can be extended across the end.

Here, $|\omega|^2$ and $|\theta|^2$ are considered as conformal flat metrics on $\mathbf{D}^*_\varepsilon$ for sufficiently small $\varepsilon > 0$. The real numbers μ and μ_* are the order of the metrics $|\omega|^2$ and $|\theta|^2$ at the origin respectively, that is,

$$\mu = \mathrm{ord}_0|\omega|^2, \quad \mu_* = \mathrm{ord}_0|\theta|^2. \tag{4.16}$$

Since $ds^2_{1,1} = |\omega|^2 + |\theta|^2$ is complete at the origin, it holds that

$$\min\{\mu, \mu_*\} = \min\left\{\mathrm{ord}_0|\omega|^2, \mathrm{ord}_0|\theta|^2\right\} \leq 1\,. \tag{4.17}$$

for a WCF-end. By (4.9), the order of the Hopf differential is

$$\mathrm{ord}_0 Q = \mu + \mu_* = \mathrm{ord}_0|\omega|^2 + \mathrm{ord}_0|\theta|^2. \tag{4.18}$$

We call the WCF-end *regular* if both G and G_* have at most poles. Then the following fact holds.

Fact 4.3 ([GMM], [KRUY1, Proposition 4.2])**.** *A WCF-end* $f\colon \mathbf{D}^* \to \mathbf{H}^3$ *of a flat front is regular if and only if the Hopf differential has a pole of order at most two at the origin, that is,* $\mathrm{ord}_0 Q \geq -2$ *holds.*

Now we investigate the value distribution of the ratio of canonical forms for weakly complete flat fronts in $\mathbf{H}^3$. We consider the case where the ratio is constant.

Proposition 4.4. *Let* $f\colon \Sigma \to \mathbf{H}^3$ *be a weakly complete flat front. If the meromorphic function* ρ *defined by (4.10) is constant, then* f *is congruent to a horosphere or a hyperbolic cylinder. Here, a surface equidistance from a geodesic is called a hyperbolic cylinder* [KUY2]*.*

Proof. In general, the function ρ is defined on the universal covering $\widetilde{\Sigma}$ of Σ. However, in this case, we can consider that ρ is constant on Σ. Then the metric $ds^2_{1,1}$ defined by (4.11) is represented as

$$ds^2_{1,1} = |\omega|^2 + |\theta|^2 = \left(1 + \left|\frac{\theta}{\omega}\right|^2\right)|\omega|^2 = (1 + |\rho|^2)|\omega|^2\,. \tag{4.19}$$

Thus the Gaussian curvature $K_{ds^2_{1,1}}$ of $ds^2_{1,1}$ vanishes identically on Σ. By Fact 4.1, Σ is biholomorphic to a closed Riemann surface of genus γ with k points removed, that is, $\Sigma = \overline{\Sigma}_\gamma \backslash \{p_1, \dots, p_k\}$. Moreover, we obtain the formula ([KRUY1, (3.2)])

$$\frac{1}{2\pi}\int_\Sigma (-K_{ds^2_{1,1}})dA = -\chi(\overline{\Sigma}_\gamma) - \sum_{j=1}^{k} \mathrm{ord}_{p_j}(ds^2_{1,1}),$$

where dA denotes the area element of $ds^2_{1,1}$ and $\chi(\overline{\Sigma}_\gamma)$ the Euler number of $\overline{\Sigma}_\gamma$. Since the metric $ds^2_{1,1}$ is complete, for each WCF-end p_j, $\mathrm{ord}_{p_j} ds^2_{1,1} \leq -1$ holds. Thus, in this case, we get $\gamma = 0$ and

$$\sum_{j=1}^{k} \operatorname{ord}_{p_j}(ds^2_{1,1}) = -2\,. \tag{4.20}$$

Since $ds^2_{1,1}$ is well-defined on Σ, the following two cases are considered:

(a) The flat front f has two WCF-ends p and q, and $\operatorname{ord}_p ds^2_{1,1} = \operatorname{ord}_q ds^2_{1,1} = -1$,

(b) The flat front f has one WCF-end p, and $\operatorname{ord}_p ds^2_{1,1} = -2$.

In the case of (a), f is congruent to a hyperbolic cylinder. In fact, the WCF-ends are asymptotic to a finite cover of a hyperbolic cylinder ([GMM], [KRUY2]). In the case of (b), then $\rho \equiv 0$. Because, if not, then it holds that $\operatorname{ord}_p Q = -4$ by (4.18). On the other hand, the identities (4.13) imply that the WCF-end p is regular. However, by Fact 4.3, it does not occur. Therefore the Hopf differential $Q = \omega\theta$ also vanishes identically on Σ, and then f is a horosphere. □

Applying the same argument as in the proof of Theorem 3.2 to the ratio ρ of weakly complete flat fronts in $\mathbf{H}^3$, we give the following ramification estimate for ρ.

Theorem 4.5. *Let $f\colon \Sigma \to \mathbf{H}^3$ be a weakly complete flat front and ρ be the meromorphic function on $\widetilde{\Sigma}$ defined by (4.10). Assume that there exist distinct values $\alpha_1, \dots, \alpha_q$ and positive integers $m_1, \dots, m_q$ such that the multiplicities of ρ on the inverse image $\rho^{-1}(\alpha_j)$ are greater than or equal to m_j for each j, and*

$$\gamma := \sum_{j=1}^{q} \left(1 - \frac{1}{m_j}\right) > 3\,.$$

Then ρ is constant, that is, f must be a horosphere or a hyperbolic cylinder.

As a corollary of Theorem 4.5, we can obtain the classification of weakly complete flat surfaces in $\mathbf{H}^3$. We remark that Sasaki [Sa], Volkov and Vladimirova [VV] have already obtained the same classification for complete flat surfaces in $\mathbf{H}^3$ (See also [GMM, Theorem 3]).

Corollary 4.6. *Any weakly complete flat surface in $\mathbf{H}^3$ must be congruent to a horosphere or a hyperbolic cylinder.*

Proof. Because a weakly complete flat surface has no singularities, the complement of the image of ρ contains at least the circle $\{|\rho| = 1\} \subset \mathbf{C} \cup \{\infty\}$. From Theorem 4.5, it is a horosphere or a hyperbolic cylinder. □

REFERENCES

[Be] S. Bernstein, *Sur un théorème de géométrie et ses applications aux équations aux dérivées partielles du type elliptique*, Comm. de la Soc. Math. de Kharkov (2éme sér.) **15** (1915–1917), 38–45.

[Ca1] E. Calabi, *Improper affine hypersurfaces of convex type and a generalization of a theorem by K.Jörgens*, Mich. Math. J., **5** (1958), 108–126

[Ca2] E. Calabi, *Examples of Bernstein problems for some non-linear equations*, Proc. Sympos. Pure Math., **15** (1970), 223–230.

[Ca3] E. Calabi, *Convex affine maximal surfaces*, Results Math., **13** (1988), 209–223.

[CM] B. Y. Chen and J. M. Morvan, *Géométrie des surfaces lagrangiennes de* $\mathbf{C}^2$, J. Math. Pures Appl., **63** (1987), 321–325.

[Fa] Y. Fang, *The Minding formula and its applications*, Arch. Math., **72** (1999), 473–480.

[FMM] L. Ferrer, A. Martínez, and F. Milán, *An extension of a theorem by K. Jörgens and a maximum principle at infinity for parabolic affine spheres*, Math. Z., **230** (1999), 471–486.

[Fu1] H. Fujimoto, *On the number of exceptional values of the Gauss map of minimal surfaces*, J. Math. Soc. Japan, **40** (1988), 235 – 247.

[Fu2] H. Fujimoto, Value Distribution Theory of the Gauss Map of Minimal Surfaces in $\mathbf{R}^m$, Vieweg, 1993.

[GMM] J. A. Gálvez, A. Martínez, and F. Milán, *Flat surfaces in hyperbolic* 3*-space*, Math. Ann., **316** (2000), 419–435.

[HL] R. Harvey and B. Lawson, *Calibrated Geometries*, Acta Math., **148** (1982), 47–157.

[Hu] A. Huber, *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv., **32** (1957), 13 – 72.

[IM] G. Ishikawa and Y. Machida, *Singularities of improper affine spheres and surfaces of constant Gaussian curvature*, Internat. J. Math. **17** (2007), 269–293.

[Jo] K. Jörgens, *Über die Lösungen der differentialgleichung* $rt - s^2 = 1$, Math. Ann., **127** (1954), 130–134.

[JR] L. Jin and M. Ru, *Algebraic curves and the Gauss map of algebraic minimal surfaces*, Differ. Geom. Appl. **25**, (2007), 701–712.

[Ka1] Y. Kawakami, *The Gauss map of pseudo-algebraic minimal surfaces in* $\mathbf{R}^4$, Math. Nachr., **282** (2009), 211–218.

[Ka2] Y. Kawakami, *Ramification estimates for the hyperbolic Gauss map*, Osaka J. Math., **46** (2009), 1059–1076.

[Ka3] Y. Kawakami, *Value distribution of the hyperbolic Gauss maps for flat fronts in hyperbolic three-space*, to appear in Houston Journal of Mathematics, arXiv:0908.1307.

[KKM] Y. Kawakami, R. Kobayashi and R. Miyaoka, *The Gauss map of pseudo-algebraic minimal surfaces*, Forum Math., **20** (2008), 1055–1069.

[Ko] R. Kobayashi, *Toward Nevanlinna theory as a geometric model of Diophantine approximation*, Sugaku Expositions **16** (2003), 39–79.

[KRSUY] M. Kokubu, W. Rossman, K. Saji, M. Umehara and K. Yamada, *Singularities of flat fronts in hyperbolic space*, Pacific J. Math., **221** (2005), 303–351.

[KRUY1] M. Kokubu, W. Rossman, M. Umehara and K. Yamada, *Flat fronts in hyperbolic 3-space and their caustics*, J. Math. Soc. Japan, **59**, (2007), 265–299.

[KRUY2] M. Kokubu, W. Rossman, M. Umehara and K. Yamada, *Asymptotic behavior of flat surfaces in hyperbolic* 3*-space*, J. Math. Soc. Japan, **61** (2009), 799–852.

[KUY1] M. Kokubu, M. Umehara and K. Yamada, *An elementary proof of Small's formula for null curves in PSL(2,* $\mathbf{C}$*) and an analogue for Legendrian curves in PSL(2,* $\mathbf{C}$*)*, Osaka J. Math., **40** (2003), 697–715.

[KUY2] M. Kokubu, M. Umehara and K. Yamada, *Flat fronts in hyperbolic* 3*-space*, Pacific J. Math., **216** (2004), 149–175.

[LSZ] A.-M. Li, U. Simon and Z. Zhao, Global affine differential geometry of hypersurfaces, Walter de Gruyter, Berlin-New York, 1993.

[Ma1] A. Martínez, *Improper affine maps*, Math. Z., **249** (2005), 755–766.

[Ma2] A. Martínez, *Relatives of Flat Surfaces in* $\mathbf{H}^3$, Proceedings of International Workshop on Integrable systems, Geometry and Visualization (November 19–24 at Kyushu university, Fukuoka, Japan), 115–132, 2005.

[Na] D. Nakajo, *A representation formula for indefinite improper affine spheres*, Result. Math., **55** (2009), 139–159.

[Ne] R. Nevanlinna, Analytic Function, Translated from the second German edition by Phillip Emig. Die Grundlehren der mathematischen Wissenschaften, Springer, New York, 1970.

[NO] J. Noguchi and T. Ochiai, Geometric Function Theory in Several Complex Variables, Transl. Math. Monog. **80**, Amer. Math. Soc., 1990.

[NS] K. Nomizu and T. Sasaki, Affine differential geometry, Cambridge University Press, 1994.

[Os1] R. Osserman, *Proof of a conjecture of Nirenberg*, Comm. on Pure and Applied Math. **12** (1959), 229–232.

[Os2] R. Osserman, *Global properties of minimal surfaces in* E^3 *and* E^n, Ann. of Math. **80** (1964), 340–364.

[Os3] R. Osserman, *A survey of minimal surfaces*, second edition, Dover Publications Inc., 1986.

[Ro] P. Roitman, *Flat surfaces in hyperbolic* 3*-space as normal surfaces to a congruence of geodesics*, Tohoku Math. J., **59** (2007), 21–37.

[Ru] M. Ru, Nevanlinna theory and its relation to Diophantine approximation, World Sci. Publ. Co. Inc., River Edge, NJ, 2001.

[Sa] S. Sasaki, *On complete flat surfaces in hyperbolic 3-space*, Kōdai Math Sem. Rep. **25** (1973), 449–457.

[Sh] K. Shiohama, *Total curvatures and minimal areas of complete open surfaces*, Proc., Amer. Math. Soc., **94** (1985), 310–316.

[SUY] K. Saji, M. Umehara and K. Yamada, *The geometry of fronts*, Ann. of Math., **169** (2009), 491–529.

[TW] N. S. Trudinger and X. J. Wang, *Affine complete locally convex hypersurfaces*, Invent. math. **150** (2002), 45–60.

[UY] M. Umehara and K. Yamada, *Applications of a complete lemma in minimal surface theory to various classes of surfaces*, preprint, arXiv:0909.1128.

[VV] Y. A. Volkov and S. M. Vladimirova, *Isometric immersions of the Euclidean plane in Lobačevskiĭ space* (Russian), Mat. Zametki **10** (1971), 327–332.

Faculty of Mathematics,
Kyushu university,
744, Motooka, Nishiku, Fukuoka-city, 819-0395, Japan
E-mail address: `kawakami@math.kyushu-u.ac.jp`

Faculty of Mathematics,
Kyushu university,
744, Motooka, Nishiku, Fukuoka-city, 819-0395, Japan
E-mail address: `nakajo@math.kyushu-u.ac.jp`

List of MI Preprint Series, Kyushu University

The Global COE Program
Math-for-Industry Education & Research Hub

MI

MI2008-1 Takahiro ITO, Shuichi INOKUCHI & Yoshihiro MIZOGUCHI
Abstract collision systems simulated by cellular automata

MI2008-2 Eiji ONODERA
The intial value problem for a third-order dispersive flow into compact almost Hermitian manifolds

MI2008-3 Hiroaki KIDO
On isosceles sets in the 4-dimensional Euclidean space

MI2008-4 Hirofumi NOTSU
Numerical computations of cavity flow problems by a pressure stabilized characteristic-curve finite element scheme

MI2008-5 Yoshiyasu OZEKI
Torsion points of abelian varieties with values in nfinite extensions over a p-adic field

MI2008-6 Yoshiyuki TOMIYAMA
Lifting Galois representations over arbitrary number fields

MI2008-7 Takehiro HIROTSU & Setsuo TANIGUCHI
The random walk model revisited

MI2008-8 Silvia GANDY, Masaaki KANNO, Hirokazu ANAI & Kazuhiro YOKOYAMA
Optimizing a particular real root of a polynomial by a special cylindrical algebraic decomposition

MI2008-9 Kazufumi KIMOTO, Sho MATSUMOTO & Masato WAKAYAMA
Alpha-determinant cyclic modules and Jacobi polynomials

MI2008-10 Sangyeol LEE & Hiroki MASUDA
Jarque-Bera Normality Test for the Driving Lévy Process of a Discretely Observed Univariate SDE

MI2008-11 Hiroyuki CHIHARA & Eiji ONODERA
A third order dispersive flow for closed curves into almost Hermitian manifolds

MI2008-12 Takehiko KINOSHITA, Kouji HASHIMOTO and Mitsuhiro T. NAKAO
On the L^2 a priori error estimates to the finite element solution of elliptic problems with singular adjoint operator

MI2008-13 Jacques FARAUT and Masato WAKAYAMA
Hermitian symmetric spaces of tube type and multivariate Meixner-Pollaczek polynomials

MI2008-14 Takashi NAKAMURA
Riemann zeta-values, Euler polynomials and the best constant of Sobolev inequality

MI2008-15 Takashi NAKAMURA
Some topics related to Hurwitz-Lerch zeta functions

MI2009-1 Yasuhide FUKUMOTO
Global time evolution of viscous vortex rings

MI2009-2 Hidetoshi MATSUI & Sadanori KONISHI
Regularized functional regression modeling for functional response and predictors

MI2009-3 Hidetoshi MATSUI & Sadanori KONISHI
Variable selection for functional regression model via the L_1 regularization

MI2009-4 Shuichi KAWANO & Sadanori KONISHI
Nonlinear logistic discrimination via regularized Gaussian basis expansions

MI2009-5 Toshiro HIRANOUCHI & Yuichiro TAGUCHII
Flat modules and Groebner bases over truncated discrete valuation rings

MI2009-6 Kenji KAJIWARA & Yasuhiro OHTA
Bilinearization and Casorati determinant solutions to non-autonomous 1+1 dimensional discrete soliton equations

MI2009-7 Yoshiyuki KAGEI
Asymptotic behavior of solutions of the compressible Navier-Stokes equation around the plane Couette flow

MI2009-8 Shohei TATEISHI, Hidetoshi MATSUI & Sadanori KONISHI
Nonlinear regression modeling via the lasso-type regularization

MI2009-9 Takeshi TAKAISHI & Masato KIMURA
Phase field model for mode III crack growth in two dimensional elasticity

MI2009-10 Shingo SAITO
Generalisation of Mack's formula for claims reserving with arbitrary exponents for the variance assumption

MI2009-11 Kenji KAJIWARA, Masanobu KANEKO, Atsushi NOBE & Teruhisa TSUDA
Ultradiscretization of a solvable two-dimensional chaotic map associated with the Hesse cubic curve

MI2009-12 Tetsu MASUDA
Hypergeometric τ-functions of the q-Painlevé system of type $E_8^{(1)}$

MI2009-13 Hidenao IWANE, Hitoshi YANAMI, Hirokazu ANAI & Kazuhiro YOKOYAMA
A Practical Implementation of a Symbolic-Numeric Cylindrical Algebraic Decomposition for Quantifier Elimination

MI2009-14 Yasunori MAEKAWA
On Gaussian decay estimates of solutions to some linear elliptic equations and its applications

MI2009-15 Yuya ISHIHARA & Yoshiyuki KAGEI
Large time behavior of the semigroup on L^p spaces associated with the linearized compressible Navier-Stokes equation in a cylindrical domain

MI2009-16 Chikashi ARITA, Atsuo KUNIBA, Kazumitsu SAKAI & Tsuyoshi SAWABE
Spectrum in multi-species asymmetric simple exclusion process on a ring

MI2009-17 Masato WAKAYAMA & Keitaro YAMAMOTO
Non-linear algebraic differential equations satisfied by certain family of elliptic functions

MI2009-18 Me Me NAING & Yasuhide FUKUMOTO
Local Instability of an Elliptical Flow Subjected to a Coriolis Force

MI2009-19 Mitsunori KAYANO & Sadanori KONISHI
Sparse functional principal component analysis via regularized basis expansions and its application

MI2009-20 Shuichi KAWANO & Sadanori KONISHI
Semi-supervised logistic discrimination via regularized Gaussian basis expansions

MI2009-21 Hiroshi YOSHIDA, Yoshihiro MIWA & Masanobu KANEKO
Elliptic curves and Fibonacci numbers arising from Lindenmayer system with symbolic computations

MI2009-22 Eiji ONODERA
A remark on the global existence of a third order dispersive flow into locally Hermitian symmetric spaces

MI2009-23 Stjepan LUGOMER & Yasuhide FUKUMOTO
Generation of ribbons, helicoids and complex scherk surface in laser-matter Interactions

MI2009-24 Yu KAWAKAMI
Recent progress in value distribution of the hyperbolic Gauss map

MI2009-25 Takehiko KINOSHITA & Mitsuhiro T. NAKAO
On very accurate enclosure of the optimal constant in the a priori error estimates for H_0^2-projection

MI2009-26 Manabu YOSHIDA
Ramification of local fields and Fontaine's property (Pm)

MI2009-27 Yu KAWAKAMI
Value distribution of the hyperbolic Gauss maps for flat fronts in hyperbolic three-space

MI2009-28 Masahisa TABATA
Numerical simulation of fluid movement in an hourglass by an energy-stable finite element scheme

MI2009-29 Yoshiyuki KAGEI & Yasunori MAEKAWA
Asymptotic behaviors of solutions to evolution equations in the presence of translation and scaling invariance

MI2009-30 Yoshiyuki KAGEI & Yasunori MAEKAWA
On asymptotic behaviors of solutions to parabolic systems modelling chemotaxis

MI2009-31 Masato WAKAYAMA & Yoshinori YAMASAKI
Hecke's zeros and higher depth determinants

MI2009-32 Olivier PIRONNEAU & Masahisa TABATA
Stability and convergence of a Galerkin-characteristics finite element scheme of lumped mass type

MI2009-33 Chikashi ARITA
Queueing process with excluded-volume effect

MI2009-34 Kenji KAJIWARA, Nobutaka NAKAZONO & Teruhisa TSUDA
Projective reduction of the discrete Painlevé system of type$(A_2 + A_1)^{(1)}$

MI2009-35 Yosuke MIZUYAMA, Takamasa SHINDE, Masahisa TABATA & Daisuke TAGAMI
Finite element computation for scattering problems of micro-hologram using DtN map

MI2009-36 Reiichiro KAWAI & Hiroki MASUDA
Exact simulation of finite variation tempered stable Ornstein-Uhlenbeck processes

MI2009-37 Hiroki MASUDA
On statistical aspects in calibrating a geometric skewed stable asset price model

MI2010-1 Hiroki MASUDA
Approximate self-weighted LAD estimation of discretely observed ergodic Ornstein-Uhlenbeck processes

MI2010-2 Reiichiro KAWAI & Hiroki MASUDA
Infinite variation tempered stable Ornstein-Uhlenbeck processes with discrete observations

MI2010-3 Kei HIROSE, Shuichi KAWANO, Daisuke MIIKE & Sadanori KONISHI
Hyper-parameter selection in Bayesian structural equation models

MI2010-4 Nobuyuki IKEDA & Setsuo TANIGUCHI
The Itô-Nisio theorem, quadratic Wiener functionals, and 1-solitons

MI2010-5 Shohei TATEISHI & Sadanori KONISHI
Nonlinear regression modeling and detecting change point via the relevance vector machine

MI2010-6 Shuichi KAWANO, Toshihiro MISUMI & Sadanori KONISHI
Semi-supervised logistic discrimination via graph-based regularization

MI2010-7 Teruhisa TSUDA
UC hierarchy and monodromy preserving deformation

MI2010-8 Takahiro ITO
Abstract collision systems on groups

MI2010-9 Hiroshi YOSHIDA, Kinji KIMURA, Naoki YOSHIDA, Junko TANAKA & Yoshihiro MIWA
An algebraic approach to underdetermined experiments

MI2010-10 Kei HIROSE & Sadanori KONISHI
Variable selection via the grouped weighted lasso for factor analysis models

MI2010-11 Katsusuke NABESHIMA & Hiroshi YOSHIDA
Derivation of specific conditions with Comprehensive Groebner Systems

MI2010-12 Yoshiyuki KAGEI, Yu NAGAFUCHI & Takeshi SUDOU
Decay estimates on solutions of the linearized compressible Navier-Stokes equation around a Poiseuille type flow

MI2010-13 Reiichiro KAWAI & Hiroki MASUDA
On simulation of tempered stable random variates

MI2010-14 Yoshiyasu OZEKI
Non-existence of certain Galois representations with a uniform tame inertia weight

MI2010-15 Me Me NAING & Yasuhide FUKUMOTO
Local Instability of a Rotating Flow Driven by Precession of Arbitrary Frequency

MI2010-16 Yu KAWAKAMI & Daisuke NAKAJO
The value distribution of the Gauss map of improper affine spheres