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O. Pironneau & M. Tabata

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Faculty of Mathematics  
Kyushu University  
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# Stability and convergence of a Galerkin-characteristics finite element scheme of lumped mass type

Olivier Pironneau<sup>†</sup> and Masahisa Tabata<sup>‡1</sup>

<sup>†</sup> *Laboratoire Jacques-Louis Lions, Paris VI university,  
75252 Paris Cedex 05, France*

<sup>‡</sup> *Faculty of Mathematics, Kyushu University,  
Motoooka, Nishi-ku, Fukuoka, 819-0395, Japan*

## Abstract

A Galerkin-characteristics finite element scheme of lumped mass type is presented for the convection diffusion problems. Under the weakly acute triangulation hypothesis the scheme is proved to be unconditionally stable and convergent in the maximum-norm. Using freefem, we show 2D and 3D numerical examples, which reflect the robustness of the scheme and the theoretical convergence result.

## 1 Introduction

Upwinding for convection-diffusion equations approximated by the Finite Element Methods was first introduced by Heinrich et al. [6] and Tabata [13] in the late seventies. After 40 years and many other methods it is still difficult to choose among the various existing schemes one which is conservative, with small numerical viscosity, unconditionally stable and positive. The most popular choices are SUPG, its Galerkin Least Square variant [7], Deconninck's PSI method[8], Discontinuous-Galerkin [4] or Galerkin-characteristics (see [9] and the bibliography therein).

Galerkin-characteristics FEM has a lot of potential but it is not so easy to find a second order variant [11] and its convergent properties are ruined by quadrature errors, at least in theory [12]. Although it is common knowledge that in practice Galerkin-characteristics FEM is an excellent method, as applied mathematicians we think it is risky to work with a method which can potentially go wrong. Even if it is fundamental, the problem is sufficiently important to be dealt with and so the object of this paper is to show -with proofs- that there is at least one case where the method can be shown to

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<sup>1</sup>Correspondence: tabata@math.kyushu-u.ac.jp

converge and even if it is an order one case, nevertheless it is a method which outperform other first order methods.

The paper gives a new proof of convergence of the method with quadrature and mass lumping, based on  $L^\infty$ -estimates and restricted to weakly acute triangulations. A numerical implementation is given in 2D and 3D using `freefem`[5] and compared with one SUPG and one Discontinuous-Galerkin method.

Throughout this paper we use  $\theta_i^n$  as a real number depending on  $i$  and  $n$  such that  $|\theta_i^n| \leq 1$ . The symbol  $c$  with or without subscripts is used for a generic positive constant independent of the discretization parameters, which may take a different value at each occurrence.

## 2 $L^\infty$ -stability

Let  $\Omega$  be a convex polygonal (or polyhedral) domain in  $\mathfrak{R}^2$  (or  $\mathfrak{R}^3$ ), and  $T$  be a positive number.

Let  $\phi : \Omega \times (0, T) \rightarrow \mathfrak{R}$  be the solution of the convection-diffusion equation,

$$\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi - \nu \Delta \phi = f \quad \text{in } \Omega \times (0, T), \quad (1a)$$

$$\phi = 0 \quad \text{on } \Gamma \times (0, T), \quad (1b)$$

$$\phi = \phi^0 \quad \text{in } \Omega, \text{ at } t = 0, \quad (1c)$$

where  $u : \Omega \times (0, T) \rightarrow \mathfrak{R}^d$  is a given velocity vanishing on  $\Gamma$ ,  $f : \Omega \times (0, T) \rightarrow \mathfrak{R}$  be a given function, and  $\phi^0 : \Omega \rightarrow \mathfrak{R}$  is a given initial function.

Let  $\mathcal{T}_h \equiv \{K\}$  be a partition of  $\Omega$  (by triangles in  $d = 2$  and by tetrahedra in  $d = 3$ ), and  $\Delta t$  be a time increment. We set  $N_T = \lfloor T/\Delta t \rfloor$ . Let  $X_h (\subset H^1(\Omega))$  be the P1-finite element space, and  $V_h$  be  $X_h \cap H_0^1(\Omega)$ . We consider the following Galerkin-characteristics finite element scheme of lumped mass type:

Find  $\{\phi_h^n\}_{n=1}^{N_T} \subset V_h$  such that, for  $n = 1, \dots, N_T$ ,

$$\left( \frac{\bar{\phi}_h^n - \bar{I}_h(\phi_h^{n-1} \circ X_h^n)}{\Delta t}, \bar{v}_h \right) + \nu a(\phi_h^n, v_h) = (\bar{I}_h f^n, \bar{v}_h), \quad \forall v_h \in V_h \quad (2a)$$

$$\phi_h^0 = I_h \phi^0. \quad (2b)$$

Here,  $(\cdot, \cdot)$  is the inner product in  $L^2(\Omega)$ ,

$$\begin{aligned} a(\phi, v) &= (\nabla\phi, \nabla v), \\ X_h^n(x) &= x - \Delta t u(x, n\Delta t), \end{aligned} \quad (3)$$

$I_h : C(\bar{\Omega}) \rightarrow X_h$  is the interpolation operator defined by

$$(I_h v)(P) = v(P), \quad (\forall \text{node } P \in \bar{\Omega}),$$

$\bar{\cdot} : V_h \rightarrow L^2(\Omega)$  is the lumping operator defined by

$$\bar{v}_h(x) = v_h(P), \quad (x \in D_P)$$

and  $D_P$  is the barycentric domain [1] associated with node  $P$  shown in Fig. 1,

$$\begin{aligned} D_P &= \bigcup_K \{D_P^K; P \in K \in \mathcal{T}_h\} \\ D_P^K &= \bigcap_{j=1}^d \{x; x \in K, \lambda_{Q(j)}(x) \leq \lambda_P(x)\}, \end{aligned}$$

where  $\{P, Q(1), \dots, Q(d)\}$  is the set of the vertices of  $K$  and  $\{\lambda_P, \lambda_{Q(1)}, \dots, \lambda_{Q(d)}\}$  is the system of the barycentric coordinates.

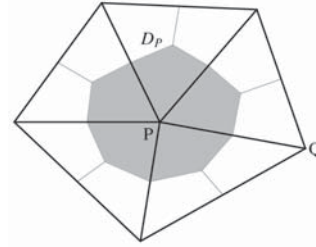


Figure 1: The barycentric domain  $D_P$  associated with  $P$ .

Let  $N$  be the number of interior nodes, and  $w_{hi}$ ,  $i = 1, \dots, N$ , be the base function associated with node  $P_i \in \Omega$ ,

$$w_{hi} \in V_h, \quad w_{hi}(P_j) = \delta_{ij}, \quad i, j = 1, \dots, N.$$

Let  $A = \{a_{ij}\}$  be the stiffness matrix with

$$a_{ij} = a(w_{hj}, w_{hi}), \quad i, j = 1, \dots, N.$$

**Hypothesis 1**  $a_{ij} \leq 0$  for  $i \neq j$ ,  $i, j = 1, \dots, N$ .

**Remark 1** A sufficient condition for Hypothesis 1 in  $d = 2$  is that the triangulation is of weakly acute type, i.e., every angle of any triangle is equal to or less than  $\pi/2$  [3]. Furthermore, a weaker sufficient condition for Hypothesis 1 in  $d = 2$  is, for any edge  $E$ ,

$$\alpha_1^E + \alpha_2^E \leq \pi,$$

where  $\alpha_i^E$ ,  $i = 1, 2$ , are two angles (of two elements sharing  $E$ ) opposite to  $E$  [10].

Let  $\vec{\phi}_h^n$  be the unknown  $N$ -vector consisting of the values of  $\phi_h^n(P_i)$  at nodes  $P_i$  and  $\psi_h^{n-1} \equiv \phi_h^{n-1} \circ X_h^n$ . Setting  $v_h = w_{hi}$ ,  $i = 1, \dots, N$ , in (2), we get a system of linear equations

$$\left( \frac{1}{\Delta t} \bar{M} + \nu A \right) \vec{\phi}_h^n = \frac{1}{\Delta t} \bar{M} \vec{\psi}_h^{n-1} + \bar{M} \vec{f}_h^{n-1}, \quad (4)$$

where  $\bar{M}$  is a diagonal matrix whose diagonal component  $m_i$  is

$$m_i = \text{meas} D_{P_i}$$

by virtue of the lumping. Similarly, setting  $v_h = w_{hi}$  in (2) and dividing the  $i$ -th equation by  $m_i$ , we obtain another equivalent equations, for  $i = 1, \dots, N$ ,

$$\frac{1}{\Delta t} (\phi_h^n(P_i) - (\phi_h^{n-1} \circ X_h^n)(P_i)) + \nu \frac{1}{m_i} \sum_{j=1}^N a_{ij} \phi_h^n(P_j) = f^n(P_i). \quad (5)$$

For a set of functions  $\{\phi_h^n\}_{n=0}^{N_T}$  we define norms,

$$\|\phi\|_{l^\infty(L^\infty)} \equiv \max_{0 \leq n \leq N_T} \|\phi^n\|_{L^\infty(\Omega_h)}, \quad \|\phi\|_{l^1(L^\infty)} \equiv \sum_{n=0}^{N_T} \Delta t \|\phi^n\|_{L^\infty(\Omega_h)}. \quad (6)$$

We have the following stability result.

**Lemma 1** *Under Hypothesis 1, scheme (2) is unconditionally  $L^\infty$ -stable, i.e., it holds*

$$\|\phi_h\|_{\ell^\infty(L^\infty)} \leq \|\phi_h^0\|_{L^\infty} + \|I_h f\|_{\ell^1(L^\infty)}. \quad (7)$$

*Proof.* Operating  $\Delta t \bar{M}^{-1}$  from left to (4), we obtain

$$(I + \nu \Delta t \bar{M}^{-1} A) \bar{\phi}_h^n = \bar{\psi}_h^{n-1} + \Delta t \bar{f}_h^{n-1}.$$

Since the matrix of the left-hand side,

$$G \equiv I + \nu \Delta t \bar{M}^{-1} A$$

is an M-matrix from Hypothesis 1 and

$$\sum_{j=1}^N g_{ij} \geq 1, \quad i = 1, \dots, N,$$

we get

$$\begin{aligned} |\phi_h^n(P_i)| &\leq \|\psi_h^{n-1}\|_{L^\infty} + \Delta t \|I_h f^n\|_{L^\infty} \\ &\leq \|\phi_h^{n-1}\|_{L^\infty} + \Delta t \|I_h f^n\|_{L^\infty}, \end{aligned}$$

which implies

$$\|\phi_h^n\|_{L^\infty} \leq \|\phi_h^{n-1}\|_{L^\infty} + \Delta t \|I_h f^n\|_{L^\infty}.$$

Summing up the above equations, we obtain (7).

**Remark 2** The result of Lemma 1 can be extended to reaction-convection-diffusion equations. Let  $b \in C(\bar{\Omega} \times [0, T])$  be a given function. We replace (1a) by

$$\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi - \nu \Delta \phi + b \phi = f \quad \text{in } \Omega \times (0, T). \quad (8)$$

We consider the scheme,

$$\begin{aligned} \left( \frac{\bar{\phi}_h^n - \bar{I}_h(\phi_h^{n-1} \circ X_h^n)}{\Delta t}, \bar{v}_h \right) + \nu a(\phi_h^n, v_h) + (\bar{I}_h(b_h^n)^+ \bar{\phi}_h^n - \bar{I}_h(b_h^n)^- \bar{\phi}_h^{n-1}, \bar{v}_h) \\ = (\bar{I}_h f^n, \bar{v}_h), \quad \forall v_h \in V_h \end{aligned} \quad (9)$$

where

$$b^+ = \max(b, 0), \quad b^- = \max(-b, 0).$$

For the scheme (9) the result

$$\|\phi_h\|_{\ell^\infty(L^\infty)} \leq c(\|\phi_h^0\|_{L^\infty} + \|I_h f\|_{\ell^1(L^\infty)}) \quad (10)$$

holds, where

$$c = c(\|b\|_{L^\infty(L^\infty)}, T)$$

is a positive constant independent of  $h$  and  $\Delta t$ . For the details including semi-linear functions  $b$  we refer to [14].

### 3 $L^\infty$ -convergence

Let  $C^{0+1}(\bar{\Omega})$  be the Lipschitz continuous function space defined by

$$C^{0+1}(\bar{\Omega}) = \{f; f \in C(\bar{\Omega}), \|f\|_{0+1,\Omega} < \infty\}$$

equipped with norm

$$\|f\|_{0+1,\Omega} \equiv \max\{|f(x)|; x \in \bar{\Omega}\} + |f|_{0+1,\Omega},$$

where

$$|f|_{0+1,\Omega} \equiv \sup\left\{\frac{|f(x) - f(y)|}{|x - y|}; x \neq y, x, y \in \Omega\right\}$$

Let  $C^{0+1,0}$  and  $C^{0,0+1}$  be function spaces composed of functions  $f \in C(\bar{\Omega} \times [0, T])$  such that  $f$  is Lipschitz continuous with respect to  $x$  and  $t$ , respectively. Let  $C^{m+1,0}$  and  $C^{0,m+1}$ ,  $m \in \mathbf{N}$ , be function spaces defined by

$$\begin{aligned} C^{m+1,0} &= \{f \in C(\bar{\Omega} \times [0, T]); D^\alpha f \in C^{0+1,0}, \forall \alpha = (\alpha_1, \dots, \alpha_d), |\alpha| \leq m\}, \\ C^{0,m+1} &= \{f \in C(\bar{\Omega} \times [0, T]); \frac{\partial^n f}{\partial t^n} \in C^{0,0+1}, \forall n, 0 < n \leq m\}. \end{aligned}$$

Similarly  $C^{m+1,n}$  and  $C^{n,m+1}$  are defined for  $m, n \in \mathbf{N}$ . We denote by  $C_0(\bar{\Omega})$  the function space consisting of continuous functions vanishing on  $\Gamma$ ,

$$C_0(\bar{\Omega}) = \{f \in C(\bar{\Omega}); f|_\Gamma = 0\}.$$

Let  $\phi_h$  be a function in  $V_h$ . Then,  $\phi_h$  has  $N$  freedoms at nodes  $P_i \in \Omega_h$ ,  $i = 1, \dots, N$ . We identify the function  $\phi_h$  with a discrete function defined on the set of the  $N$  nodes  $P_i$ . We use the same notation  $V_h$  to represent the discrete function space. The space  $V_h$  is equipped with the norm

$$\|\phi_h\|_{\ell^\infty} \equiv \max\{|\phi_h(P_i)|; i = 1, \dots, N\}. \quad (11)$$

We introduce a finite difference operator  $L_h : V_h \rightarrow V_h$  defined by

$$(L_h \phi_h)(P_i) = \frac{1}{m_i} \sum_{j=1}^N a_{ij} \phi_h(P_j), \quad i = 1, \dots, N, \phi_h \in V_h. \quad (12)$$

Then, (5) can be regarded as a generalized difference equation in  $V_h$ ,

$$\frac{1}{\Delta t} (\phi_h^n - (\phi_h^{n-1} \circ X_h^n)) + \nu L_h \phi_h^n = f_h^n, \quad (13)$$

where  $\phi_h^{n-1} \circ X_h^n$ , and  $f_h^n$  are discrete functions in  $V_h$ . Corresponding to Lemma 1 we have

**Lemma 2** *Under Hypothesis 1, scheme (13) is unconditionally stable, i.e., it holds*

$$\|\phi_h\|_{\ell^\infty(\ell^\infty)} \leq \|\phi_h^0\|_{\ell^\infty} + \|f_h\|_{\ell^1(\ell^\infty)}. \quad (14)$$

**Remark 3** Here we treat only the case of homogeneous Dirichlet boundary conditions. For the more general discussion on the inhomogeneous boundary conditions and the Neumann boundary conditions on the generalized difference equations we refer to [15].

We use the same notation  $I_h$  to represent an operator from  $C(\bar{\Omega})$  to the discrete function space  $V_h$  defined by

$$(I_h\phi)(P_i) = \phi(P_i), \quad i = 1, \dots, N.$$

Let  $L \equiv -\Delta$ . It is well-known that  $L_h$  has no local consistency [14, 16], that is, for  $\phi \in C_0^\infty(\Omega)$

$$\|(L_h I_h - I_h L)\phi\|_{\ell^\infty} \not\rightarrow 0, \quad (h \rightarrow +0).$$

To recover the local consistency we introduce the operator  $J_h$  [15],

$$J_h : C^{2+1}(\bar{\Omega}) \cap C_0(\bar{\Omega}) \rightarrow V_h$$

defined by

$$a(J_h\phi, v_h) = (L\phi, v_h), \quad \forall v_h \in V_h,$$

where  $V_h$  is considered to be the finite element space.  $J_h\phi$  is nothing but the discrete function whose nodal values are those of the P1-finite element solution for the Poisson equation subject to the homogeneous Dirichlet boundary conditions.

**Proposition 1** Suppose  $\phi \in C^{2+1}(\bar{\Omega}) \cap C_0(\bar{\Omega})$ . Then, there exists a constant  $c_1$  independent of  $h$  and  $\phi$ , and for any  $\epsilon > 0$  there exists a constant  $c_\epsilon$  independent of  $h$  and  $\phi$  such that

$$\|(L_h J_h - I_h L)\phi\|_{\ell^\infty} \leq c_1 h \|\phi\|_{2+1, \Omega}, \quad (15)$$

$$\|(I_h - J_h)\phi\|_{\ell^\infty} \leq c_\epsilon h^{2-\epsilon} \|\phi\|_{1+1, \Omega}. \quad (16)$$

In the case  $d = 3$  we can take  $\epsilon = 0$ .

*Proof.* Let  $\phi \in C^{2+1}(\bar{\Omega}) \cap C_0(\bar{\Omega})$ . From the definition of  $L_h$  and  $J_h$  we have for any  $P_i$ ,  $i = 1, \dots, N$ ,

$$\begin{aligned} (L_h J_h \phi)(P_i) &= \frac{1}{m_i} a(J_h \phi, w_{hi}) \\ &= \frac{1}{m_i} (L\phi, w_{hi}) \\ &= (L\phi)(P_i) + c_1 \theta_i h \|\phi\|_{2+1}. \end{aligned} \quad (17)$$

Here we have used the property  $m_i = (1, w_{hi})$  and the Taylor expansion to obtain the last estimate. From (17) we get (15). (16) is nothing but the  $L^\infty$ -error estimate for the  $P1$ -finite element method [2]. In the case of  $d = 2$  the power of  $-\epsilon$  is necessary to eliminate the logarithmic term  $|\ln h|$ , which appears in the estimate only in  $d = 2$ .

To prove the convergence we assume the regularity of the functions  $\phi$ ,  $u$ , and  $f$ .

### Hypothesis 2

$$\begin{aligned} \phi &\in C^{2+1,0} \cap C^{0+1,1} \cap C^{1,0+1} \cap C^{0,1+1}, \\ u &\in C(\bar{\Omega} \times [0, T]), \quad f \in C(\bar{\Omega} \times [0, T]). \end{aligned} \quad (18)$$

**Theorem 1** *Assume Hypotheses 1 and 2. Let  $\phi_h$  be the solution of (2). Then, for any  $\epsilon \in (0, 1)$  we have*

$$\|\phi_h - I_h \phi\|_{\ell^\infty(L^\infty)} \leq c_\epsilon \left( h + \Delta t + \frac{h^{2-\epsilon}}{\Delta t} \right), \quad (19)$$

where  $c_\epsilon$  is independent of  $\Delta t$  and  $h$ . By taking  $\Delta t = O(h)$  we have

$$\|u_h - I_h u\|_{\ell^\infty(L^\infty)} \leq c_\epsilon h^{1-\epsilon}. \quad (20)$$

In  $d = 3$  we can take  $\epsilon = 0$ .

*Proof.* Since  $\phi \in C^{2+1,0} \cap H_0^1(\Omega)$ , we set  $e_h \equiv \phi_h - J_h \phi \in V_h$ . From (16) we have

$$\begin{aligned} \|\phi_h - I_h \phi\|_{\ell^\infty(L^\infty)} &\leq \|e_h\|_{\ell^\infty(L^\infty)} + \|(J_h - I_h)\phi\|_{\ell^\infty(L^\infty)} \\ &\leq \|e_h\|_{\ell^\infty(L^\infty)} + c_\epsilon h^{2-\epsilon} \|\phi\|_{C^{1+1,0}}. \end{aligned} \quad (21)$$

From (13) we have

$$\begin{aligned}
& \frac{1}{\Delta t} (e_h^n - e_h^{n-1} \circ X_h^n) + \nu L_h e_h^n \tag{22} \\
&= f_h^n - \left\{ \frac{1}{\Delta t} (J_h \phi^n - (J_h \phi^{n-1}) \circ X_h^n) + \nu L_h J_h \phi^n \right\} \\
&= I_h f^n - \left\{ \frac{1}{\Delta t} (I_h \phi^n - (I_h \phi^{n-1}) \circ X_h^n) + \nu I_h L \phi^n \right\} \\
&\quad - \frac{1}{\Delta t} \{ (J_h - I_h) \phi^n - (J_h - I_h) \phi^{n-1} \circ X_h^n \} - \nu (L_h J_h - I_h L) \phi^n \\
&\equiv I_1 + I_2 + I_3 + I_4 \equiv F_h^n. \tag{23}
\end{aligned}$$

At node  $P_i$ ,  $I_1 + I_2$  is evaluated as

$$\begin{aligned}
& f^n(P_i) - \frac{1}{\Delta t} \{ \phi^n(P_i) - \phi^{n-1}(P_i - \Delta t u^n(P_i)) \} + \nu \Delta \phi^n(P_i) \\
&= f(P_i, n\Delta t) - \frac{\partial \phi}{\partial t}(P_i, n\Delta t) - (u \cdot \nabla \phi)(P_i, n\Delta t) + \nu \Delta \phi(P_i, n\Delta t) \\
&\quad + c \theta_i^n \Delta t (\|\phi\|_{(C^{0,1+1} \cap C^{1,0+1})(Q^n)} + \|u^n\|_{L^\infty} \|\phi\|_{(C^{1+1,0} \cap C^{0+1,1})(Q^n)}),
\end{aligned}$$

which implies

$$|I_1 + I_2| \leq c \Delta t (\|\phi\|_{(C^{0,1+1} \cap C^{1,0+1})(Q^n)} + \|u^n\|_{L^\infty} \|\phi\|_{(C^{1+1,0} \cap C^{0+1,1})(Q^n)}), \tag{24}$$

where  $Q^n \equiv \bar{\Omega} \times [(n-1)\Delta t, n\Delta t]$ . By virtue of (15) and (16),  $I_3$  and  $I_4$  are evaluated as

$$|I_3| \leq c_\epsilon \frac{h^{2-\epsilon}}{\Delta t} (\|\phi^n\|_{C^{1+1,0}(\bar{\Omega})} + \|\phi^{n-1}\|_{C^{1+1,0}(\bar{\Omega})}), \tag{25}$$

$$|I_4| \leq ch \|\phi^n\|_{C^{2+1,0}(\bar{\Omega})}. \tag{26}$$

Combining (24), (25) and (26), we obtain

$$\begin{aligned}
\|F_h^n\|_{\ell^1(\ell^\infty)} &\leq c_\epsilon \frac{h^{2-\epsilon}}{\Delta t} \|\phi\|_{C^{1+1,0}(Q^n)} + c(h \|\phi\|_{C^{2+1,0}(Q^n)} \\
&\quad + \Delta t (\|\phi\|_{(C^{0,1+1} \cap C^{1,0+1})(Q^n)} + \|u\|_{L^\infty} \|\phi\|_{(C^{1+1,0} \cap C^{0+1,1})(Q^n)})). \tag{27}
\end{aligned}$$

Applying Lemma 2 to (22), we have

$$\begin{aligned}
\|e_h\|_{\ell^\infty(\ell^\infty)} &\leq \|e_h^0\|_{\ell^\infty} + \|F_h^n\|_{\ell^1(\ell^\infty)} \\
&\leq c_\epsilon \frac{h^{2-\epsilon}}{\Delta t} \|\phi\|_{C^{1+1,0}} + c(h \|\phi\|_{C^{2+1,0}} \\
&\quad + \Delta t (\|\phi\|_{C^{0,1+1} \cap C^{1,0+1}} + \|u\|_{L^\infty} \|\phi\|_{C^{1+1,0} \cap C^{0+1,1}})).
\end{aligned}$$

The above estimate and (21) imply (19).

**Remark 4**  $X_h^n(x)$  defined by (3) is the first-order approximation of the characteristic curve. Now we replace it by the second-order approximation,

$$X_h^n(x) = x - u^n \left( x - u^n(x) \frac{\Delta t}{2} \right) \Delta t,$$

which improves the estimate (24) to  $\Delta t^2$ . Furthermore, if the triangulation is uniform, e.g., of Friedrichs-Keller type [16], we have a better estimate

$$\|(L_h J_h - I_h L)\phi\|_{\ell^\infty} \leq c_1 h^2 \|\phi\|_{3+1, \Omega}$$

in place of (15) [15]. In this case we have an improved result,

$$\|\phi_h - I_h \phi\|_{\ell^\infty(L^\infty)} \leq c_\epsilon \left( h^2 + \Delta t^2 + \frac{h^{2-\epsilon}}{\Delta t} \right). \quad (28)$$

By taking  $\Delta t = O(h^{2/3})$  we have

$$\|u_h - I_h u\|_{\ell^\infty(L^\infty)} \leq c_\epsilon h^{4/3-\epsilon}. \quad (29)$$

In  $d = 3$  we can take  $\epsilon = 0$ .

**Remark 5** The result of Theorem 1 can be extended to the reaction-convection-diffusion equation (8) and the estimate (19) holds for the scheme (9).

## 4 Numerical tests

### 4.1 The rotating hill

A point  $x^0 = (x_1^0, x_2^0)^T$  convected by  $u(x) = (x_2, -x_1)^T$  is in fact rotated at time  $t$  to  $x^0(t) = (x_1^0 \cos t + x_2^0 \sin t, -x_1^0 \sin t + x_2^0 \cos t)^T$ . So consider

$$\phi_\epsilon(x, t) = e^{-\lambda t + |x - x^0(t)|^2 r(t)}.$$

It verifies

$$\frac{\partial \phi_\epsilon}{\partial t} = (-\lambda + |x - x^0(t)|^2 \dot{r} - 2r \dot{x}^0 \cdot (x - x^0(t))) \phi_\epsilon,$$

$$u \cdot \nabla \phi_\epsilon = 2u \cdot (x - x^0(t)) r \phi_\epsilon, \quad \Delta \phi_\epsilon = 4r \phi_\epsilon + 4r^2 |x - x^0(t)|^2 \phi_\epsilon.$$

Since  $\dot{x}^0(t) \cdot (x - x^0(t)) = u \cdot (x - x^0(t))$ , we have

$$\frac{\partial \phi_\epsilon}{\partial t} + u \cdot \nabla \phi_\epsilon - \nu \Delta \phi_\epsilon + (\lambda + 4r\nu) \phi_\epsilon = |x - x^0(t)|^2 \phi_\epsilon (\dot{r} - 4r^2 \nu).$$

With  $r(t) = -1/(4\nu t + t_0)$  the right hand side is zero .

## 4.2 Convergence study

We consider the following problem: find  $\phi$  such that  $\phi = \phi_e$  initially and on the boundary and

$$\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi - \nu \Delta \phi + (\lambda + 4r\nu)\phi = 0.$$

Here we assume  $\lambda \geq 4\nu t_0^{-1}$ , which implies  $\lambda + 4r\nu \geq 0$ . Obviously the solution is  $\phi = \phi_e$ .

The problem is discretized by (2) with a second order scheme for  $X_h^n$ , namely

$$X_h^n(x) = x - u \left( x - u(x) \frac{\Delta t}{2} \right) \Delta t.$$

A Delaunay-Voronoi mesh generator is used for the triangulations of the unit circle. It does not necessarily gives a mesh of weakly acute type but it usually comes fairly close (Figure 2-left).

We tested meshes with 942, 2,023 and 7,876 vertices, corresponding respectively to 75, 150 and 300 boundary vertices. The time step is controlled by  $N_T$  which for each mesh is 23 then 45 then 90. The other parameters are

$$\begin{aligned} x_1^0 &= 0.35, \quad x_2^0 = 0.35, \quad T = 2\pi, \quad \Delta t = T/N_T, \\ \nu &= 0 \text{ or } 0.01, \quad t_0 = 0.2, \quad \lambda = 4\nu/t_0. \end{aligned}$$

Figure 2-right shows a log-log plot of the errors in both cases. The left (resp. right) plots of Figure 3 shows the exact and computed solutions after one turn of convection-diffusion-dissipation for the coarsest of the 3 meshes in the case  $\nu = 0$  (resp.  $\nu = 0.01$ ).

## 4.3 Comparison with two other methods

For comparison we have also implemented the SUPG/Least square Galerkin method and a Discontinuous-Galerkin method.

SUPG reads:

$$\int_{\Omega} \left( \frac{\phi^m - \phi^{m-1}}{\Delta t} + u \cdot \nabla \phi \right) (w_h + \alpha u \cdot \nabla w_h) + \int_{\Omega} \nu \nabla \phi^m \cdot \nabla w_h = 0.$$

Results are shown on Figure 4

With homogeneous Dirichlet conditions the dual Discontinuous-Galerkin methods reads:

$$\int_{\Omega} \left( \frac{\phi^m - \phi^{m-1}}{\Delta t} + u \cdot \nabla \phi^m \right) w_h + \nu \int_{\Omega} \nabla \phi^m \cdot \nabla w_h + \int_E w_h \left( \alpha |n \cdot u| - \frac{1}{2} n \cdot u \right) [\phi^m] = 0$$

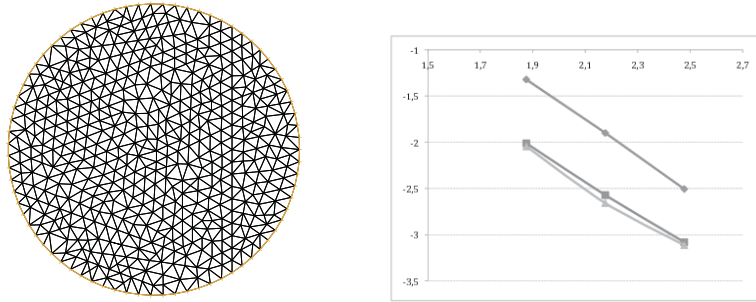


Figure 2: The coarsest of the 3 meshes (left) and a log-log plot of the error in the case  $\nu = 0$  and  $\nu = 0.01$ ; both seems somewhat better than order one

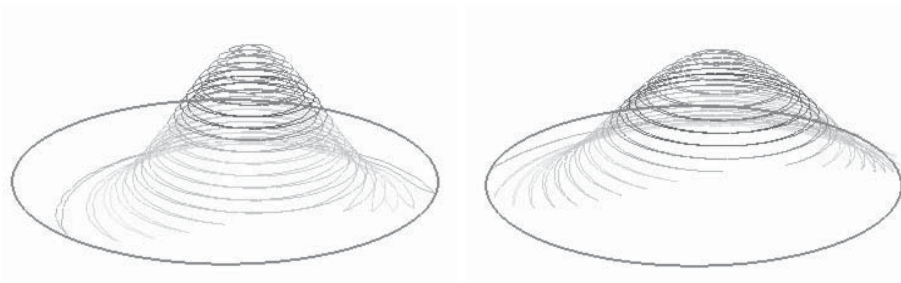


Figure 3: Exact and computed solutions after one turn of convection-diffusion-dissipation for the coarsest of the 3 meshes in the case  $\nu = 0$  (left) and  $\nu = 0.01$  (right)

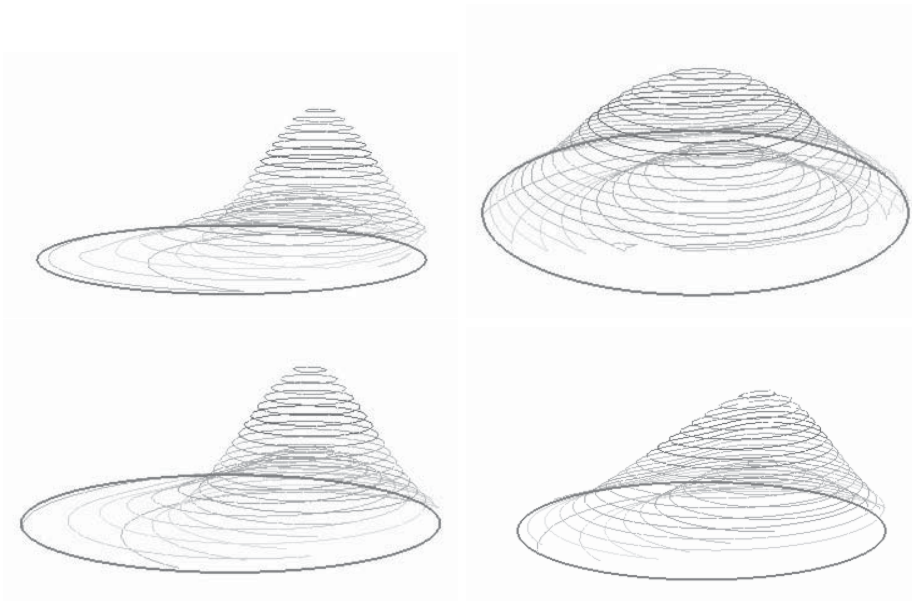


Figure 4: Top row: Performance of a SUPG-Galerkin least-square method with  $\alpha = 0.1$  in the case  $\nu = 0$  (left) and  $\nu = 0.01$  (right). Bottom row: Performance of a Discontinuous Galerkin method in the case  $\nu = 0$  (left) and  $\nu = 0.01$  (right). Note that the views are not from the same angle so as to enhance the vision of the difference between the exact and the computed curves (exact curves forms the higher hill of the two).

for all  $w_h \in V_h$ .

Here  $E$  is the set of inner edges and  $[b]$  is the jump of  $b$  from the local triangle to triangle on the other side of  $E$ . In the test shown on Figure 4  $\alpha = \frac{1}{2}$ . Table 1 shows the errors for the problem with diffusion  $\nu = 0.01$  integrated numerically without upwinding, with SUPG upwinding, and with the Galerkin-characteristics method.

Table 1:  $L^2$ -errors to compare several methods with and without diffusion. In both cases the Galerkin-characteristics method (G-C) out-perform the others. The numbers of time steps are proportional to the number of mesh points  $N$  on the boundary and the total number of mesh points is proportional to  $N^2$ .

N	$\nu = 0.01$			$\nu = 0$		
	Centered	SUPG	G-C	D-G	SUPG	G-C
75	6.96 e-2	6.65 e-2	9.74 e-3	2.88 e-1	2.87 e-1	4.06 e-2
150	4.47 e-2	4.29 e-2	2.27 e-3	2.14 e-1	2.13 e-1	1.02 e-2
300	2.66 e-2	2.56 e-2	8.31 e-4	1.48 e-1	1.46 e-1	2.54 e-3

#### 4.4 Tests in 3 dimensions

The same test can be performed in 3D with version 3 of **freefem++**. The geometry is the cube  $(-1, 1)^3$ . The convection field is a rotation about the  $x_3$ -axis,  $u = (x_2, -x_1, 0)^T$  and the initial solution is the analytical solution  $\phi_e$  at  $t = 0$ :

$$\phi_e(x, t) = e^{-\lambda t + |x - x^0(t)|^2 r(t)},$$

which satisfies

$$\frac{\partial \phi_e}{\partial t} + u \cdot \nabla \phi_e - \nu \Delta \phi_e + (\lambda + 6r\nu)\phi_e = 0,$$

when  $r(t) = -1/(4\nu t + t_0)$ .

The numerical implementation with **freefem++** is simple; the script is

```
int M=30;
real x0=0.35, y0=0.35, z0=0, dt=2*pi/M, nu=0, t0=0.2, lam=6*nu/t0, t=0;
mesh Th2=square(M,M, [x*2-1, y*2-1]);
mesh3 th=buildlayers(Th2,M, zbound=[-1,1]);
fespace Vh(th,P13d);
func ue = exp(-lam*t - ((x - (x0*cos(t) + y0*sin(t)))^2
```

```

    +(y-(-x0*sin(t) + y0*cos(t)))^2+(z-z0)^2)/(4*nu*t+t0));
Vh u,v,vo=ue
for( t=0;t<=T;t+=dt){
  solve onestep(u,v,solver=CG)
  = int3d(th)( nu*dt*(dx(u)*dx(v)+dy(u)*dy(v)+dz(u)*dz(v)))
  + int3d(th,qfV=qfV1lump)(u*v*(1+dt*(lam- 6*nu/(4*nu*t+t0))))
  - int3d(th,qfV=qfV1lump)( vo(x*(1-dt^2)-y*dt,y*(1-dt^2)+x*dt,z)*v)
  + on(1,u=ue);
  vo=u;
}

```

The results for the same parametric values as in 2D and  $z_0 = 0$ ,  $\nu = 0$  and  $\nu = 0.01$  are shown on Figures 5 and 6, respectively, for a uniform mesh of the unit cube corresponding to  $N = 30$  points on each edge of the cube. The decreasing of  $L^2$  errors  $e(N)$  as a function of  $N$  is as follows:

$$e(10) = 4.77 \times 10^{-2}, \quad e(20) = 9.99 \times 10^{-3}, \quad e(40) = 2.32 \times 10^{-3}.$$

## 5 Concluding remarks

We have presented a Galerkin-characteristics finite element scheme of lumped mass type for the convection diffusion problems. In this scheme no numerical quadrature is required. Under the weakly acute triangulation hypothesis we have proved the scheme is unconditionally stable and convergent in the  $L^\infty$ -norm. For the P1-element the convergence order is essentially  $O(h)$  by choosing  $\Delta t = O(h)$ . 2D and 3D numerical results have shown the robustness of the scheme and reflected the theoretical convergence result.

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We are grateful to Frédéric Hecht for his extension of `freefem++` to handle mass lumping in finite element integrals in 3D. The second author was supported by the Japan Society for the Promotion of Science under Grant-in-Aid for Scientific Research (S), No.20224001 and by the Ministry of Education, Culture, Sports, Science and Technology of Japan under Global COE Program, Mathematics for Industry.

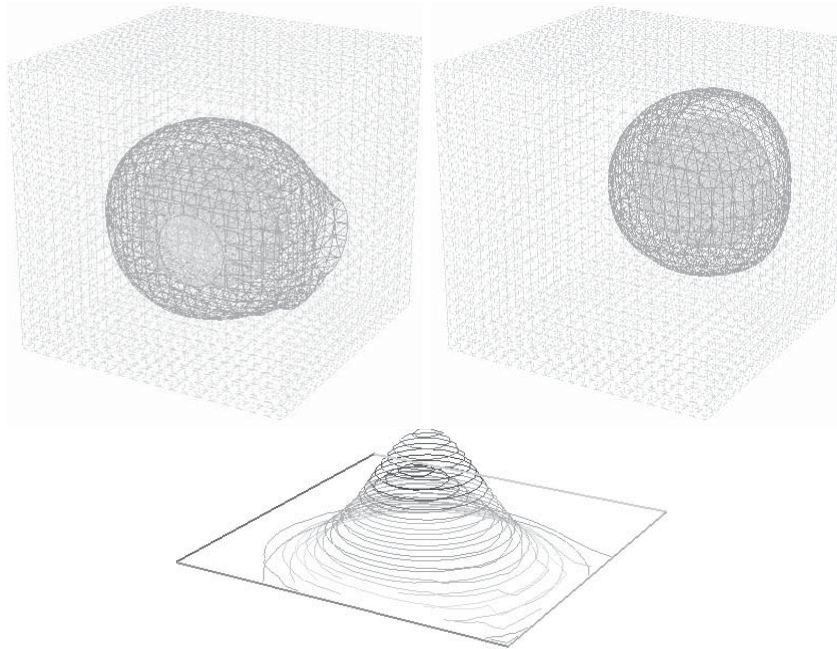


Figure 5: Galerkin-characteristics with mass lumping for the rotating hill in 3D with  $\nu = 0$ . Display of the solution at  $t = 4.0$  and after one turn; finally comparison with the analytical solution after one turn in the plane  $x_3 = 0$ .

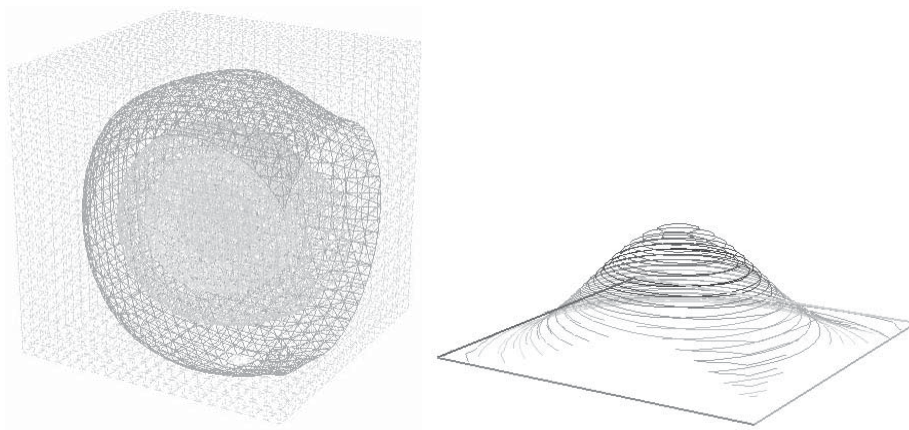


Figure 6: Galerkin-characteristics with mass lumping for the rotating hill in 3D with  $\nu = 0.01$ . Display of the solution after one turn and comparison with the analytical solution in the plane  $x_3 = 0$ .

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